Stochastic Modeling and Applications
Vol. 22 No.2 (December, 2018)

ON TOPOLOGICAL SPACE OF TERNARY SEMIGROUPS

MACHCHHINDRA GOPHANE, DATTATRAY SHINDE*, AND VINOD KHARAT

Abstract. In this paper, we introduce and study the topological space of ternary semigroups formed by the set of prime ideals. We investigate the various properties of the topological space of a ternary semigroup. We also study interesting properties such as compactness, connectedness and separation axioms of this topological space. Furthermore, we define irreducible topological spaces, Noetherian ternary semigroups and prime full ideals in ternary semigroups and study their properties.

1. Introduction

In the 19^{th} century, D. H. Lehmer [6] studied the literature of a ternary algebraic system. The ternary semigroup is a particular case of n-ary semigroups. So many results on ternary semigroups has an analogous version for n-ary semigroups. The ideal theory in ternary semigroups was introduced by F. M. Sioson [9] in 1965. In [7], Y. Sarala et al. studied the properties of the ideals of ternary semigroups. M. Shabir and S. Bashir [8] introduced and studied the notion of prime, semiprime and irreducible ideals in ternary semigroups.

The notion of the structure space of Γ- semigroups was introduced by S. Kar and S. Chattopadhyay in [3]. In [5], some special classes of all proper prime k-ideals, prime ideals and strongly irreducible ideals in Γ- semirings is introduced. They also obtained the topological spaces of these ideals of Γ- semirings. R. D. Jagtap and Y. S. Pawar [2] studied the space of prime ideals of a Γ- semiring and properties of the space of prime ideals of a Γ- semiring.

In this article, we introduce and study the concept of the topological space of ternary semigroups. We consider the set $\mathcal{P}$ of all prime ideals of a ternary semigroup T and build the topology τ on $\mathcal{P}$ using the closure operator defined in terms of intersection and inclusion relations among these ideals of ternary semigroup T. We investigate various topological properties of space $(\mathcal{P}, \tau)$. This topological space $(\mathcal{P}, \tau)$ is referred as the structure space of the ternary semigroup T. We also studied the compactness, connectedness and separation axioms in this topological space $(\mathcal{P}, \tau)$.

2000 *Mathematics Subject Classification.* Primary 06F99, 20M12; Secondary 20N99.
Key words and phrases. Ternary Semigroup, Prime Ideal, Hull-kernel Topology, Prime full ideal.
*Corresponding Author.

2. Preliminaries

Definition 2.1. [6] A non-empty set T with a ternary operation $[\,] : T \times T \times T \longrightarrow T$ is called a ternary semigroup if $[\,]$ satisfies the associative law, $[[a\ b\ c]\ d\ e] = [a\ [b\ c\ d]\ e] = [a\ b\ [c\ d\ e]]$, for all $a, b, c, d, e \in T$.

Definition 2.2. [1] An element $0 \in T$ is said to be a zero element of T if $0xy = x0y = xy0 = 0$ for all $x, y \in T$.

Definition 2.3. [1] An element $e \in T$ is said to be an identity element of T if $exx = xxe = xex = x$ for all $x \in T$. It can prove that $eea = eae = aee = a$

Definition 2.4. [9] A non-empty subset I of a ternary semigroup T is said to be a left (respectively, right, lateral) ideal of T if $TTI \subseteq I$ (respectively, $ITT \subseteq I$, $TIT \subseteq I$).

A non-empty subset I of T is said to be ideal of T if it is a left ideal, a right ideal and a lateral ideal of T. An ideal I of T is called a proper ideal of T if $I \neq T$.

Definition 2.5. [9] Let X be the non-empty subset of T. The intersection of all ideals of T containing X is called as ideal of T generated by X and it is denoted by $\langle X \rangle$. The ideal generated by $\{a\}$ for some $a \in T$ is denoted by $\langle a \rangle$.

Definition 2.6. [8] A proper ideal I of T is said to be a prime ideal of T provided I_1, I_2, I_3 are ideals of T and $I_1 I_2 I_3 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$.

Definition 2.7. [8] A proper ideal I of T is said to be a semiprime ideal of T provided P is ideal of T and $P^3 \subseteq I$ implies $P \subseteq I$.

In this article, we write T for a ternary semigroup with zero, unless otherwise specified.

3. Topological Space of Ternary Semigroups

Let $\mathcal{P}$ be the family of all prime ideals of T. For any subset A of $\mathcal{P}$, we define $\overline{A} = \{I \in \mathcal{P} : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}$.

Theorem 3.1. *If A is a subset of $\mathcal{P}$ then the function $A \longrightarrow \overline{A}$ is a closure operator on $\mathcal{P}$.*

Proof. (i) Obviously $\overline{\emptyset} = \emptyset$.
(ii) By the definition of $\overline{A}$, for every $\alpha, I_\alpha \in A$. Therefore $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ implies $I_\alpha \in \overline{A}$. Hence $A \subseteq \overline{A}$.
(iii) By (ii), we have $\overline{A} \subseteq \overline{\overline{A}}$. Let $I \in \overline{\overline{A}}$. Then $\bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I$. Now, $I_\alpha \in \overline{A}$ implies that $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$ for all $\alpha \in \Delta$, where Δ denotes the indexing set. Thus $\bigcap_{I_\gamma \in A} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I$. Therefore $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I$. So $I \in \overline{A}$ and hence $\overline{\overline{A}} \subseteq \overline{A}$. Thus

$\overline{\overline{A}} = \bar{A}$.

