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Fixed points for weak φ-contractions on partial metric spaces

Cristina Di Bari and Pasquale Vetro

Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34 - 90123 Palermo, Italy

dibari@math.unipa.it

vetro@math.unipa.it

Abstract. In this paper, following [W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory. 4 (2003) 79-89], we give a fixed point result for cyclic weak φ-contractions on partial metric space. A Maia type fixed point theorem for cyclic weak φ-contractions is also given.

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1 Introduction

Matthews [5] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. In [1, 3, 7, 9, 10] we have some generalizations of the result of Matthews. In this paper, we give a fixed point result for cyclic weak φ-contractions on partial metric space. A Maia type fixed point theorem for cyclic weak φ-contractions is also given. Our results generalize some interesting results of [4].

2 Preliminaries

First, we recall some definitions and some properties of partial metric spaces that can be found in [5, 7, 9, 10]. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty[$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $([0, +\infty[, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, +\infty[$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 2.1. Let (X,p) be a partial metric space.
(i) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.
(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.
(iv) A sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n,x_m) = 0$. We say that (X,p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0,+\infty[, p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x,y) = \max\{x,y\}$, provides an example of a 0-complete partial metric space which is not complete.

It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 2.2. *Let (X,p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to x \in X$ and $p(x,x) = 0$, then $\lim_{n\to+\infty} p(x_n,z) = p(x,z)$ for all $z \in X$.*

Proof. By the triangle inequality

$$p(x,z) - p(x_n,x) \le p(x_n,z) \le p(x,z) + p(x_n,x).$$

Letting $n \to +\infty$, we obtain that $p(x_n,z) \to p(x,z)$. □

Define $p(x,A) = \inf\{p(x,a) : a \in A\}$. Then $a \in \overline{A} \Leftrightarrow p(a,A) = p(a,a)$, where $\overline{A}$ denotes the closure of A.

3 Fixed point results for cyclic mappings

Let X be a nonempty set, m a positive integer and $T : X \to X$ a mapping. By definition a finite family $A_1, \ldots, A_m$ of nonempty subsets of X is a cyclic representation of X with respect to T if

(i) $\bigcup_{j=1}^m A_j = X$;

(ii) $T(A_1) \subset A_2,\ T(A_2) \subset A_3,\ \ldots,\ T(A_m) \subset A_1$.

Let (X,p) be a partial metric space, m a positive integer, $A_1, \ldots, A_m$ closed nonempty subsets of X and $Y = \bigcup_{j=1}^m A_j$. A mapping $T : Y \to Y$ is a cyclic weak φ-contraction if

(i) $A_1, \ldots, A_m$ is a cyclic representation of Y with respect to T;

(ii) there exists a nondecreasing function $\varphi : [0,+\infty[\to [0,+\infty[$, with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$, such that

$$p(Tx,Ty) \le p(x,y) - \varphi(p(x,y)) \tag{3.1}$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1, \ldots, m$, where $A_{m+1} = A_1$.

Example 3.1. Let $X = [0, +\infty[$ and $p : X \times X \to \mathbb{R}$ defined by $p(x, y) = \max\{x, y\}$, then (X, p) is a partial metric space. Let $A_1 = A_2 = \cdots = A_m = [0, 2]$ and $Y = \bigcup_{j=1}^{m} A_j$. Define $T : Y \to Y$ by $Tx = \frac{x}{1+x}$ for all $x \in Y$ and $\varphi : [0, +\infty[\to [0, +\infty[$ such that $\varphi(t) = \frac{t^2}{1+t}$. It is easy to show that T is a cyclic weak φ-contraction.

Denote with Φ the family of nondecreasing function $\varphi : [0, +\infty[\to [0, +\infty[$ continuous at 0, such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for each $t > 0$. Let $T : X \to X$ a mapping and set $Fix(T) = \{x \in X : x = Tx\}$.

