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SYMMETRY OF JUMP-DIFFUSION STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we define admitted Lie symmetry of jump-diffusion stochastic differential equations. Determining equations are derived in an Itô calculus context and were found to be non-stochastic even though they represent a stochastic process. Applications to some stochastic differential equations are presented and later showed the Lie bracket relations between the infinitesimals generators.

1. Introduction

Lie symmetry theory of ordinary differential equations is well understood in literature [1, 2, 3, 4] and can used for many important applications in the context of differential equations. Lie's classical approach is based on finding a symmetry group associated with the differential equation, it is a local Lie group of point transformations on the space of independent and dependent variables of differential equation that maps solutions to solutions.

In contrast to the deterministic differential equations, only a few attempts have been made to extend Lie group theory to the stochastic differential equation. It is worth noting that the theory is still developing. Lie symmetries of Wiener process stochastic differential equation were discussed in [5, 6, 9, 7, 12, 8] which is based on the standard method of the random time change of Brownian motion [10]. That is, the Wiener process is transformed as

$$d\overline{W}(\bar{t}) = \sqrt{\frac{d\bar{t}(t)}{dt}}dW(t). \tag{1.1}$$

In [13], we derived a similar random time change formula for Poisson processes in the context of Lie point symmetries by ensuring the instantaneous mean and variance of the Poisson process remained invariant under Lie group transformations. i.e.,

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{1.2}$$

In this paper, we extended the Lie symmetry methods to the class of Itô stochastic differential equations (SDE) driven by both Wiener and Poisson processes [15];

$$dX_i(t) = f_i(t, X(t))dt + G_{il}(t, X(t))dW_l(t) + J_i(t, X(t))dN(t) \tag{1.3}$$

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i.e., jump-diffusion stochastic differential equation. $f_i(t, X(t))$ and $J_i(t, X(t))$ are $n \times 1$ dimensional drift vector coefficients and jump diffusion coefficients respectively. $G_{il}(t, X(t))$ is the Wiener diffusion matrix coefficient of $n \times M$ dimensions, $dW(t)$ is called the infinitesimal increment of the Wiener process, while $dN(t)$ is called the infinitesimal increment of the Poisson process.

To ensure the existence and uniqueness of the solution of (1.3), the instantaneous drift coefficient $f_i(t, X(t))$, Wiener diffusion coefficient $G_{ik}(t, X(t))$ and the jump diffusion coefficient $J_i(t, X(t))$ are assumed to comply with Ikeda-Watanabe conditions [14].

The Lie point symmetries of (1.3) are discussed by considering infinitesimals involving the spatial variable x and time variable t, using the generating operator

$$H = \tau(t, x)\frac{\partial}{\partial t} + \xi_i(t, x)\frac{\partial}{\partial x_i}. \tag{1.4}$$

For an arbitrary function $F(t, X(t))$ which is twice contentiously differential with respect to spatial coordinates x, and differentiable once with respect to time t, then by the Itô lemma for jump diffusion process, the Itô process $F(t, X(t))$ of (1.3) exists as

$$\begin{aligned} dF_j(t, X(t)) = &\left(\frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t, X(t))G_{mk}(t, X(t))\frac{\partial^2 F_j}{\partial x_i \partial x_m}\right)dt \\ &+G_{il}(t, X(t))\frac{\partial F_j}{\partial x_i}dW(t) + \Big(F_j\Big(t, X_i(t) + J(t, X_i(t))\Big) - F_j(t, X_i(t))\Big)dN(t). \end{aligned} \tag{1.5}$$

The Einstein summation convention is assumed through out. For the matter of convenience let introduce the following operators;

$$\Gamma_{(F)_j} = \frac{\partial F_j}{\partial t}dt + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_{ik}(t, X(t))G_{mk}(t, X(t))\frac{\partial^2 F_j}{\partial x_i \partial x_m}, \tag{1.6}$$

$$\Gamma^*_{(F)_j} = G_{il}(t, X(t))\frac{\partial F_j}{\partial x_i} \tag{1.7}$$

and

$$\Gamma^{**}_{(F)_j} = F_j(t, X_i(t) + J(t, X_i(t))) - F_j(t, X_i(t)). \tag{1.8}$$

Therefore, the Itô process (1.5) can be rewritten as

$$dF_j(t, X(t)) = \Gamma_{(F)_j}dt + \Gamma^*_{(F)_j}dW(t) + \Gamma^{**}_{(F)_j}dN(t). \tag{1.9}$$