(iv) Let B be any subset of $\mathcal{P}$ such that $A \subseteq B$. Let $I \in \overline{A}$. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I$. Since $A \subseteq B$, it follows that $\bigcap_{I_\alpha \in B} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha \subseteq I$. This shows that $I \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(v) Let B be any subset of $\mathcal{P}$. To prove that, $\overline{A \cup B} = \overline{A} \cup \overline{B}$, firstly we prove that $\overline{A \cup B \cup C} = \overline{A} \cup \overline{B} \cup \overline{C}$ for any subset C of $\mathcal{P}$. From (iv), $\overline{A} \subseteq \overline{A \cup B \cup C}, \overline{B} \subseteq \overline{A \cup B \cup C}$ and $\overline{C} \subseteq \overline{A \cup B \cup C}$. This implies that, $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cup B \cup C}$. Now let $I \in \overline{A \cup B \cup C}$. Then $\bigcap_{I_\alpha \in A \cup B \cup C} I_\alpha \subseteq I$. Obviously, $\bigcap_{I_\alpha \in A \cup B \cup C} I_\alpha = \left(\bigcap_{I_\alpha \in A} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in B} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in C} I_\alpha\right)$ Since $\bigcap_{I_\alpha \in A} I_\alpha$, $\bigcap_{I_\alpha \in B} I_\alpha$ and $\bigcap_{I_\alpha \in C} I_\alpha$ are ideals of T,

also $\left(\bigcap_{I_\alpha \in A} I_\alpha\right)\left(\bigcap_{I_\alpha \in B} I_\alpha\right)\left(\bigcap_{I_\alpha \in C} I_\alpha\right) \subseteq \left(\bigcap_{I_\alpha \in A} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in B} I_\alpha\right) \cap \left(\bigcap_{I_\alpha \in C} I_\alpha\right) = \bigcap_{I_\alpha \in A \cup B \cup C} I_\alpha$

$\subseteq I$. As I is a prime ideal of T, we get $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I$ or $\bigcap_{I_\alpha \in B} I_\alpha \subseteq I$ or $\bigcap_{I_\alpha \in C} I_\alpha \subseteq I$, i.e. either $I \in \overline{A}$ or $I \in \overline{B}$ or $I \in \overline{C}$. Hence $I \in \overline{A} \cup \overline{B} \cup \overline{C}$. This shows that $\overline{A \cup B \cup C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ and hence $\overline{A \cup B \cup C} = \overline{A} \cup \overline{B} \cup \overline{C}$. Since $\overline{\emptyset} = \emptyset$, putting $C = \emptyset$, we get $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

Definition 3.2. The closure operator $A \longrightarrow \overline{A}$ induces a topology τ on $\mathcal{P}$. This topology τ is called the hull-kernel topology and the topological space $(\mathcal{P}, \tau)$ is called structure space of a ternary semigroup T.

Let I be an ideal of T, we define $X(I) = \{J \in \mathcal{P} : I \subseteq J\}$ and $Y(I) = \mathcal{P} \setminus X(I) = \{J \in \mathcal{P} : I \nsubseteq J\}$.

Theorem 3.3. *Any closed set in $\mathcal{P}$ is of the form $X(I)$ where I is an ideal of T.*

Proof. Let $\overline{A}$ be any closed set in $\mathcal{P}$, where $A \subseteq \mathcal{P}$. Let $A = \{I_\alpha : \alpha \in \Delta\}$ where Δ is an index set and $I = \bigcap_{I_\alpha \in A} I_\alpha$. Then I is an ideal of T. Let $J \in \overline{A}$ then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq J \Rightarrow I \subseteq J$. Therefore $J \in X(I)$ and so $\overline{A} \subseteq X(I)$. Now, let $J \in X(I)$ then $I \subseteq J \Rightarrow \bigcap_{I_\alpha \in A} I_\alpha \subseteq J$. Therefore $J \in \overline{A}$ and hence $X(I) \subseteq \overline{A}$. Thus $\overline{A} = X(I)$. □

Corollary 3.4. *Any open set in $\mathcal{P}$ is of the form $Y(I)$ where I is an ideal of T.*

Let I be an ideal of T, we define for any $a \in T$, $X(a) = \{I \in \mathcal{P} : a \in I\}$ and $Y(a) = \mathcal{P} \setminus X(a) = \{I \in \mathcal{P} : a \notin I\}$.

