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A NEW DISCRETE WEIBULL DISTRIBUTION AND ITS APPLICATION IN IMMUNOGOLD ASSAY DATA

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Abstract. In the present scenario, discretization of continuous distribution has got special attention in reliability and life testing experiments. This is because the existing models may be incompatible with continuous measurements. For discretizing continuous distributions, various methods are available in literature. Among them a newly developed method is the two stage composite discretization method which includes three different methodologies. Using these methods, we discretized Weibull distribution thereby some new discretized distributions are obtained and that are the proposed new distributions here. A comparative study based on the hazard rate functions of the newly proposed distributions with the existing distributions is also made. A real data analysis is also carried out using immunogold assay data.

1. Introduction

Recently discrete distributions play a vital role in modeling survival data instead of continuous measurements. Usually almost all lifetimes are measured on continuous scale and based on which many continuous lifetime distributions are proposed by many researchers. But in practice, it is not always possible to consider the lifetime data as continuous. For example, although the time spend by a patient in a hospital ward is treated as continuous, it can also be considered in a discrete manner as the number of days or weeks. In certain cases the discrete measurement will be more suitable for appropriate modeling of the data. In reliability analysis, for modeling data much emphasizes has been given to continuous lifetime distributions. But when deriving the reliability of a system using continuous distributions the processes results in derivational difficulty and lack of closed analytic form. More over if we use approximation procedures that will result in discretization. Also the existing discrete distributions are not able to accommodate many practical situations. Hence we have to propose new discrete lifetime distributions and it can be done by discretizing continuous distributions. Now a days, there are various methods available for discretizing continuous distributions. Alzaatrech et al. (2012) and Chakraborty (2015) conducted a detailed study of these methods and using them, researchers developed discrete analogues of various continuous distributions so far, for details, see Lisman and van Zuylen (1972), Stein and Dattero (1984), Kemp (1997), Das Gupta (1993), Gupta and Kundu (1999), Szablowski (2001), Dilip Roy (2003, 2004), Inusah and Kozubowski (2006), Kozubowski and

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Inusah (2006), Krishna and Pundir (2009), Jazi et al. (2010), Nekoukhou et al. (2012), Lekshmi and Sebastian (2014), Nekoukhou and Hamed Bidram (2015), Chakraborty (2015), Alamastaz et. al (2016), Chakraborty and Chakravarthy (2016), Josmar et al.(2017), Berhane Abebe and Rama Shanker (2018), Ganji and Gharari (2018), Jayakmar and Sankaran (2018), Yari and Tondpour (2018), Abebe and Shukla(2019), Krishnakumari and Dais (2020) and Morshedy et.al (2020). So we concentrate our study in the development of some new discretized distributions from their continuous counterpart.

The remaining sections of the manuscript is as follows. The discretization method introduced by Yari and Tondpour (2018) is discussed in section 2. In section 3, some new discrete life time distributions are proposed and studied. In section 4, we compare the newly proposed distributions with the existing distributions using hazard rate functions. The parameters of the distribution is estimated in section 5. A real data analysis is carried out in section 6. We concluded the article by section 7.

2. Discretization Methods

Discrete distributions play an extensive role in modeling real life situations. A large number of discrete distributions are so far proposed and studied by many researchers. For details, see the books by Balakrishnan and Nevzorov (2003), Jonson et al. (2005) and Consul and Famoye (2006). In this section instead of considering discrete distributions we are concentrating on the discretization of Weibull distribution using the methods proposed by Yari and Tondpour (2018). Actually three methods are proposed and each method consists of two stages. In the first stage a new continuous random variable is constructed from the underlying continuous random variable and in the second stage, this new random variable is discretized maintaining the same hazard rate function. The following are the three methodologies used.

2.1. Methodology I. In this methodology, in the first stage, we construct a new continuous random variable X_1 with hazard rate function

$$h_{X_1}(x) = e^{-F(x)},\ x \geq 0. \tag{2.1}$$

from X, another variable which is also continuous having support $[0, \infty)$ and distribution function $F_X(x)$. In the second stage, with the help of the following methodology, Y is derived by discretizing X_1 by maintaining the form of the hazard rate function of Y as that X_1. If $S_{X_1}(x)$, the survival function and $h_{X_1}(x)$, the hazard rate function of $S_{X_1}(x)$ then survival function of Y is obtained as

$$S_Y(k) = (1 - h_{X_1}(1))(1 - h_{X_1}(2))...(1 - h_{X_1}(k-1)); k = 1, 2, ..., m$$

and the corresponding probability mass function is

$$P(Y = k) = \begin{cases} h_{X_1}(0));\ k = 0 \\ \\ ((1 - h_{X_1}(1))(1 - h_{X_1}(2))...(1 - h_{X_1}(k-1)))h_{X_1}(k); \\ \ k = 1, 2, ..., m \\ 0;\ otherwise. \end{cases} \tag{2.2}$$