2. Lie Group Transformations

Consider a one parameter group of transformations of time index t and the spatial variable x respectively,

$$\overline{t} = \theta_1(x, t, \epsilon), \qquad \overline{x} = \theta_2(x, t, \epsilon)$$

with the infinitesimals

$$\frac{\partial \theta_1}{\partial \epsilon} = \tau(t, x), \qquad \frac{\partial \theta_2}{\partial \epsilon} = \xi(t, x).$$

Satisfying the initial conditions below, where ϵ is the parameter of the group

$$\bar{t}\Big|_{\epsilon=0} = t, \qquad \overline{X}(\bar{t})\Big|_{\epsilon=0} = X(t).$$

The one parameter Lie group of infinitesimal transformations is therefore

$$\bar{t} = t + \epsilon\tau(t,x), \tag{2.1}$$

$$\overline{X_j}(\bar{t}) = X_j(t) + \epsilon\xi_j(t,x), \tag{2.2}$$

with the corresponding infinitesimal generator of the Lie algebra

$$H = \tau(t, X(t))\frac{\partial}{\partial t} + \xi_i(t, X_i(t))\frac{\partial}{\partial x_i}.$$

The differential group transformation of the temporal, spatial, Wiener process and the jump process variables are

$$d\bar{t} = dt + \epsilon d\tau + O(\epsilon), \tag{2.3}$$

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon d\xi_j + O(\epsilon), \tag{2.4}$$

$$d\overline{W_j}(\bar{t}) = dW_j(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}dW_j(t) + O(\epsilon), \tag{2.5}$$

and

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{2.6}$$

While using the Itô formula (1.9), the spatial and temporal infinitesimals in Itô forms can be written respectively as

$$d\xi_j = \Gamma_{(\xi)_j}(t, X(t))dt + \Gamma^{*}_{(\xi)_j}(t, X(t))dW(t) + \Gamma^{**}_{(\xi)_j}(t, X(t))dN(t) \tag{2.7}$$

and

$$d\tau = \Gamma_{(\tau)}(t, X(t))dt + \Gamma^{*}_{(\tau)}(t, X(t))dW(t) + \Gamma^{**}_{(\tau)}(t, X(t))dN(t). \tag{2.8}$$

Substituting the Itô forms of the spatial infinitesimal (2.7) and the temporal infinitesimal (2.8) into equations (2.3), (2.4) and (2.5), the point group transformation of the spatial, temporal and Wiener processes respectively can be rewrite as

$$d\bar{t} = dt + \epsilon\Big(\Gamma_{(\tau)}(t, X(t))dt + \Gamma^{*}_{(\tau)}(t, X(t))dW(t) + \Gamma^{**}_{(\tau)}(t, X(t))dN(t)\Big) + O(\epsilon), \tag{2.9}$$

$$d\overline{X_j}(\bar{t}) = dX_j(t) + \epsilon\Big(\Gamma_{(\xi)_j}dt + \Gamma^{*}_{(\xi)_j}dW_j(t) + \Gamma^{**}_{(\xi)_j}dN(t)\Big)(t, X(t)) + O(\epsilon) \tag{2.10}$$

and

$$d\overline{W_j}(\bar{t}) = dW_j(t) + \frac{\epsilon}{2}\Big(\Gamma_{(\tau)}(t, X(t))dW_j(t) + \Gamma^{*}_{(\tau)}(t, X(t))\Big) + O(\epsilon). \tag{2.11}$$

Similarly, substituting the Itô temporal infinitesimal of the (2.8) into (2.6), the jump process variables group transformation can be rewritten in Itô form as

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{\Gamma_{(\tau)}dt + \Gamma^{*}_{(\tau)}dW_j(t) + \Gamma^{**}_{(\tau)}dN(t)}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{2.12}$$

After transforming the temporal, spatial, Wiener process and Poisson process infinitesimals using one parameter group of transformation in Itô context, we are now going to proceed next to find the transformed drift vector, Wiener diffusion and Poisson process coefficients using our infinitesimal generator in the subsequent section.

2.1. Invariance Form Of The Spatial Process. To ensure the recovery of the finite transformations from the infinitesimal transformation, we need to transform (1.3) into

$$d\overline{X_j}(\bar{t}) = \overline{f_j}(\bar{t}, \overline{X}(\bar{t}))d\bar{t} + \overline{G_{jk}}(\bar{t}, \overline{X}(\bar{t}))d\overline{W}(\bar{t}) + \overline{J_j}(\bar{t}, \overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{2.13}$$

In order to accomplish this, we need to transform the drift vector coefficient $f_j(t, X(t))$, the Wiener diffusion coefficient $G_{jk}(t, X(t))$ as well as the jump diffusion coefficient $J_j(t, X(t))$ using the infinitesimal generator (1.4) i.e.,

$$H = \tau(t, x)\frac{\partial}{\partial t} + \xi_i(t, x)\frac{\partial}{\partial x_i}.$$