Theorem 3.5. *The set $\{Y(a) : a \in T\}$ forms a base for open sets for the topology τ on $\mathcal{P}$.*

Proof. Let G be any open set in τ i.e. $G \in \tau$. Then by Corollary 3.4, we have $G = Y(I)$ where I is an ideal of T. For any $J \in G = Y(I)$ we have $I \nsubseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Hence $J \in Y(a)$. Therefore $G \subseteq Y(a)$. Now to show that $Y(a) \subseteq G$. Let $K \in Y(a)$. Then $a \notin K$. This gives that, $I \nsubseteq K$. Therefore $K \in Y(I) = G$. Hence $Y(a) \subseteq G$. Thus we get $J \in Y(a) \subseteq G$. Then $G = \bigcup_{a \in T} Y(a)$. Therefore $\{Y(a) : a \in T\}$ forms an open base for the hull-kernel topology τ on $\mathcal{P}$. □

Theorem 3.6. *The topological space $(\mathcal{P}, \tau)$ is a T_0-space.*

Proof. Let I and J be two distinct elements of $\mathcal{P}$. Then there is an element a either in $I \setminus J$ or in $J \setminus I$. Assume that $a \in I \setminus J$. But then $J \in Y(a)$ and $I \notin Y(a)$ i. e. $Y(a)$ is a neighborhood of J not containing I. Hence $(\mathcal{P}, \tau)$ is a T_0-space. □

Theorem 3.7. *The topological space $(\mathcal{P}, \tau)$ is a T_1-space if and only if no element of $\mathcal{P}$ is contained in any other element of $\mathcal{P}$.*

Proof. Let $(\mathcal{P}, \tau)$ is a T_1-space. Suppose that I and J be any two distinct elements of $\mathcal{P}$. Then each of I and J has a neighborhood not containing the other. Since I and J are arbitrary elements of $\mathcal{P}$, it follows that no element of $\mathcal{P}$ is contained in any other element of $\mathcal{P}$.

Conversely, suppose that no element of $\mathcal{P}$ is contained in any other element of $\mathcal{P}$. Let I and J be any two distinct elements of $\mathcal{P}$. Then by assumption either $I \not\subset J$ and $J \not\subset I$. This shows that there exist $a, b \in T$ such that $a \in I$ but $a \notin J$ and $b \in J$ but $b \notin I$. Then we have $I \in Y(b)$ but $I \notin Y(a)$ and $J \in Y(a)$ but $J \notin Y(b)$, it means that, each of I and J has a neighborhood not containing the other. Hence $(\mathcal{P}, \tau)$ is a T_1-space. □

Corollary 3.8. *If $(\mathcal{P}, \tau)$ is a Hausdorff space, then no prime ideal contains any other prime ideal. Alternatively, If the space $(\mathcal{P}, \tau)$ is a Hausdorff space then the set of all minimal prime ideals and maximal ideals coincide.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a Hausdorff space. Since every Hausdorff space is T_1-space. Hence by Theorem 3.7, it gives that no prime ideal contains any other prime ideal. □

Definition 3.9. [8] An proper ideal I of T is said to be a maximal ideal if I is not properly contained in any other proper ideal of T.

Corollary 3.10. *Let M be the set of all proper maximal ideals of a ternary semigroup T with identity. Then $(\mathsf{M}, \tau_{\mathsf{M}})$ is a T_1-space, where τ_{M} is the induced topology on M from $(\mathcal{P}, \tau)$.*

Theorem 3.11. *The topological space $(\mathcal{P}, \tau)$ is a Hausdorff space if and only if for any two distinct pair of elements I and J of $\mathcal{P}$, there exist $a, b \in T$ such that $a \notin I$, $b \notin J$ and there does not exist any element K of $\mathcal{P}$ such that $a \notin K$ and $b \notin K$.*

Proof. Suppose that the topological space $(\mathcal{P}, \tau)$ is a Hausdorff space. Then for any two distinct pair of elements I and J of $\mathcal{P}$ there exists two basic open sets $Y(a)$ and $Y(b)$ such that $I \in Y(a), J \in Y(b)$ and $Y(a) \cap Y(b) = \emptyset$. Now $I \in Y(a)$ and $J \in Y(b)$ imply that $a \notin I$ and $b \notin J$. Let if possible there exist K in $\mathcal{P}$ such that $a \notin K$ and $b \notin K$. Then $K \in Y(a) \cap Y(b)$, a contradiction, since $Y(a) \cap Y(b) = \emptyset$. Thus there does not exist any element K of $\mathcal{P}$ such that $a \notin K$ and $b \notin K$.

Conversely, Suppose that the given condition holds. To show the space $(\mathcal{P}, \tau)$ is a Hausdorff space. Let I and J be two distinct elements of $\mathcal{P}$. Then by assumption there exists $a, b \in T$ such that $a \notin I, b \notin J$ and there does not exist any K of $\mathcal{P}$ such that $a \notin K$ and $b \notin K$. Then $I \in Y(a), J \in Y(b)$ and $Y(a) \cap Y(b) = \emptyset$. Hence $(\mathcal{P}, \tau)$ is a Hausdorff space. □

