Minima Domain Intervals and the S-convexity, as well as the Convexity, Phenomenon

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Abstract. In this note, we propose refinement in the analytical definition of the s_2-convex classes of functions aiming progressing further in the direction of including S_2-convexity properly in the body of Real Analysis.

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1   Introduction

We have been working on refining and improving, as well as fixing, the definition of the s_2-convex classes of functions for a while now. The concept $S-$convexity is supposed to be a proper extension of the concept Convexity, what then implies that some of our proposed modifications may, if 'universally' accepted, affect the definition of the convex class of functions.

This piece of work regards argumenting in favor of an explicit reference to a minimum necessary interval of domain in the analytical definition of the s_2-convex classes of functions in order to both test the s_2-convexity pertinence requirement accurately and state it in a scientific manner, obeying the rigors of Real Analysis.

2   Analytical definition of the phenomenon S_2-convexity and symbology

Symbols [1]

- K_s^2 stands for the set of $s-$convex classes of type 2;

- $0 < s \leq 1$ is the real number to replace s_2 in the expression "s_2-convex", therefore the number to actually select one class amongst the so many possible s_2-convex classes of functions;

- K_1^2 is a mathematical synonym for the class convex functions.

Definitions [2]

Definition 2.1. A function $f : X \to \mathbb{R}$, where $|f(x)| = f(x)$, is told to belong to K_s^2 if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all $\lambda \in [0, 1]$, $\{x, y\} \subset X$, $s = s_2 \,|\, 0 < s_2 \leq 1$, $X \subseteq \mathbb{R}_+$.

Definition 2.2. A function $f : X \to \mathbb{R}$, where $|f(x)| = -f(x)$, is told to belong to K_s^2 if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^{\frac{1}{s}} f(x) + (1 - \lambda)^{\frac{1}{s}} f(y)$$

holds for all $\lambda \in [0, 1]$, $\{x, y\} \subset X$, $s = s_2 \,|\, 0 < s \leq 1$, $X \subseteq \mathbb{R}_+$.

Remark 2.3. If[1] the inequality is obeyed in the reverse situation by f then f is said to be s_2-concave.

3   Geometric definition of the phenomenon S_2-convexity

Definition 3.1. A real function $f : X \to Y$ is called convex if and only if for all choices (x_1, y_1) and (x_2, y_2), where $\{x_1, x_2\} \subset X$, $\{y_1, y_2\} \subset Y$, $Y = Im_f$, $x_1 \neq x_2$, and $y_1 \neq y_2$, it happens that the chord drawn between (x_1, y_1) and (x_2, y_2) does not contain any point with height, measured against the vertical cartesian axis, that is inferior to its horizontal equivalent in the curve representing the ordered pairs of f in the interval considered for the chord in terms of distance from the origin of the cartesian axis.

Remark 3.2. Notice that, to extend the above geometric definition, one needs to come up with a limiting geometric line connecting the same set of points $((x_1, y_1), (x_2, y_2))$, but a line that lies above the chords in at least some piece of it. Furthermore, it is clearly the case that we must have a continuous line as curve for f (otherwise, we give margin to doubts in the mathematical decisions to be made when comparing the chord with the line), what then implies that if a real function f is s_2-convex in $[a, b]$ then it is continuous in $[a, b]$.

4   Argumentation defending explicit reference to a minimum interval of domain in the analytical definition of the s_2-convex class of functions

One of the major criticisms to $S-$convexity is that clear inconsistency is found when putting the geometrical definition against the analytical one.

The geometric definition of Convexity, therefore also of $S-$convexity, DOES imply that the right side of the inequality forms a line, not mattering if curved or straight. The only way to generate this line, minimum condition of existence for it, is that $x \neq y$, therefore $X \neq \{x\}$ for all $x \in \mathbb{R}$. One needs more than one point to form a line; In fact, an infinite number of points is needed, therefore an interval, a *minimum interval*. Once we do not find mention to such an interval so far in the definitions, there is a clear omission in the analytical definition of $S-$convexity, therefore in the definition of convexity.

The necessity of the minimum interval is delivered by two facts: The first fact is that x must be different from y in all definitions (coherence with the geometric definition) and the second fact is that the mention to a piece X, from the real numbers, usually excludes the possibility of X being a degenerated interval.

Nothing could be more appropriate than adding a 'distance element' to the current analytical definition in order to make it more scientifically accurate, therefore. Let's call this distance element δ.

Given the nature of δ, read from the previous paragraphs, we may guarantee that $\delta > 0$.

To accept the replacement of y with $(x + \delta)$, notice that the current definition of $S-$convexity deals with a vector in $\mathbb{R}^2$ ((x, y)), but its current domain is in $\mathbb{R}$ and so we want it to be in

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[1]This remark applies to both definitions preceding it.

order to have the analytical definition matching the geometrical one.

To take away one dimension from the definition, it suffices that we replace y with $(x + \delta)$, action that will also address our minima interval problem and, therefore, will be our choice of action.

5 Proposed definition

Definition 5.1. A function $f : X \to \mathbb{R}$, where $|f(x)| = f(x)$, is told to belong to K_s^2 if the inequality

$$f(\lambda x + (1 - \lambda)(x + \delta)) \leq \lambda^s f(x) + (1 - \lambda)^s f(x + \delta)$$

holds for all $\lambda \in [0, 1]$, $x \in X$, $s = s_2 \,|\, 0 < s_2 \leq 1$, $X \subseteq \mathbb{R}_+ \wedge X = [a, b]$; for all $0 < \delta \leq (b - x)$.

Definition 5.2. A function $f : X \to \mathbb{R}$, where $|f(x)| = -f(x)$, is told to belong to K_s^2 if the inequality

$$f(\lambda x + (1 - \lambda)(x + \delta)) \leq \lambda^{\frac{1}{s}} f(x) + (1 - \lambda)^{\frac{1}{s}} f(x + \delta)$$

holds for all $\lambda \in [0, 1]$, $x \in X$, $s = s_2 \,|\, 0 < s_2 \leq 1$, $X \subseteq \mathbb{R}_+ \wedge X = [a, b]$; for all $0 < \delta \leq (b - x)$.

Remark 5.3. If[2] the inequality is obeyed in the reverse situation by f then f is said to be s_2-concave.

6 Conclusion

In this short note, we have proposed more modifications to the analytical definition of the phenomenon $S-$convexity.

This time, our proposed modifications, if accepted, will affect the number of variables and the domain interval in the definition of S_2-convexity.

Our proposed wording for the definition of the phenomenon S_2-convexity brings $(x + \delta)$ in place of y and limits δ in order to both guarantee that the domain interval be not degenerated and $(x + \delta)$ be inside of the boundaries of the domain.

References

[1] M.R. Pinheiro, Convexity Secrets, Trafford Publishing (2008), ISBN: 1425138217.

[2] M.R. Pinheiro, First Note on the Definition of S_2-convexity, Advances in Pure Mathematics. 1 (1) (2011) 1-2.

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[2]This remark applies to both definitions preceding it.