The drift vector coefficient, the Wiener diffusion coefficient, and the jump diffusion coefficient can be transformed respectively as follows

$$\begin{aligned} \overline{f_j}(\bar{t}, \overline{X}(\bar{t})) &= (f_j + \epsilon H(f_j))\,(t, X(t)) \\ &= f_j(t, X(t)) + \epsilon\left(\tau\frac{\partial f_j}{\partial t} + \xi_i\frac{\partial f_j}{\partial x_i}\right)(t, X(t)), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \overline{G^l_j}(\bar{t}, \overline{X}(\bar{t})) &= (G_{jk} + \epsilon H(G_{jk}))\,(t, X(t)) \\ &= G_{jk}(t, X(t)) + \epsilon\left(\tau\frac{\partial G_{jk}}{\partial t} + \xi_i\frac{\partial G_{jk}}{\partial x_i}\right)(t, X(t)). \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \overline{J_j}(\bar{t}, \overline{X}(\bar{t})) &= (J_j + \epsilon H(J_j))\,(t, X(t)) \\ &= J_j(t, X(t)) + \epsilon\left(\tau\frac{\partial J_j}{\partial t} + \xi_i\frac{\partial J_j}{\partial x_i}\right)(t, X(t)). \end{aligned} \tag{2.16}$$

In the next sections, we ensure the properties of both Wiener and Poisson process moments remains invariant under Lie group transformation. This will help us to obtain extra conditions for the determining equations of jump-diffusion stochastic differential equation (1.3) [11], and ensure (1.3) remains unchanged under the Lie group of transformations.

2.2. Wiener Invariance Properties. We apply the invariance to the moments of the Wiener process to make sure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Wiener process which are:

$$E_Q\left[dW(t)|W = w\right] = 0 \tag{2.17}$$

and

$$E_Q\left[dW_l(t)dW_m(t)|W = w\right] = \delta^l_m dt. \tag{2.18}$$

The invariance of the instantaneous mean of the transformed Wiener process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{W}(t)|W = w\right] = 0. \tag{2.19}$$

Expanding (2.19) using (2.11) gives

$$E_{\overline{Q}}\left[dW(t) + \frac{\epsilon}{2}\left(\Gamma_{(\tau)}(t, X(t))dW(t) + \Gamma^*_{(\tau)}(t, X(t))\right) + O(\epsilon)\right] = 0, \tag{2.20}$$

simplifying (2.20) and the use of instantaneous mean property (2.17) yields

$$\Gamma^*_{(\tau)}(t, X(t)) = 0. \tag{2.21}$$

Next, we apply the invariance form to instantaneous variance of the transformed Wiener process measure i.e., using (2.11) and (2.21) we get

$$\begin{aligned} E_{\overline{Q}}\left[d\overline{W}_l(t)d\overline{W}_m(t)|W = w\right] = E_{\overline{Q}}\left[dW_l(t)dW_m(t)|W = w\right] \\ +\epsilon E_{\overline{Q}}\left[\left(\frac{\Gamma_{(\tau)}dt + \Gamma^*_{(\tau)}dW(t) + \Gamma^{**}_{(\tau)}dN}{dt}\right)d\overline{W}_l(t)d\overline{W}_m(t)|W = w\right]. \end{aligned} \tag{2.22}$$

Expanding (2.22) gives

$$E_{\overline{Q}}\left[d\overline{W}_l(t,w)d\overline{W}_m(t,w)|W = w\right] = \delta^l_m d\bar{t}. \tag{2.23}$$

Equation (2.22) implies that instantaneous variance remain invariant under the Lie group of transformations.

Remark 2.1. We have seen that applying the invariance transformation to the mean and variance of the Wiener process lead to additional conditions for the determining equations. This is the same extra condition obtained by [11, 5, 8] when investigating the symmetry of stochastic equations driven by Wiener processes.

2.3. Poisson Invariance Properties. Similarly, we apply the invariance to moments of the Poisson process to make sure it remains invariant under the Lie group transformations, *viz* the instantaneous mean and variance of the Poisson process which are;

$$E_Q\left[dN(t)\right] = \lambda dt \tag{2.24}$$

and

$$E_Q\left[dN(t)dN(t)\right] = \lambda dt. \tag{2.25}$$

The invariance of the instantaneous mean of the transformed Poisson process under new measure $\overline{Q}$ is

$$E_{\overline{Q}}\left[d\overline{N}(t)\right] = \lambda d\bar{t}. \tag{2.26}$$

Expanding (2.26) using (2.12) and (2.9) gives

$$\Gamma^{**}_{(\tau)}(t, X(t)) = 0. \tag{2.27}$$

Next, we apply the invariant form to instantaneous variance of the transformed Poisson process measure (2.25) from which using (2.12) and (2.27) yields

$$E_{\overline{Q}}\left[d\overline{N}(t)d\overline{N}(t)\right] = \lambda d\bar{t}. \tag{2.28}$$