Theorem 3.12. *If $(\mathcal{P}, \tau)$ is a Hausdorff space containing more than one element then there exist $a, b \in T$ such that $\mathcal{P} = Y(a) \cup Y(b) \cup X(I)$ where I is the ideal generated by a, b in T.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a Hausdorff space containing more than one element. Let $J, K \in \mathcal{P}$ such that $J \neq K$. Since $(\mathcal{P}, \tau)$ is a Hausdorff space, there exists two basic open sets $Y(a)$ and $Y(b)$ such that $J \in Y(a), K \in Y(b)$ and $Y(a) \cap Y(b) = \emptyset$. Let I be the ideal generated by $a, b \in T$. Then I is the smallest ideal containing a and b. Let $L \in \mathcal{P}$. Then either $a \in L, b \notin L$ or $a \notin L, b \in L$ or $a, b \in L$. The case, $a \notin L, b \notin L$ is not possible, since $a \notin L, b \notin L$ implies that $L \in Y(a)$ and $L \in Y(b)$ that is $L \in Y(a) \cap Y(b)$ which is not possible because $Y(a) \cap Y(b) = \emptyset$. Now in the first case, $L \in Y(b)$ and hence $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. In the second case, $L \in Y(a)$ and hence $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. In the third case, $L \in X(I)$ and hence $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. Therefore $\mathcal{P} \subseteq Y(a) \cup Y(b) \cup X(I)$. But $Y(a) \cup Y(b) \cup X(I) \subseteq \mathcal{P}$. Hence $\mathcal{P} = Y(a) \cup Y(b) \cup X(I)$. □

Theorem 3.13. *The topological space $(\mathcal{P}, \tau)$ is a regular space if and only if for any $I \in \mathcal{P}$ and $a \notin I$ for $a \in T$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a regular space. Let $I \in \mathcal{P}$ and $a \notin I$ for $a \in T$. As $a \notin I$, we have $I \in Y(a)$ and $Y(a)$ is an open set of $\mathcal{P}$ implies $X(a) = \mathcal{P} \setminus Y(a)$ is a closed set of $\mathcal{P}$ not containing I. As $(\mathcal{P}, \tau)$ is a regular space, there exist two disjoints open sets say G and H such that $I \in G, \mathcal{P} \setminus Y(a) \subseteq H$ and $G \cap H = \emptyset$. $\mathcal{P} \setminus Y(a) \subseteq H$ implies that $\mathcal{P} \setminus H \subseteq Y(a)$. Since H is an open set of $\mathcal{P}$ implies $\mathcal{P} \setminus H$ is a closed set and hence there exist an ideal J of T such that, $\mathcal{P} \setminus H = X(J)$, by Theorem 3.3. So we find that $X(J) \subseteq Y(a)$. Again $G \cap H = \emptyset$, we have $H \subseteq \mathcal{P} \setminus G$. Since G is open set, $\mathcal{P} \setminus G$ is closed and hence there exists an ideal K of T such that $\mathcal{P} \setminus G = X(K)$ i. e. $H \subseteq X(K)$. Since $I \in G, I \notin \mathcal{P} \setminus G = X(K)$. This implies that $K \nsubseteq I$. Thus there exists $b \in K (\subset T)$ such that $b \notin I$. So $I \in Y(b)$. Now we show that $H \subseteq X(b)$. Let $M \in H \subseteq X(K)$. Then $K \subseteq M$. Since $b \in K$, it gives that $b \in M$ and hence $M \in X(b)$. Therefore $H \subseteq X(b)$. This implies that $\mathcal{P} \setminus X(b) \subseteq \mathcal{P} \setminus H = X(J) \Rightarrow Y(b) \subseteq X(J)$. Thus we get for any $I \in \mathcal{P}$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$.

Conversely, suppose that for any $I \in \mathcal{P}$ and $a \notin I, a \in T$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$. To show the space $(\mathcal{P}, \tau)$ is a regular space. Let $I \in \mathcal{P}$ and $X(K)$ be any closed set not containing I. Since $I \notin X(K)$, we have $K \nsubseteq I$. This implies that there exists $a \in K$ such that $a \notin I$. Now by the given condition, there exists an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$. Since $a \in K, Y(a) \cap X(K) = \emptyset$. This implies that $X(K) \subseteq \mathcal{P} \setminus Y(a) \subseteq \mathcal{P} \setminus X(J)$. Since $X(J)$ is a closed set, $\mathcal{P} \setminus X(J)$ is an open set containing the closed set $X(K)$. Therefore $Y(b) \cap (\mathcal{P} \setminus X(J)) = \emptyset$. So we find that $Y(b)$ and $\mathcal{P} \setminus X(J)$ are two disjoints open sets containing I and $X(K)$ respectively. Therefore $(\mathcal{P}, \tau)$ is a regular space. □

Corollary 3.14. *The topological space $(\mathcal{P}, \tau)$ is a T_3-space if and only if for any $I \in \mathcal{P}$ and $a \notin I$ for $a \in T$ there exist an ideal J of T and $b \in T$ such that $I \in Y(b) \subseteq X(J) \subseteq Y(a)$.*

Theorem 3.15. *The topological space $(\mathcal{P}, \tau)$ is a compact space if and only if for any collection $\{a_i\}_{i \in \Delta}$ (where Δ is indexing set) of T there exists a finite subcollection $\{a_1, a_2, \ldots, a_n\}$ in T such that $I \in \mathcal{P}$ there exist a_i such that $a_i \notin I$.*