Lemma 3.2. *Let (X, p) be a partial metric space, m a positive integer, $A_1, \ldots, A_m$ closed nonempty subsets of X and $Y = \bigcup_{j=1}^{m} A_j$. If $T : Y \to Y$ is a cyclic weak φ-contraction, then*

(i) *$p(Tx, Ty) \leq p(x, y)$ for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1, \ldots, m$;*

(ii) *$p(T^n x, T^{n+1} y) \to 0$ for all $x, y \in A_j$, $j = 1, \ldots, m$;*

(iii) *$p(T^{m(n+1)} x, T^{mn} x) \to 0$ for all $x \in A_j$, $j = 1, \ldots, m$;*

(iv) *If $z \in Fix(T)$, then $p(z, z) = 0$.*

Proof. Only properties (ii) and (iv) are nontrivial. First, we prove (ii). Let $x, y \in A_j$ and define $t_n = p(T^n x, T^{n+1} y)$, since T is a cyclic weak φ-contraction, we have

$$t_{n+1} \leq t_n - \varphi(t_n) \leq t_n, \qquad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Thus the sequence $\{t_n\}$ is nonincreasing and hence there exists $\alpha \geq 0$ such that $t_n \to \alpha$. We show that $\alpha = 0$. Assume $\alpha > 0$, then there exists n_0 such that $t_1 < n\varphi(\alpha)$ for all $n \geq n_0$. Now, by the monotonicity of φ for all $n \geq n_0$, we have

$$t_{n+1} \leq t_n - \varphi(\alpha) \leq t_{n-1} - 2\varphi(\alpha) \leq \cdots \leq t_1 - n\varphi(\alpha)$$

which is a contradiction and so $\alpha = 0$.

Property (iv) follows from

$$p(z, z) = p(Tz, Tz) \leq p(z, z) - \varphi(p(z, z))$$

which is possible only if $p(z, z) = 0$. □

Lemma 3.3. *Let (X, p) be a partial metric space, m a positive integer, $A_1, \ldots, A_m$ closed nonempty subsets of X and $Y = \bigcup_{j=1}^{m} A_j$. If $T : Y \to Y$ is a cyclic weak φ-contraction, given $x_0 \in A_j$ $(j = 1, \ldots, m)$, then for every $\varepsilon > 0$ there exists n_ε such that $p(T^{ms} x_0, T^{mn+1} x_0) < \varepsilon$ for all $s > n \geq n_\varepsilon$.*

Proof. Suppose the contrary. Then there exists $\varepsilon > 0$ such that for each $k \geq 1$, there exist $s_k > n_k \geq k$ so that

$$p(T^{ms_k} x_0, T^{mn_k+1} x_0) \geq \varepsilon \quad \text{ and } p(T^{m(s_k-1)} x_0, T^{mn_k+1} x_0) < \varepsilon.$$

From

$$\begin{aligned} \varepsilon &\leq p(T^{ms_k} x_0, T^{mn_k+1} x_0) \\ &\leq p(T^{ms_k} x_0, T^{m(s_k-1)} x_0) + p(T^{m(s_k-1)} x_0, T^{mn_k+1} x_0) \\ &\leq p(T^{ms_k} x_0, T^{m(s_k-1)} x_0) + \varepsilon, \end{aligned}$$

by Lemma 3.2, it follows that $\lim_{k \to +\infty} p(T^{ms_k} x_0, T^{mn_k+1} x_0) = \varepsilon$.

Since, by Lemma 3.2, $p(T^{m(s_k+1)}x_0, T^{m(n_k+1)+1}x_0) \leq p(T^{ms_k+1}x_0, T^{mn_k+2}x_0)$, we have

$$\begin{aligned}
&p(T^{ms_k}x_0, T^{mn_k+1}x_0)\\
&\leq p(T^{ms_k}x_0, T^{m(s_k+1)}x_0) + p(T^{m(s_k+1)}x_0, T^{m(n_k+1)+1}x_0) + p(T^{m(n_k+1)+1}x_0, T^{mn_k+1}x_0)\\
&\leq p(T^{ms_k}x_0, T^{m(s_k+1)}x_0) + p(T^{ms_k+1}x_0, T^{mn_k+2}x_0) + p(T^{m(n_k+1)+1}x_0, T^{mn_k+1}x_0)\\
&\leq p(T^{ms_k}x_0, T^{m(s_k+1)}x_0) + p(T^{ms_k}x_0, T^{mn_k+1}x_0) - \varphi(\varepsilon) + p(T^{m(n_k+1)+1}x_0, T^{mn_k+1}x_0).
\end{aligned}$$

Letting $k \to +\infty$, by Lemma 3.2, we obtain

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon),$$

which is a contradiction. Consequently, for every $\varepsilon > 0$ there exists n_ε such that

$$p(T^{ms}x_0, T^{mn+1}x_0) < \varepsilon$$

for all $s > n \geq n_\varepsilon$. □

Lemma 3.4. *Let (X, p) be a partial metric space, m a positive integer, $A_1, \ldots, A_m$ closed nonempty subsets of X, $Y = \bigcup_{j=1}^m A_j$ and $T : Y \to Y$ a cyclic weak φ-contraction. Assuming that there exist a sequence $\{y_n\} \subset Y$ such that $p(y_n, Ty_n) \to 0$ as $n \to +\infty$ and $z \in Fix(T)$, then $y_n \to z$, as $n \to +\infty$. Moreover, T has at most one fixed point.*