Thus using (2.21) and (2.27) we have derived the following generalised random time change formula

$$\bar{t} = \int^t \Gamma_{(\tau)}(s)ds. \tag{2.29}$$

With

$$\Gamma_{(\tau)} = constant = c_1 \tag{2.30}$$

obtained using the probabilistic invariance property of the transformed time index differential i.e.

$$E_{\overline{Q}}\left[d\bar{t}(t,w)|W = w\right] = d\bar{t}. \tag{2.31}$$

Remark 2.2. In [6, 9, 7] while discussing symmetries of stochastic differential equations driven by the Brownian motion, they restrict their work such that the temporal infinitesimal $\tau(t,x)$ depends only on t not x in the beginning i.e., fiber-preserving transformations

$$H = \tau(t)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i}. \tag{2.32}$$

Invariance of the instantaneous mean of the Poisson process under the group transformation yields the same result in this case. Hence, we can conclude that the temporal infinitesimal $\tau(t,x)$ does not depend on x, therefore $\tau(x,t) = \tau(t)$.

Definition 2.3. A symmetry of jump-diffusion stochastic differential equation (1.3) i.e.,

$$dX_i(t) = f_i(t, X(t))dt + G_{ik}(t, X(t))dW_k(t) + J_i(t, X(t))dN(t) \tag{2.33}$$

is a one parameter group of transformations that leaves (2.33) and infinitesimal moments of the Wiener and the Poisson processes invariant.

3. Determining Equations

In this section, we concentrate on finding determining equations for the admitted Lie group symmetries of (1.3).

The intention is to transform (1.3) into

$$d\overline{X_j}(\bar{t}) = \overline{f_j}(\bar{t}, \overline{X}(\bar{t}))d\bar{t} + \overline{G_{jk}}(\bar{t}, \overline{X}(\bar{t}))d\overline{W}(\bar{t}) + \overline{J_j}(\bar{t}, \overline{X}(\bar{t}))d\overline{N}(\bar{t}). \tag{3.1}$$

Substituting (2.9), (2.11), (2.12), (2.14), (2.15) and (2.16) into (3.1) by considering equations (2.21) and (2.27), gives

$$d\overline{X_j}(\bar{t}) = dX_j + \epsilon\left(f_j\Gamma_{(\tau)}(t, X(t)) + \frac{\lambda J_j}{2}\Gamma_{(\tau)}(t, X(t)) + H(f_j)\right)dt$$
$$+\left(\frac{G_{jk}}{2}\Gamma_{(\tau)}(t, X(t)) + H(G_{jk})\right)dW(t) + \left(\frac{J_j}{2}\Gamma_{(\tau)}(t, X(t)) + H(J_j)\right)dN(t). \tag{3.2}$$

Therefore, by comparing (2.10) and (3.2) we successfully obtain the following determining equations

$$\left(f_j\Gamma_{(\tau)} + \frac{\lambda J_j}{2}\Gamma_{(\tau)} + H(f_j) - \Gamma_{(\xi)_j}\right)(t, X(t)) = 0, \tag{3.3}$$

$$\left(\frac{G_{jk}}{2}\Gamma_{(\tau)} + H(G_{jk}) - \Gamma^{*}_{(\xi)_j}\right)(t, X(t)) = 0 \tag{3.4}$$

and

$$\left(\frac{J_j}{2}\Gamma_{(\tau)} + H(J_j) - \Gamma^{**}_{(\xi)_j}\right)(t, X(t)) = 0. \tag{3.5}$$

With additional conditions obtained from the invariance of both Wiener (2.21) and Poisson momenta (2.27) respectively as;

$$\Gamma^{*}_{(\tau)}(t, X(t)) = 0 \tag{3.6}$$

and

$$\Gamma^{**}_{(\tau)}(t, X(t)) = 0. \tag{3.7}$$

Using (2.30) and (3.7) we get the temporal infinitesimal as

$$\tau(t) = c_1 t + c_2. \tag{3.8}$$

Where the operators $\Gamma(t,x)$, $\Gamma^*(t,x)$ and $\Gamma^{**}(t,x)$ are defined as in (1.6), (1.7) and (1.8), while $\lambda > 0$ is called the intensity of the jump process, then the spatial and temporal infinitesimals $\xi(t,x)$ and $\tau(t,x)$ are called the admitted symmetries of (1.3), if and only if they satisfied the determining equations (3.3) - (3.7).