Proof. Suppose that $(\mathcal{P}, \tau)$ is a compact space. Then the open cover $\{Y(a_i) : a_i \in T\}$ of $(\mathcal{P}, \tau)$ has a finite subcover $\{Y(a_i) : i = 1, 2, \ldots, n\}$. Let I be any element of $\mathcal{P}$. Then $I \in \{Y(a_i) : i = 1, 2, \ldots, n\}$. Therefore $I \in Y(a_i)$ for some $a_i \in T$. Hence $a_i \notin I$. Thus $\{a_1, a_2, \ldots, a_n\}$ is the required finite subcollection of elements of T such that for any $I \in \mathcal{P}$ there exist a_i such that $a_i \notin I$.

Conversely, suppose that the given condition holds. To show the space $(\mathcal{P}, \tau)$ is a compact space. Let $\{Y(a_i) : a_i \in T\}$ be an open cover of $(\mathcal{P}, \tau)$. Assume that no finite subcollection of $\{Y(a_i) : a_i \in T\}$ be forms a cover of $\mathcal{P}$. This means that for any finite set $\{a_1, a_2, \ldots, a_n\}$ of elements of T, $Y(a_1) \cup Y(a_2) \cup \ldots \cup Y(a_n) \neq \mathcal{P} \Rightarrow \mathcal{P} \setminus [Y(a_1) \cup Y(a_2) \cup \ldots \cup Y(a_n)] \neq \emptyset \Rightarrow X(a_1) \cap X(a_2) \cap \ldots \cap X(a_n) \neq \emptyset$. This implies there exist $I \in \mathcal{P}$ such that $I \in X(a_1) \cap X(a_2) \cap \ldots \cap X(a_n)$ gives that $a_1, a_2, \ldots, a_n \in I$. Which is a contradiction to our hypothesis. Hence our assumption $\{Y(a_i) : a_i \in T\}$ has no finite subcover which covers $\mathcal{P}$ is wrong. Therefore $\{Y(a_i) : a_i \in T\}$ has finite subcover which covers $\mathcal{P}$. Hence $(\mathcal{P}, \tau)$ is a compact space. □

Corollary 3.16. *If T is finitely generated, then the space $(\mathcal{P}, \tau)$ is compact.*

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be a finite set of generators of T. Then for any $I \in \mathcal{P}$ there exist a_i such that $a_i \notin I$. Hence by Theorem 3.15, $(\mathcal{P}, \tau)$ is a compact space. □

The arbitrary intersection of all prime ideals of T is a semiprime ideal of T, provided it is non-empty. We give a necessary condition for the intersection of prime ideals of T to be a prime ideal in the following theorem,

Theorem 3.17. *Let $\{I_i\}_{i \in \Delta}$ (where Δ is any indexing set) be a family of all prime ideals of T such that $\{I_i\}_{i \in \Delta}$ forms a chain of ideals then $\bigcap_{i \in \Delta} I_i$ is a prime ideal of T.*

Proof. Let $\{I_i\}_{i \in \Delta}$ (where Δ is any indexing set) be a family of all prime ideals of T. It is clear that $\bigcap_{i \in \Delta} I_i$ is an ideal of T. Let I_1, I_2 and I_3 be any three ideals of T such that $I_1I_2I_3 \subseteq \bigcap_{i\in\Delta} I_i$. If either $I_1 \subseteq I_i\ \forall i \in \Delta$ or $I_2 \subseteq I_i\ \forall i \in \Delta$ or $I_3 \subseteq I_i\ \forall i \in \Delta$ then either $I_1 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_2 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_3 \subseteq \bigcap_{i\in\Delta} I_i$. If possible, let $I_1, I_2, I_3 \not\subseteq \bigcap_{i\in\Delta} I_i$ then there exist i, j and k such that $I_1 \not\subseteq \bigcap_{i\in\Delta} I_i, I_2 \not\subseteq \bigcap_{j\in\Delta} I_j$ and $I_3 \not\subseteq \bigcap_{k\in\Delta} I_k$. Since $\{I_i\}_{i\in\Delta}$ form a chain of ideals, let $I_i \subseteq I_j \not\subseteq I_k$. This implies that $I_2, I_3 \not\subseteq I_i$. Since $I_1I_2I_3 \subseteq I_i$ and I_i is prime ideal of T, we must have either $I_1 \subseteq I_i$ or $I_2 \subseteq I_i$ or $I_3 \subseteq I_i$. Which is a contradiction. Therefore, either $I_1 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_2 \subseteq \bigcap_{i\in\Delta} I_i$ or $I_3 \subseteq \bigcap_{i\in\Delta} I_i$. Hence $\bigcap_{i\in\Delta} I_i$ is a prime ideal of T. □

Definition 3.18. The topological space $(\mathcal{P}, \tau)$ of T is called irreducible if for any decomposition $\mathcal{P} = \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$, where $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ are closed subsets of $\mathcal{P}$ then either $\mathcal{P} = \mathcal{U}$ or $\mathcal{P} = \mathcal{V}$ or $\mathcal{P} = \mathcal{W}$.