Proof. Assume that the sequence $\{y_n\}$ doesn't converge to z, then $\limsup_{n\to+\infty} p(y_n, z) = \alpha > 0$. Let $N = \{n : p(y_n, Ty_n) < \varphi(\frac{\alpha}{2})$ *and* $p(y_n, z) > \frac{\alpha}{2}\}$. For all $n \in N$, we have

$$\begin{aligned}
p(y_n, z) &\leq p(y_n, Ty_n) + p(Ty_n, Tz) - p(Ty_n, Ty_n)\\
&\leq p(y_n, Ty_n) + p(y_n, z) - \varphi(\frac{\alpha}{2})\\
&< p(y_n, z),
\end{aligned}$$

which is a contradiction and so the sequence $\{y_n\}$ converges to z. Lemma 3.2 ensures that there exist sequences $\{y_n\} \subset Y$ such that $p(y_n, Ty_n) \to 0$ as $n \to +\infty$. We show that T has at most one fixed point. Assume the contrary and let $w \in Fix(T)$. From

$$\begin{aligned}
p(z, w) &\leq p(z, y_n) + p(y_n, w) - p(y_n, y_n)\\
&\leq p(z, y_n) + p(y_n, w),
\end{aligned}$$

letting $n \to +\infty$, since $p(z, y_n), p(y_n, w) \to 0$, we get $p(z, w) \leq 0$ and so $z = w$. □

The following theorem of fixed point in a partial metric space is our main results.

Theorem 3.5. *Let (X, p) be a partial metric space, m a positive integer, $A_1, \ldots, A_m$ 0-complete nonempty subsets of X and $Y = \bigcup_{j=1}^m A_j$. If $T : Y \to Y$ is a cyclic weak φ-contraction with $\varphi \in \Phi$, then T has a unique fixed point $z \in \bigcap_{j=1}^m A_j$.*

Proof. Let $x_0 \in Y = \bigcup_{j=1}^m A_j$ and $\varepsilon > 0$. Let $\{x_n\}$ be the Picard iteration defined by $x_n = Tx_{n-1}$ for all n. By Lemmas 3.2 and 3.3, there exists n_ε such that

$$p(x_{mn+1}, x_{mn}) < \frac{\varepsilon}{2} \quad \textit{and} \quad p(x_{ms}, x_{mn+1}) < \frac{\varepsilon}{2}$$

for all $s > n \geq n_\varepsilon$. This implies

$$\begin{aligned}
p(x_{ms}, x_{mn}) &\leq p(x_{ms}, x_{mn+1}) + p(x_{mn+1}, x_{mn}) - p(x_{mn+1}, x_{mn+1})\\
&\leq p(x_{ms}, x_{mn+1}) + p(x_{mn+1}, x_{mn}) < \varepsilon
\end{aligned}$$

for all $s > n \geq n_\varepsilon$. Consequently, $\lim_{s,n\to+\infty} p(x_{ms}, x_{mn}) = 0$ and hence $\{x_{mn}\}$ is a 0-Cauchy sequence. Now, also, $\{x_n\}$ is a 0-Cauchy sequence, by Lemma 3.2. Since Y is 0-complete there exists $z \in Y$ such that $p(x_n, z) \to p(z, z) = 0$. Also, $x_{mn+j} \to z$ for $j = 0, 1, \ldots, j-1$. This implies that $z \in \bigcap_{j=1}^m A_j$, since each A_j is 0-complete. We show that $z = Tz$. From

$$\begin{aligned} p(z, Tz) &\leq p(z, x_{n+1}) + p(Tx_n, Tz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(x_n, z) - \varphi(p(x_n, z)) \end{aligned}$$

and $\lim_{n\to+\infty} \varphi(p(x_n, z)) = 0$, letting $n \to +\infty$, we get $p(z, Tz) \leq 0$. This implies that $z = Tz$ and hence z is a fixed point of T. The uniqueness of the fixed point is obvious. □