Remark 3.1. Note that by removing the jump diffusion term i.e., substituting $J(t,x) = 0$ in (1.3), the determining equations was partially covered in [5, 6] when considering stochastic differential equations driven by Wiener processes using the so called fiber preserving transformation and the ignoring extra condition found in (3.6). Similarly, for jump diffusion term $J(t,x) = 0$, the determining equations (3.3), (3.4) and (3.6) are derived in [5, 11] while studying Wiener stochastic differential equations.

4. Applications

In this section, we apply the determining equations obtained for some jump-diffusion models to find their respective infinitesimals.

Example 4.1. Consider a stochastic model driven by both Wiener and Poisson processes

$$dX(t) = -kt^2 dt + \sqrt{D} dW + b dN(t) \tag{4.1}$$

with initial condition $X(0) = x_0$, where D is non-negative constant and $b \neq 0$

From the jump-diffusion model (4.1), we have the drift vector, Wiener diffusion and jump coefficients respectively as

$$f(t,x) = -kt^2, \quad G(x,t) = \sqrt{D}, \quad D > 0 \quad and \quad J(t,x) = b \quad b \neq 0. \tag{4.2}$$

Using the determining equations (3.3), (3.4) and (3.5) respectively, we get

$$\begin{aligned} -kt^2\left(\frac{\partial \tau}{dt} - kt^2\frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + \frac{b\lambda}{2}\left(\frac{\partial \tau}{\partial t} - kt^2\frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) - 2kt\tau \\ = \frac{\partial \xi}{\partial t} - kt^2\frac{\partial \xi}{\partial x} + \frac{D}{2}\frac{\partial^2 \xi}{\partial x^2}, \end{aligned} \tag{4.3}$$

$$\frac{\sqrt{D}}{2}\left(\frac{\partial \tau}{dt} - kt^2\frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) = \sqrt{D}\frac{\partial \xi}{\partial x}, \tag{4.4}$$

and

$$\frac{b}{2}\left(\frac{\partial \tau}{\partial t} - kt^2\frac{\partial \tau}{\partial x} + \frac{D}{2}\frac{\partial^2 \tau}{\partial x^2}\right) = \xi(t, x+b) - \xi(t,x). \tag{4.5}$$

Substituting the temporal infinitesimal (3.8) into (4.3), (4.4) and (4.5) we respectively get

$$\left(-kt^2 + \frac{b\lambda}{2} - 2kt^2\right)c_1 - 2ktc_2 = \frac{\partial \xi}{\partial t} - kt^2\frac{\partial \xi}{\partial x} + \frac{D}{2}\frac{\partial^2 \xi}{\partial x^2}, \tag{4.6}$$

$$\frac{\partial \xi}{\partial x} = \frac{c_1}{2} \tag{4.7}$$

and

$$\xi(t, x+b) - \xi(t, x) = \frac{bc_1}{2}. \tag{4.8}$$

Solving the differential equation in (4.7) yields the spatial infinitesimal

$$\xi(t, x) = \frac{c_1 x}{2} + f(t). \tag{4.9}$$

Substituting the spatial infinitesimal (4.9) in (4.6) gives

$$\left(-kt^2 + \frac{b\lambda}{2} - 2kt^2\right)c_1 - 2ktc_2 = \frac{df(t)}{dt} - \frac{kt^2 c_1}{2}. \tag{4.10}$$

Solving (4.10) for $f(t)$ yields

$$f(t) = \left(\frac{tb\lambda}{2} - \frac{5kt^3}{6}\right)c_1 - kt^2 c_2 + c_3. \tag{4.11}$$

Substituting (4.11) into (4.9) and by using (4.8) the spatial infinitesimal becomes

$$\xi(t, x) = \left(\frac{x}{2} + \frac{tb\lambda}{2} - \frac{5kt^3}{6}\right)c_1 - kt^2 c_2 + c_3. \tag{4.12}$$

Therefore, jump-diffusion stochastic differential equation (4.1) has admit three dimensional Lie symmetry infinitesimal generators

$$H_1 = t\frac{\partial}{\partial t} + \left(\frac{x}{2} + \frac{tb\lambda}{2} - \frac{5kt^3}{6}\right)\frac{\partial}{\partial x}, \tag{4.13}$$

$$H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}, \qquad H_3 = \frac{\partial}{\partial x} \tag{4.14}$$

with corresponding Lie bracket relations given by

Table 1. Commutator table for the Lie algebra generators (4.13) and (4.14)

$[H_i, H_j]$	H_1	H_2	H_3
H_1	0	$-H_4$	$-\frac{H_3}{2}$
H_2	H_4	0	0
H_3	$\frac{H_3}{2}$	0	0

where H_4 is a linear combination of H_2 and H_3, which is given as $H_4 = H_2 + \frac{b\lambda_0 H_3}{2}$. The commutative table shows that the infinitesimals generators (4.13) and (4.14) are closed under Lie bracket relations and hence form a Lie algebra.