If P(Y=k) is such that the total probability is not equal to one, then for ensuring it as one we multiply every P(y) with w, a positive constant. Hence the probability mass function becomes

$$P(Y=y) = \begin{cases} w;\ y=0 \\ wh_{X_1}(y)\Pi_{i=1}^{y-1}(1-h_{X_1}(i));\ y=1,2,...,m \\ 0;\ otherwise. \end{cases} \quad . \tag{2.3}$$

where m can be finite or infinite since $h_{X_1}(x)$ is always between zero and one. Now, by using (2.3) the resulting pmf of Y in the new methodology is

$$P(Y=y) = \begin{cases} w;\ y=0 \\ \\ we^{-F_X(x)}\Pi_{i=1}^{y-1}(1-e^{-F_X(i)});\ y=1,2,...,m \\ \\ 0;\ otherwise. \end{cases} \quad . \tag{2.4}$$

2.2. Methodology II. In this method, we construct X_1 a new continuous random variable with hazard rate function

$$h_{X_1}(x) = \frac{2F_X(x)}{1+F_X(x)}$$

using the variable X which is also continuous with support $[0,\infty)$ and having distribution function $F_X(x)$. Then in the second stage a discrete analogue of X_1 say Y is derived with the help of (2.3). Also note that discrete distributions obtained in this methodology has increasing hazard rate function.

2.3. Methodology III. Here, in the first stage a new continuous random variable X_1 is constructed from another variable X (continuous) with support $[0,\infty)$ and distribution function $F_X(x)$ .The hazard rate function of X_1 is

$$h_{X_1}(x) = \frac{1}{f_X(x)+1}\ x \geq 0.$$

Then in the second stage, by using (2.3), Y, a discrete analogue of X_1 is derived . Here the hazard rate function of Y is decreasing or increasing on (a,b) where a, b$\in R^+$ according as $f_X(x)$ is increasing or decreasing on the same interval.

In the first two methods, we obtain respectively the decreasing and increasing hazard rate functions for Y and in the third method they can be increasing, U shaped or modified unimodal. An important advantage of the method is that discrete analogues obtained have monotonic and non-monotonic hazard rate functions.

3. Some New Discrete Lifetime Distributions

Weibull distribution is the most commonly used distribution for modeling data related to reliability engineering and life time analysis because of its adaptability. It is one of the popular distribution for modeling phenomenon with monotonic failure rate. A variety of life behaviors can be modeled by Weibull distribution by varying the values of parameters. A discrete analogue of continuous Weibull distribution is derived by maintaining one or more discriminative property of the continuous distribution. Discrete Weibull distribution is more useful in acceptance sampling, corrosion modeling, wear modeling, electronic failure modeling etc. While considering reliability analysis, it can be used for modeling failure data arose from different fields. For details see Jayakumar and Sankaran (2018), Kundu and Nekokhou (2019). In this section, we introduce some new discretized distributions derived from the continuous Weibull distribution namely, New Discrete Weibull (Type I, Type II and Type III) distributions using the method explained in section 2.

3.1. New Discrete Weibull Type I Distribution. A two parameter Weibull distribution with parameters λ and $\beta(\lambda > 0$ and $\beta > 0)$, is considered here with pdf

$$f(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{(\beta-1)} e^{-(\frac{x}{\lambda})^{\beta}};\ x \geq 0 \tag{3.1}$$

and distribution function

$$F(x) = 1 - e^{-(\frac{x}{\lambda})^{\beta}};\ x \geq 0.$$

Applying methodology I of section 2, the discrete version of (3.1) takes the form

$$P(Y=y) = \begin{cases} w;\ y=0 \\ w e^{e^{-(\frac{y}{\lambda})^{\beta}}-1} \Pi_{i=1}^{y-1}(1-e^{e^{-(\frac{i}{\lambda})^{\beta}}-1});\ y=1,2,...,m \\ 0;\ otherwise. \end{cases} \tag{3.2}$$

The graph of New Discrete Weibull Type 1 distribution is given in Figure 1.