Theorem 3.19. *Let $\mathcal{U}$ be a closed subset of $\mathcal{P}$. Then $\mathcal{U}$ is irreducible if and only if $\bigcap_{I_i\in\mathcal{U}} I_i$ is a prime ideal of T.*

Proof. Assume that $\mathcal{U}$ is irreducible. To prove that $\bigcap_{I_i\in\mathcal{U}} I_i$ is a prime ideal of T. Let A, B and C be any three ideals of T such that $ABC \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$. Then $ABC \subseteq I_i$, $\forall\, i$. As I_i is a prime ideal of T, we have $A \subseteq I_i$ or $B \subseteq I_i$ or $C \subseteq I_i$, $\forall\, i$. Then $I_i \in \mathcal{U}\cap\overline{A}$ or $I_i \in \mathcal{U}\cap\overline{B}$ or $I_i \in \mathcal{U}\cap\overline{C}$ give $I_i \in (\mathcal{U}\cap\overline{A})\cup(\mathcal{U}\cap\overline{B})\cup(\mathcal{U}\cap\overline{C})$. Therefore $\mathcal{U} = (\mathcal{U}\cap\overline{A})\cup(\mathcal{U}\cap\overline{B})\cup(\mathcal{U}\cap\overline{C}) = \big[(\mathcal{U}\cap\overline{A})\cup(\mathcal{U}\cap\overline{B})\big]\cup(\mathcal{U}\cap\overline{C})$. But $(\mathcal{U}\cap\overline{A}), (\mathcal{U}\cap\overline{B})$ and $(\mathcal{U}\cap\overline{C})$ are closed subsets of $\mathcal{U}$ and $\mathcal{U}$ is irreducible imply, $\mathcal{U} = (\mathcal{U}\cap\overline{A})\cup(\mathcal{U}\cap\overline{B})$ or $\mathcal{U} = (\mathcal{U}\cap\overline{C}) \Rightarrow \mathcal{U} = (\mathcal{U}\cap\overline{A})$ or $\mathcal{U} = (\mathcal{U}\cap\overline{B})$ or $\mathcal{U} = (\mathcal{U}\cap\overline{C})$. Hence $\mathcal{U} \subseteq \overline{A}$ or $\mathcal{U} \subseteq \overline{B}$ or $\mathcal{U} \subseteq \overline{C}$. This shows that, $A \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$ or $B \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$ or $C \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$. Therefore $\bigcap_{I_i\in\mathcal{U}} I_i$ is a prime ideal of T.

Conversely, suppose that $\bigcap_{I_i\in\mathcal{U}} I_i$ is a prime ideal of T. To show that $\mathcal{U}$ is irreducible. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are closed subsets of $\mathcal{U}$ such that $\mathcal{U} = \mathcal{X}\cup\mathcal{Y}\cup\mathcal{Z}$. Then $\bigcap_{I_i\in\mathcal{U}} I_i \subseteq \bigcap_{I_i\in\mathcal{X}} I_i$, $\bigcap_{I_i\in\mathcal{U}} I_i \subseteq \bigcap_{I_i\in\mathcal{Y}} I_i$ and $\bigcap_{I_i\in\mathcal{U}} I_i \subseteq \bigcap_{I_i\in\mathcal{Z}} I_i$. We have, $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{X}\cup\mathcal{Y}\cup\mathcal{Z}} I_i = \left(\bigcap_{I_i\in\mathcal{X}} I_i\right)\cap\left(\bigcap_{I_i\in\mathcal{Y}} I_i\right)\cap\left(\bigcap_{I_i\in\mathcal{Z}} I_i\right)$

Now, $\left(\bigcap_{I_i\in\mathcal{X}} I_i\right)\left(\bigcap_{I_i\in\mathcal{Y}} I_i\right)\left(\bigcap_{I_i\in\mathcal{Z}} I_i\right) \subseteq \left(\bigcap_{I_i\in\mathcal{X}} I_i\right)\cap\left(\bigcap_{I_i\in\mathcal{Y}} I_i\right)\cap\left(\bigcap_{I_i\in\mathcal{Z}} I_i\right) = \bigcap_{I_i\in\mathcal{X}\cup\mathcal{Y}\cup\mathcal{Z}} I_i = \bigcap_{I_i\in\mathcal{U}} I_i.$