Example 4.2. Consider the system of stochastic differential equations studied by Giuseppe Gaeta [12] with additional jump term

$$\begin{aligned} dX_1(t) &= X_2 dt \\ dX_2(t) &= -k^2 X_2 dt + \sqrt{2k^2} dW + \alpha t dN(t) \end{aligned} \tag{4.15}$$

with k^2 a positive constant and $X(0) = x_0$.

Therefore, the drift, jump vector and Wiener diffusion matrix are respectively

$$f_j(t,x) = \begin{pmatrix} X_2 \\ -k^2X_2 \end{pmatrix}, \quad J_j(t,x) = \begin{pmatrix} 0 \\ \alpha t \end{pmatrix}, \quad G_{ij}(t,x) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix}. \tag{4.16}$$

Using determining equation (3.3) for $j = 1$ and $j = 2$ gives

$$x_2\left(\frac{\partial \tau}{dt} + x_2\frac{\partial \tau}{\partial x_1} - k^2x_2\frac{\partial \tau}{\partial x_2}\right) + \xi_2(t,x_1,x_2) = \frac{\partial \xi_1}{\partial t} + x_2\frac{\partial \xi_1}{\partial x_1} - k^2x_2\frac{\partial \xi_1}{\partial x_2} + k^2\frac{\partial^2 \xi_1}{\partial x_2^2} \tag{4.17}$$

and

$$\begin{gathered} -k^2x_2\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2x_2\frac{\partial \tau}{\partial x_2}\right) - k^2\xi_2(t,x_1,x_2) \\ +\frac{\alpha\lambda t}{2}\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2x_2\frac{\partial \tau}{\partial x_2}\right) = \frac{\partial \xi_2}{\partial t} + x_2\frac{\partial \xi_2}{\partial x_1} - k^2x_2\frac{\partial \xi_2}{\partial x_2} + k^2\frac{\partial^2 \xi_2}{\partial x_2^2}. \end{gathered} \tag{4.18}$$

While equation (3.4) for $j = 1$ and $j = 2$ gives

$$\frac{\partial \xi_1}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2x_2\frac{\partial \tau}{\partial x_2}\right) \tag{4.19}$$

and

$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + x_2\frac{\partial \tau}{\partial x_1} - k^2x_2\frac{\partial \tau}{\partial x_2}\right). \tag{4.20}$$

Substituting temporal infinitesimal (3.8) into (4.17) and (4.18) respectively gives

$$x_2c_1 + \xi_2 = \frac{\partial \xi_1}{\partial t} + x_2\frac{\partial \xi_1}{\partial x_1} - k^2x_2\frac{\partial \xi_1}{\partial x_2} + k^2\frac{\partial^2 \xi_1}{\partial x_2^2}, \tag{4.21}$$

$$\left(\frac{\alpha\lambda_0 t}{2} - k^2x_2\right)c_1 - k^2\xi_2 = \frac{\partial \xi_2}{\partial t} + x_2\frac{\partial \xi_2}{\partial x_1} - k^2x_2\frac{\partial \xi_2}{\partial x_2} + k^2\frac{\partial^2 \xi_2}{\partial x_2^2}. \tag{4.22}$$

Similarly, substituting (3.8) into (4.19) and (4.20) yields

$$\frac{\partial \xi_1}{\partial x_2} = \frac{c_1}{2} \tag{4.23}$$

and

$$\frac{\partial \xi_2}{\partial x_2} = \frac{c_1}{2}. \tag{4.24}$$

Solving (4.23) and (4.24) respectively gives

$$\xi_1 = \frac{c_1x_2}{2} + f(t,x_1) \tag{4.25}$$

and

$$\xi_2 = \frac{c_1x_2}{2} + g(t,x_1). \tag{4.26}$$

Substituting (4.25) and (4.26) into (4.21) and (4.22) respectively gives

$$x_2c_1 + \frac{c_1x_2}{2} + g(t,x_1) = \frac{\partial f(t,x_1)}{\partial t} + x_2\frac{\partial f(t,x_1)}{\partial x_1} \tag{4.27}$$

and

$$\left(\frac{\alpha\lambda_0 t}{2} - k^2x_2\right)c_1 - \frac{c_1k^2x_2}{2} - k^2g(t,x_1) = \frac{\partial g(t,x_1)}{\partial t} + x_2\frac{\partial g(t,x_1)}{\partial x_1}. \tag{4.28}$$