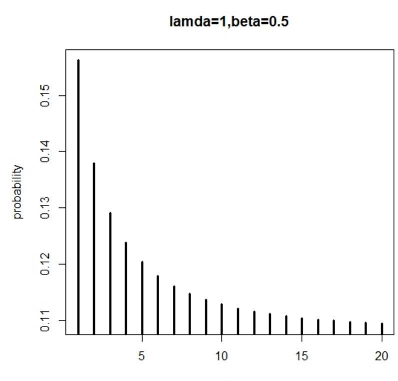

FIGURE 1. New Discrete Weibull Type I Distribution

Its hazard rate function is

$$h_Y(y) = e^{e^{-(\frac{y}{\lambda})^{\beta}}-1} \tag{3.3}$$

and it is decreasing.

3.2. New Discrete Weibull Type II Distribution. The discrete pmf of (3.1) by applying methodology II in section 2 is obtained as

$$P(Y=y)=\begin{cases} w;\ y=0 \\ \\ w\frac{2(1-e^{-(\frac{y}{\lambda})^{\beta}})}{2-e^{-(\frac{y}{\lambda})^{\beta}}}\Pi_{i=1}^{y-1}(1-\frac{2(1-e^{-(\frac{i}{\lambda})^{\beta}})}{2-e^{-(\frac{i}{\lambda})^{\beta}}};\ y=1,2,3,...,m. \\ \\ 0;\ otherwise. \end{cases} \quad . \tag{3.4}$$

The graph of New Discrete Weibull Type II distribution is given in Figure 2.

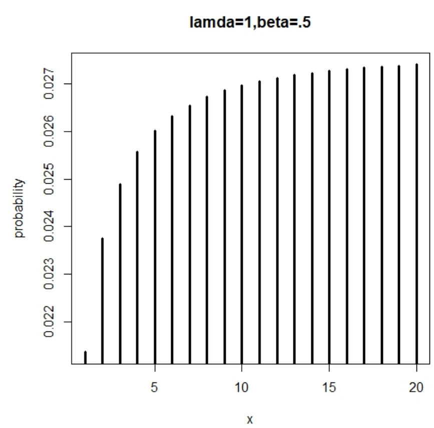

FIGURE 2. New Discrete Weibull Type II Distribution

Its hazard rate function is

$$h_Y(y)=\frac{2(1-e^{-(\frac{y}{\lambda})^{\beta}})}{2-e^{-(\frac{y}{\lambda})^{\beta}}} \tag{3.5}$$

and is an increasing function.

3.3. New Discrete Weibull Type III Distribution. New Discrete Weibull distribution of Type III obtained from (3.1) by applying methodology III is

$$P(Y=y)=\begin{cases} w;\ y=0 \\ \\ w\frac{1}{1+\frac{\beta}{\lambda}(\frac{y}{\lambda})^{(\beta-1)}e^{-(\frac{y}{\lambda})^{\beta}}}\Pi_{i=1}^{y-1}(1-\frac{1}{1+\frac{\beta}{\lambda}(\frac{x}{\lambda})^{(\beta-1)}e^{-(\frac{x}{\lambda})^{\beta}}});\ y=1,2,3,...,m \\ \\ 0;\ otherwise \end{cases} \quad . \tag{3.6}$$

having hazard rate function

$$h_Y(y)=\frac{1}{1+\frac{\beta}{\lambda}(\frac{y}{\lambda})^{(\beta-1)}e^{-(\frac{y}{\lambda})^{\beta}}}. \tag{3.7}$$

Here we obtain a U shaped hazard rate function for $\beta > 1$ and $\lambda > 3$.

The graph of New Discrete Weibull Type III distribution is given in Figure 3.

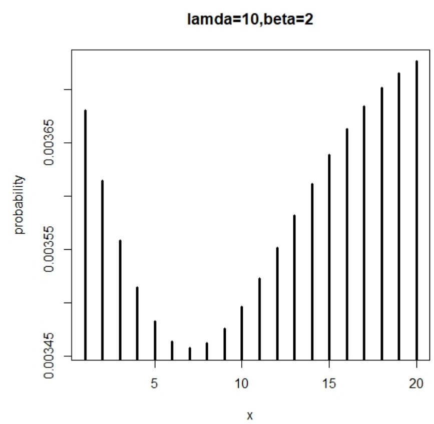

FIGURE 3. New Discrete Weibull Type III Distribution

4. Comparison

Though most of life time data are expressed in continuous measurements, there exists some situations where it can be treated in a discrete manner. In reliability and life data analysis hazard rate is an inevitable characteristic. Different shapes of hazard rate functions can give different distributions. Depending on the conditions imposed on the parameters, the shape of the hazard rate functions may be increasing, decreasing and bathtub. Discrete analogues derived from the continuous distributions have the same shape as that of the continuous one for the same parametric values. Here we discuss about the acquiescence of this fact.