Theorem 3.6. *Let (X, p) be a partial metric space, m a positive integer, $A_1, \ldots, A_m$ 0-complete nonempty subsets of X, $Y = \bigcup_{j=1}^m A_j$ and $T : Y \to Y$ a cyclic weak φ-contraction, with $\varphi \in \Phi$. Assuming that there exists a sequence $\{y_n\} \subset Y$ such that $p(y_n, y) \to p(y, y) = 0$ and $p(y_{n+1}, Ty_n) \to 0$ as $n \to +\infty$, then for all $x \in Y$ we have that $\lim_{n\to+\infty} p(y_n, T^n x) = 0$.*

Proof. By Theorem 3.5, T has a unique fixed point z such that $p(z, z) = 0$. Now, by Lemma 2.2, $\lim_{n\to+\infty} p(y_n, z) = p(y, z)$. If $y \neq z$, then $p(y, z) > 0$ and thus there is $\overline{n}$ such that $p(y_n, z) \geq p(y, z)/2$ for all $n \geq \overline{n}$.

From

$$\begin{aligned} p(y_{n+1}, z) &\leq p(y_{n+1}, Ty_n) + p(Ty_n, Tz) - p(Ty_n, Ty_n) \\ &\leq p(y_{n+1}, Ty_n) + p(y_n, z) - \varphi(\frac{p(y, z)}{2}), \end{aligned}$$

for all $n \geq \overline{n}$, letting $n \to +\infty$ we deduce that

$$p(y, z) \leq p(y, z) - \varphi(\frac{p(y, z)}{2}),$$

which is possible only if $p(y, z) = 0$, that is if $y = z$. For all $x \in Y$, by Lemmas 3.2 and 3.4, we have

$$p(y_{n+1}, T^n x) \leq p(y_{n+1}, z) + p(z, T^n x) \to 0, \text{ as } n \to +\infty.$$

□

Now, if we choose the function $\varphi : [0, +\infty[\to [0, +\infty[$ such that $\varphi(t) = (1 - k)t$ for all t, where $k \in]0, 1[$, from Theorem 3.5, we obtain the following corollary.

Corollary 3.7. ([4], Theorem 1.3). *Let $A_1, \ldots, A_m$ be a finite family of nonempty closed subsets of a complete metric space (X, d), and suppose $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ satisfies the following conditions:*

(i) $T(A_1) \subset A_2,\ T(A_2) \subset A_3,\ \ldots,\ T(A_m) \subset A_1$

(ii) *there exists $k \in]0, 1[$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A_i, y \in A_{i+1}$ for $1 \leq i \leq m$, where $A_{m+1} = A_1$.*

Then T has a unique fixed point.

Denote with Ψ the family of functions $\psi : [0, +\infty[\to [0, +\infty[$ such that the function $\varphi : [0, +\infty[\to [0, +\infty[$ defined by $\varphi(t) = t - \psi(t)$ belongs to Φ. From Theorem 3.5, we obtain the following result of Boyd-Wong type [2].

Corollary 3.8. *Let $A_1, \ldots, A_m$ be a finite family of nonempty closed subsets of a complete metric space (X, d), and suppose $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ satisfies the following conditions:*

(i) $T(A_1) \subset A_2,\ T(A_2) \subset A_3,\ \ldots,\ T(A_m) \subset A_1$

(ii) *there exists* $\psi \in \Psi$ *such that* $d(Tx, Ty) \leq \psi(d(x, y))$ *for all* $x \in A_i, y \in A_{i+1}$ *for* $1 \leq i \leq m$, *where* $A_{m+1} = A_1$.

Then T *has a unique fixed point.*

Maia type result regarding cyclic weak φ-contractions with $\varphi \in \Phi$ is given in the following theorem.

Theorem 3.9. *Let* X *be a nonempty set,* p *and* ρ *two partial metrics on* X, m *a positive integer,* $A_1, \ldots, A_m$ *closed nonempty subsets of* (X, p), $Y = \bigcup_{j=1}^m A_j$ *and* $T : Y \to Y$. *Assuming that*

(i) $A_1, \ldots, A_m$ *is a cyclic representation of* Y *with respect to* T*;*

(ii) $p(x, y) \leq \rho(x, y)$, *for any* $x, y \in Y$*;*

(iii) (Y, p) *is a 0-complete partial metric space;*

(iv) $T : (Y, p) \to (Y, p)$ *is continuous;*

(v) $T : (Y, \rho) \to (Y, \rho)$ *is a cyclic weak* φ*-contraction with* $\varphi \in \Phi$.

Then T *has a unique fixed point.*

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