It is clear to see that, since $f(t,x_1)$ and $g(t,x_1)$ does not depend on x_2 we have from (4.27) and (4.28)

$$c_1 = 0, \qquad \frac{\partial f(t,x_1)}{\partial x_1} = 0 \qquad and \qquad \frac{\partial g(t,x_1)}{\partial x_1} = 0. \tag{4.29}$$

Substituting from (4.29) into (4.27) and (4.28) we respectively obtained

$$\frac{\partial f(t,x_1)}{\partial t} - g(t,x_1) = 0 \tag{4.30}$$

and

$$\frac{\partial g(t,x_1)}{\partial t} + k^2 g(t,x_1) = 0. \tag{4.31}$$

Solving (4.31) gives

$$g(t,x_1) = c_3 e^{-k^2 t}. \tag{4.32}$$

Substituting (4.32) into (4.30) leads to

$$f(t,x_1) = -\frac{c_3 e^{-k^2 t}}{k^2} + c_4. \tag{4.33}$$

Therefore, by using (4.29) in (3.8), (4.25) and (4.26) respectively gives the following infinitesimals

$$\tau(t) = c_2, \qquad \xi_1 = \frac{-c_3 e^{-k^2 t}}{k^2} + c_4 \qquad and \qquad \xi_2 = c_3 e^{-k^2 t}. \tag{4.34}$$

We can clearly see that ξ_1 satisfied (3.5) automatically for $j = 1$. For $j = 2$, substituting $\xi_2 = c_3 e^{-k^2 t}$ and $\tau(t) = c_2$ into (3.5) we get

$$c_2 \alpha = 0. \tag{4.35}$$

Which implies $c_2 = 0$, since $\alpha \neq 0$.

Therefore, by substituting $c_2 = 0$ into (4.34) we have the infinitesimals reduced to

$$\tau(t) = 0, \qquad \xi_1 = \frac{-c_3 e^{-k^2 t}}{k^2} + c_4 \qquad and \qquad \xi_2 = c_3 e^{-k^2 t}. \tag{4.36}$$

Hence the symmetries of the infinitesimal generators are two dimensional given as:

$$H_1 = \frac{-e^{-k^2 t}}{k^2} \frac{\partial}{\partial x_1} + e^{-k^2 t} \frac{\partial}{\partial x_2}, \qquad H_2 = \frac{\partial}{\partial x_1} \tag{4.37}$$

with corresponding Lie bracket relations $[H_1, H_2] = [H_2, H_1] = 0$. Which shows that the symmetries generator (4.37) forms an abelian group.

Remark 4.3. After considering the model that involves both Wiener and Poisson diffusion, we recover two of the three symmetries obtained by Giuseppe Gaeta [12]. This is to be expected since the jump term adds uncertainty to the model.

Example 4.4. Consider the jump SDE, linear in the state process $X(t)$, with constant coefficients;

$$dX(t) = X(t)\Big(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t)\Big) \tag{4.38}$$

with initial condition $X(t_0) = x_0 > 0$. The coefficient $u_0(t)$ is called the drift or deterministic coefficient, $v_0(t)$ is called the jump amplitude coefficient of the jump term and $\alpha_0(t)$ is called Wiener diffusion coefficient, with jump intensity $\lambda = \lambda_0$.

Therefore, we have the drift, Brownian motion diffusion and jump coefficients as

$$f(t,x) = u_0 x, \qquad g(t,x) = \alpha_0 x \qquad and \qquad J(t,x) = v_0 x. \tag{4.39}$$

with $u_0, \quad \alpha_0 \quad v_0$ non-zero.

Using the determining equations (3.3), (3.4) and (3.5) we respectively get

$$\begin{aligned} u_0 x\left(\frac{\partial \tau}{dt} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + \frac{v_0 x\lambda_0}{2}\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) \\ + u_0\xi(t,x) = \frac{\partial \xi}{\partial t} + u_0 x\frac{\partial \xi}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \xi}{\partial x^2}, \end{aligned} \tag{4.40}$$

$$\frac{\alpha_0 x}{2}\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + \alpha_0\xi(t,x) = \alpha_0 x\frac{\partial \xi}{\partial x} \tag{4.41}$$

and

$$\frac{v_0 x}{2}\left(\frac{\partial \tau}{\partial t} + u_0 x\frac{\partial \tau}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \tau}{\partial x^2}\right) + v_0\xi = \xi(t, x + v_0 x) - \xi(t,x). \tag{4.42}$$