Now the comparison of two parameter Weibull distribution (continuous), its discrete analogue proposed by Nakagawa and Osaki (1975) and the newly proposed discrete analogues of the same two parameter Weibull distribution is considered here. The hazard rate functions of these three distributions along with their graphs are given below. The hazard rate function of the Weibull distribution (continuous) is

$$h_X(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{(\beta-1)} \tag{4.1}$$

and its graph is given in Figure 4.

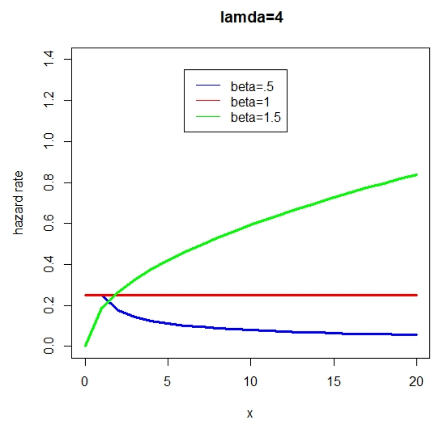

FIGURE 4. Hazard function-Two parameter Weibull Distribution

Next we consider the hazard rate function of discrete Weibull distribution introduced by Nakagawa and Osaki (1975) and it is

$$h_X(x) = 1 - (q)^{(x+1)^\beta - x^\beta}. \tag{4.2}$$

Figure 5 shows the graph of hazard rate function for q=0.6 and various values of β.

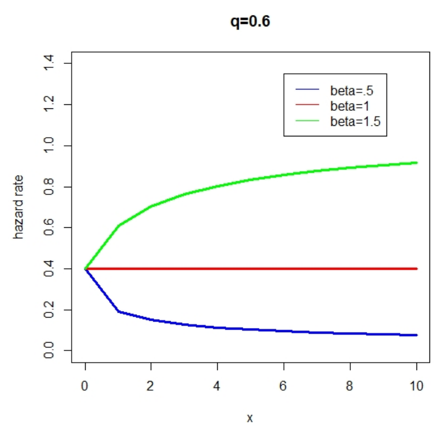

FIGURE 5. Hazard function-Discrete Weibull Distribution-Nakagawa and Osaki

The hazard rate functions of New discrete Weibull (Type I, Type II and Type

III) distributions are obtained earlier and it is given in (3.3), (3.5) and (3.7) respectively. The hazard plots are given in Figure 6.

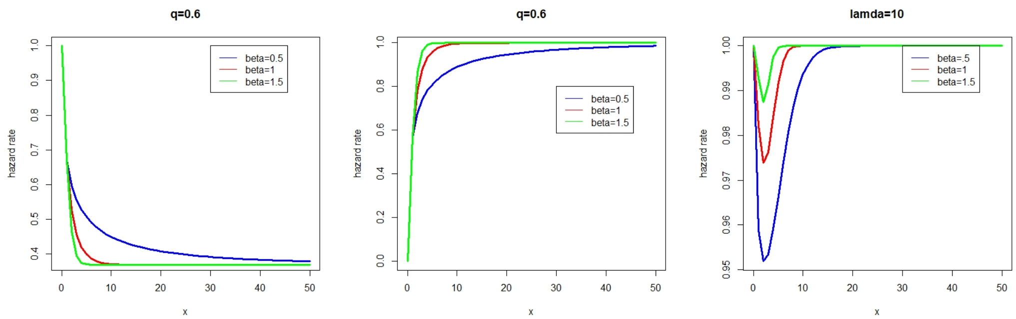

FIGURE 6. Graph of hazard rate (i) New Discrete Weibull Type I (ii)New Discrete Weibull Type II (iii) New Discrete Weibull Type III

From Figure 4 and Figure 5 , the following results were obtained.
(a) For $\beta > 1$, increasing hazard rate.
(b) For $\beta = 1$, constant hazard rate.
(c) For $\beta < 1$, decreasing hazard rate.

In many cases the hazard rate function of Discrete Weibull distributions has the same nature as that of the continuous counter part. But apart from figure 4 and 5 (nature varies on β), from figure 6 , it is seen that for all values of β, hazard function is decreasing for Type I New discrete Weibull distribution and increasing for New discrete Weibull distribution of Type II. But in the case of New discrete Weibull distribution of Type III, the hazard function attains U shape for $\beta > 1$ and $\lambda > 3$.