Since, $\bigcap_{I_i\in\mathcal{U}} I_i$ is prime ideal of T, then we have $\bigcap_{I_i\in\mathcal{X}} I_i \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$ or $\bigcap_{I_i\in\mathcal{Y}} I_i \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$ or $\bigcap_{I_i\in\mathcal{Z}} I_i \subseteq \bigcap_{I_i\in\mathcal{U}} I_i$. Therefore $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{X}} I_i$ or $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{Y}} I_i$ or $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{Z}} I_i$. Now for any $I_k \in \mathcal{U}$. Then we have $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{X}} I_i \subseteq I_k$ or $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{Y}} I_i \subseteq I_k$ or $\bigcap_{I_i\in\mathcal{U}} I_i = \bigcap_{I_i\in\mathcal{Z}} I_i \subseteq I_k$. Since $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are closed subsets of $\mathcal{U}$, so either $I_i \subseteq I_k$ for all $I_i \in \mathcal{X}$ or $I_i \subseteq I_k$ for all $I_i \in \mathcal{Y}$ or $I_i \subseteq I_k$ for all $I_i \in \mathcal{Z}$. Thus $I_k \in \overline{\mathcal{X}} = \mathcal{X}$ or $I_k \in \overline{\mathcal{Y}} = \mathcal{Y}$ or $I_k \in \overline{\mathcal{Z}} = \mathcal{Z}$, since $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are closed subsets of $\mathcal{U}$. Therefore $\mathcal{U} \subseteq \mathcal{X}$ or $\mathcal{U} \subseteq \mathcal{Y}$ or $\mathcal{U} \subseteq \mathcal{Z}$. Hence $\mathcal{U} = \mathcal{X}$ or $\mathcal{U} = \mathcal{Y}$ or $\mathcal{U} = \mathcal{Z}$. Consequently, $\mathcal{U}$ is irreducible. □

For any subset $\mathcal{U}$ of $\mathcal{P}$ we define $R(\mathcal{U}) = \bigcap_{I_j\in\mathcal{U}} I_j$. Clearly $R(\mathcal{P}) = \bigcap_{I_j\in\mathcal{P}} I_j$ is $\mathcal{P}$-radical of T. Always $R(\mathcal{P}) \subseteq R(\mathcal{U})$. We know that $\mathcal{U} \subseteq \mathcal{P}$ is dense in $\mathcal{P}$ if $\overline{\mathcal{U}} = \mathcal{P}$.

Theorem 3.20. *The subset $\mathcal{U}$ of $\mathcal{P}$ is dense in $\mathcal{P}$ if and only if $R(\mathcal{U}) = R(\mathcal{P})$.*

Proof. Assume that the subset $\mathcal{U}$ of $\mathcal{P}$ is dense in $\mathcal{P}$. As $\mathcal{U} \subseteq \mathcal{P}$, we have $R(\mathcal{P}) \subseteq R(\mathcal{U})$. To show that $R(\mathcal{U}) \subseteq R(\mathcal{P})$. As $\overline{\mathcal{U}} = \mathcal{P}$ gives $\overline{\mathcal{U}} = \{I \in \mathcal{P} : \bigcap_{I_\alpha\in\mathcal{U}} I_\alpha \subseteq I\} = \mathcal{P}$. $A \in \mathcal{P}$ implies $A \in \overline{\mathcal{U}}$. Then $R(\mathcal{U}) \subseteq A$. As this is true for each $A \in \mathcal{P}$, we get $R(\mathcal{U}) = \bigcap_{I_\alpha\in\mathcal{U}} I_\alpha \subseteq \bigcap_{I_\alpha\in\mathcal{P}} I_\alpha = R(\mathcal{P})$. Hence $R(\mathcal{U}) = R(\mathcal{P})$.

Conversely, suppose that $R(\mathcal{U}) = R(\mathcal{P})$. To show that $\overline{\mathcal{U}} = \mathcal{P}$. Assume that $\mathcal{P} \setminus \overline{\mathcal{U}} \neq \emptyset$. Then there is a prime ideal say A of T such that $A \in \mathcal{P} \setminus \overline{\mathcal{U}}$ that is $A \in \mathcal{P}$ and $A \notin \overline{\mathcal{U}}$. $A \notin \overline{\mathcal{U}}$ implies there exists any open set say $Y(I)$ containing A such that $Y(I) \cap \big(\overline{\mathcal{U}} \setminus \{A\}\big) = \emptyset$. That is open set of $\mathcal{P}$ containing A does not contains any other element of $\mathcal{U}$ other than A. Therefore $R(\mathcal{P}) = \bigcap_{I_\alpha\in\mathcal{P}} I_\alpha \subset R(\mathcal{U}) = \bigcap_{I_\alpha\in\mathcal{U}} I_\alpha$. Then $R(\mathcal{U}) \neq R(\mathcal{P})$, which contradicts our hypothesis. Thus $\mathcal{P} \setminus \overline{\mathcal{U}} = \emptyset$. Hence $\overline{\mathcal{U}} = \mathcal{P}$ i.e. $\mathcal{U}$ is dense in $\mathcal{P}$. □

Definition 3.21. A ternary semigroup T is called a Noetherian ternary semigroup if it satisfies the ascending chain condition for ideals of T, for any sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ of ideals of T, then there exists a positive integer m such that $I_m = I_{m+1} = \dots$

Theorem 3.22. [4] *A topological space is compact if and only if each family of closed sets which has the finite intersection property has a non-void intersection.*

Theorem 3.23. *If T is a Noetherian ternary semigroup then the structure space $(\mathcal{P}, \tau)$ is countably compact.*

Proof. Let $\{X(I_n)\}_{n=1}^{\infty}$ be a countable collection of closed sets in $\mathcal{P}$ with finite intersection property. Let us consider the following ascending chain of prime ideals of T,

$$\langle I_1\rangle \subseteq \langle I_1 \cup I_2\rangle \subseteq \langle I_1 \cup I_2 \cup I_3\rangle \subseteq \dots$$