Substituting temporal infinitesimal (3.8) into (4.40), (4.41) and (4.42) we respectively have

$$\left(u_0 x + \frac{v_0 x\lambda_0}{2}\right)c_1 + u_0\xi(t,x) = \frac{\partial \xi}{\partial t} + u_0 x\frac{\partial \xi}{\partial x} + \frac{\alpha_0^2 x^2}{2}\frac{\partial^2 \xi}{\partial x^2}, \tag{4.43}$$

$$\frac{c_1 x}{2} + \xi(t,x) = x\frac{\partial \xi}{\partial x}, \tag{4.44}$$

and

$$\frac{v_0 c_1 x}{2} + v_0\xi = \xi(t, x + v_0 x) - \xi(t,x). \tag{4.45}$$

By solving (4.44), we get the spatial infinitesimal as

$$\xi(t,x) = \frac{x c_1 \ln|x|}{2} + f(t)x, \quad for \quad x > 0. \tag{4.46}$$

By substituting spatial infinitesimal (4.46) into (4.45), we finally obtain

$$c_1 = 0. \tag{4.47}$$

Therefore, substituting (4.47) into (4.46) and (3.8), the temporal and spatial infinitesimals respectively reduce to

$$\tau(t) = c_2 \tag{4.48}$$

and

$$\xi(t,x) = f(t)x. \tag{4.49}$$

Substituting (4.49) in (4.43) gives

$$\frac{df(t)}{dt} = 0. \tag{4.50}$$

Which implies

$$\xi(t,x) = c_3 x. \tag{4.51}$$

Therefore, the symmetries algebra is two dimensional given as

$$H_1 = \frac{\partial}{\partial t}, \qquad H_2 = x\frac{\partial}{\partial x}. \tag{4.52}$$

The Lie bracket relation of the generator (4.52) is $[H_1, H_2] = [H_2, H_1] = 0$, which shows that the symmetries algebra is also an abelian group .

Remark 4.5. Symmetry algebra of geometric Brownian motion driven stochastic differential equation was discussed by Ebrahim and Mahomed F. M. [5] and the generators are found to be generated by three-dimensional algebra. In this example, we see that adding Poisson diffusion to the model reduces the symmetry by one dimension. Interestingly, the two generators found are the only ones that leave a stochastic differential invariant [5].

5. Conclusion

Lie symmetry of jump-diffusion stochastic differential equations was discussed, by considering infinitesimals of the spatial and temporal variables. This was achieved by utilising the random time formula for standard Brownian motion used in [9, 7, 8] to study symmetry of Wiener process stochastic differential equations i.e.,

$$d\overline{W}(\bar{t}) = \sqrt{\frac{d\bar{t}(t)}{dt}}dW(t) \tag{5.1}$$

and the random time formula for Poisson processes [13] i.e.,

$$d\overline{N}(\bar{t}) = dN(t) + \frac{\epsilon}{2}\frac{d\tau}{dt}\Big(\lambda dt + dN(t)\Big) + O(\epsilon). \tag{5.2}$$

The determining equations of the jump-diffusion stochastic differential equation

$$dX_i(t) = f_i(t, X(t))dt + G_{il}(t, X(t))dW_k(t) + J_i(t, X(t))dN(t) \tag{5.3}$$

were derived and are found to be deterministic after applying the invariance methodology to the moments of both Wiener and Poisson processes. The determining equations found are similar to the one used in [5, 11, 9, 8] if the Poisson terms are removed. We apply the determining equations to several jump-diffusion models to show how they can be used to find the admitted Lie infinitesimals transformation of the respective models. Finally, a Lie bracket relation was found to show the relationship between the infinitesimals generators, which show that the infinitesimals generators are closed under Lie relations and hence form a Lie algebra. The Lie group classification of the given examples is given in *Table 2* below.

TABLE 2. Lie Group Classification

Group Dimension	Basis Operators	Equations
3	$H_1 = t\frac{\partial}{\partial t} + \left(\frac{x}{2} + \frac{tb\lambda_0}{2} - \frac{5kt^3}{6}\right)\frac{\partial}{\partial x}$, $H_2 = \frac{\partial}{\partial t} - kt^2\frac{\partial}{\partial x}$, $H_3 = \frac{\partial}{\partial x}$.	$dX = -kt^2dt + \sqrt{D}dW + bdN(t)$
2	$H_1 = \frac{-e^{-k^2t}}{k^2}\frac{\partial}{\partial x_1} + e^{-k^2t}\frac{\partial}{\partial x_2}$, $H_2 = \frac{\partial}{\partial x_1}$.	$dX_1 = X_2dt$ $dX_2 = -k^2X_2dt + \sqrt{2k^2}dW + \alpha tdN(t)$
2	$H_1 = \frac{\partial}{\partial t}$, $H_2 = x\frac{\partial}{\partial x}$.	$dX(t) = X(t)(u_0(t)dt + \alpha_0(t)dW + v_0(t)dN(t))$

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