5. Estimation

Here, we estimate the parameters of New Discrete Webull distribution Type I Distribution using the method of least squares. The hazard rate function of this distribution is given by (3.3). Taking the logarithm of (3.3) on both sides we get

$$logh(y) = e^{-(\frac{y}{\lambda})^{\beta}} - 1$$

This can be expressed as a linear model as

$$W = A + bY$$

where

$$W = log(w)$$
$$A = log(a)$$
$$Y = log(y)$$
$$w = log(\frac{1}{1 + log(h)})$$
$$a = (\frac{1}{\lambda})^{\beta}$$

and b=β . Now for each sample observation y_i, we can estimate the corresponding hazard rate value $h(y_i)$ as the ratio of the sample frequency of values equal to y_i and the sample frequency of values equal to or greater than y_i, see Bracquemond and Gaudoin (2002) . The estimated values of the parameters are obtained as

$$\hat{b} = \frac{cov(Y,W)}{var(Y)} = \hat{\beta}$$

and

$$\hat{\lambda} = a^{\frac{-1}{b}}$$

6. Real Data Analysis

As an application of New Discrete Webull Type I Distribution, we consider the data set used by Cullen, Walsh, Nicholson, and Harris (1990). Here we give the counts of sites with 1, 2, 3, 4, and 5 particles from immunogold assay data and 122, 50, 18, 4 and 4 were the counts. Figure 7 shows the observed data.

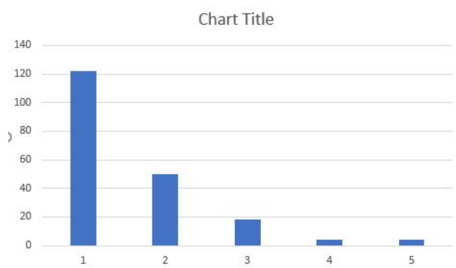

FIGURE 7. Observed data

Since it has the same shape as that of New Discrete Webull Type I Distribution, the empirical data is fitted for New Discrete Webull Type I Distribution. The parametric values estimated through least square method are $\hat{\lambda} = 12.997$ and $\hat{\beta} = 0.231203$. Embeded graph of the theoretical pmf over the empirical pmf is shown in Figure 8.

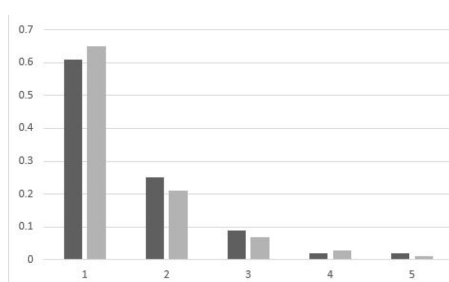

FIGURE 8.Empiriical and Theoretical pmf of New Discrete Weibull Type I

Also we check the goodness of fit using chi-square statistic and found that the proposed model is a good fit. In addition, for comparing its suitability with Discrete

Burr XII (Type I) distribution (Yari and Tondpour (2018)), the LL, AIC and BIC values are tabulated and are shown in Table 1.

TABLE 1: Values of MLE's ,Chi square, Log likelihood, AIC, and BIC values of Immunogold Assay data.

Distribution fitted	estimated values	Chi-square	LL	AIC	BIC
Discrete Burr XII (Type I)	$\hat{p} = 0.9554$ $\hat{c} = 0.3596$	1.7031	-206.7	417.48	418.07
New Discrete Weibull Type I	$\hat{\lambda} = 12.997$ $\hat{\beta} = 0.231203$	1.447	-203.6	411.2	411.79

Table 1 shows that the New Discrete Weibull Type I distribution is a better model when compared to Discrete Burr XII (Type I)distribution .

7. Summary

We usually come across situations where lifetimes are suitable to measure as discrete random variable rather than continuous one. In the past decades distributions like Geometric and Negative binomial are used to model discrete lifetime data. But the need for more distributions and the availability of various discretization methods lead to a number of discrete distributions including discrete Weibull distributions suitable to various situations. In reliability and lifetime analysis, hazard function is a chief characteristic. Hazard rate function of the existing discrete Weibull distributions has the same nature as that of the continuous counter part and the nature varies according to the values of the parameter β. In this work, we derive some new discrete life time distributions viz. New Discrete Weibull (Type I, II and III) with the property that for all values of β , hazard function is decreasing for Type I, increasing for Type II and for Type III it attains U shape for $\beta > 1$ and $\lambda > 3$. This property is illustrated through a comparative study based on hazard rate functions of these newly proposed distributions with the existing distributions . A real data analysis is also carried out using immunogold assay data and finds that our newly proposed New Discrete Weibull Type I distribution provides a better model than the existing ones.

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