Since T is a Noetherian ternary semigroup there exist a positive integer m such that,

$$\langle I_1 \cup I_2 \cup \dots \cup I_m\rangle = \langle I_1 \cup I_2 \cup \dots \cup I_{m+1}\rangle = \dots$$

Thus it follows that $\langle I_1 \cup I_2 \cup \dots \cup I_m\rangle \in \bigcap\limits_{n=1}^{\infty} X(I_n)$. Hence $\bigcap\limits_{n=1}^{\infty} X(I_n) \neq \emptyset$ and hence $(\mathcal{P}, \tau)$ is countably compact. □

Corollary 3.24. *If T is a Noetherian ternary semigroup and $(\mathcal{P}, \tau)$ is second countable then $(\mathcal{P}, \tau)$ is compact.*

Proof. Proof follows from Theorem 3.23 and the fact that a second countable space is compact if it is countably compact. □

Remark 3.25. The set of all idempotent elements of T is denoted by $E(T)$, i.e. $E(T) = \{a \in T : aaa = a\}$.

Definition 3.26. An ideal I of T is said to be full ideal if $E(T) \subseteq I$.

Definition 3.27. An ideal I of T is said to be a prime full ideal if it is both prime and full ideal.

Let $\mathcal{F}$ be the family of all prime full ideals of T. Then we see that $\mathcal{F}$ is a subset of $\mathcal{P}$ and $(\mathcal{F}, \tau_{\mathcal{F}})$ is a topological space where $\tau_{\mathcal{F}}$ is the subspace topology.

Theorem 3.28. *The space $(\mathcal{F}, \tau_{\mathcal{F}})$ is a compact space if $E(T) \neq \{0\}$.*

Proof. Let $\{X(I_i)\}_{i\in\Delta}$ (where Δ is any indexing set) be any collection of closed sets in $\mathcal{F}$ with finite intersection property. Let I be the prime full ideal generated by $E(T)$. Since any prime full ideal J of T contains $E(T)$, then J contains I. Hence $I \in \bigcap\limits_{i\in\Delta} X(I_i) \neq \emptyset$. Consequently, the space $(\mathcal{F}, \tau_{\mathcal{F}})$ is a compact space. □

Theorem 3.29. *The space $(\mathcal{F}, \tau_{\mathcal{F}})$ is a connected space if $E(T) \neq \{0\}$.*

Proof. Let I be the prime ideal generated by $E(T)$. Since any prime full ideal J contains $E(T)$, J contains I. Hence I belongs to any closed set $X(K)$ of $\mathcal{F}$. Consequently, any two closed sets of $\mathcal{F}$ are not disjoint. Hence $(\mathcal{F}, \tau_{\mathcal{F}})$ is a connected space. □

Acknowledgment. The authors express gratitude to the anonymous reviewers for their invaluable feedback and suggestions, which have significantly enhanced the paper's overall quality and clarity of presentation.

References

1. Dixit, V. N. and Dewan, S.: A note on quasi and bi-ideals in ternary semigroups, *International Journal of Mathematics and Mathematical Sciences* **18(3)** (1995) 501 – 508.
2. Jagtap, R. D. and Pawar, Y. S.: Structure space of prime ideals of Γ-semirings, *Palestine Journal of Mathematics* **5(1)** (2016) 166–170.
3. Kar, S. and Chattopadhyay, S.: On structure space of Γ-semigroups, *Acta Univ. Palack. Olomuc., Fac. Rer. Nat., Mathematica* **47(1)** (2008) 37–46.
4. Kelly, J. L.: *General Topology*, New York, Van Nostrand, 1969.
5. Kostaq, H., Vardhami, I. and Gjino, K.: On the topological structures on Γ-semirings, *European Scientific Journal* **9(21)** (2013) 17–30.
6. Lehmer, D. H.: A ternary analogue of abelian groups, *Amer. J. Math.* **54** (1932) 329–338.
7. Sarala, Y., Anjaneyulu, A. and Madhusudhanarao, D.: Ideals in ternary semigroups, *International e-Journal of Mathematics and Engineering* **203** (2013) 1950 – 1968.
8. Shabir, M. and Bashir, S.: Prime ideals in ternary semigroups, *Asian European Journal of Mathematics* **2(1)** (2009) 141–154.
9. Sioson, F. M.: Ideal theory in ternary semigroups, *Math. Japan* **10** (1965) 63 – 84.

Machchhindra Gophane: Department of Mathematics, Shivaji University, Kolhapur-416004, Maharashtra, India.

Email address: mtg _ maths@unishivaji.ac.in

Dattatray Shinde: Department of Mathematics, Shivaji University, Kolhapur-416004, Maharashtra, India.

Email address: shindednmaths@gmail.com

Vinod Kharat: Department of Mathematics, N. B. Navale Sinhagad College of Engineering, Kegaon, Solapur, Maharashtra, India.

Email address: vvkvinod9@gmail.com