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VERSAL DEFORMATIONS OF AFFINE VECTOR FIELDS ON TORUS

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ABSTRACT. We study a classical problem of describing the versal deformations of a centrally extended metrized Lie algebra generated by the direct sum of affine vector fields and differential forms on torus

1. UNIVERSAL DEFORMATIONS OF VECTOR FIELDS AND DIFFERENTIAL FORMS

1.1. **Deformations.** We consider a smooth vector field $A \in \Gamma(T(\mathbb{T}^n))$ on the n-dimensional torus $\mathbb{T}^n$. A deformation of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ we will call a vector field $A(\tau) \in \Gamma(T(\mathbb{T}^n))$, which depends analytically on the parameter $\tau \in \mathbb{C}^k, k \in \mathbb{Z}_+$, in some vicinity of the point $\tau = 0 \in \mathbb{C}^k$, and such that $A(0) = A$. The space of parameters $\Upsilon\{\tau \in \mathbb{C}^k\}$ is ofen called a base of the deformation. Similarly will consider a differential 1-form $l \in \Lambda^1(\mathbb{T}^n)$ on the n-dimensional torus $\mathbb{T}^n$ its related deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$, which depends analytically on the parameter $\tau \in \mathbb{C}^k, k \in \mathbb{Z}_+$, in some vicinity of the point $\tau = 0 \in \mathbb{C}^k$ and such that $l(0) = l$.

Definition 1.1. Two vector fields deformations $A(\tau)$ and $B(\tau) \in \Gamma(T(\mathbb{T}^n))$ are called equivalent, if there exists such a deformation $g(\tau) \in Diff(\mathbb{T}^n)$ of the identity $Id \in Diff(\mathbb{T}^n)$, that $Ad_{g(\tau)}A(\tau) = B(\tau)$, where $ad : Diff(\mathbb{T}^n) \times \Gamma(T(\mathbb{T}^n)) \to \Gamma(T(\mathbb{T}^n))$ is the usual [2, 4, 6] adjoint mapping of the space $Diff(\mathbb{T}^n)$ on $\Gamma(T(\mathbb{T}^n))$. Similarly, two 1-form deformations $l(\tau)$ and $p(\tau) \in \Lambda^1(\mathbb{T}^n)$ are called equivalent, if there exists such a deformation $g(\tau) \in Diff(\mathbb{T}^n)$ of the identity $I \in Diff(\mathbb{T}^n)$, that $Ad^*_{g(\tau)}l(\tau) = p(\tau)$, where $Ad^* : Diff(\mathbb{T}^n) \times \Lambda^1(\mathbb{T}^n) \to \Lambda^1(\mathbb{T}^n)$ is the usual adjoint mapping of the space $Diff(\mathbb{T}^n)$ on $\Lambda^1(\mathbb{T}^n)$.

Let φ - a germ of a holomorphic at zero mapping $\mathbb{C}^m \to \mathbb{C}^k$, that is a set of converging at $0 \in \mathbb{C}^m$ degree series of complex varibles, and assume that $\varphi(0) = 0$. The mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^m\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ defines evidently a new deformation $\check{\varphi}l(\sigma) \in \Lambda^1(\mathbb{T}^n)$ of the 1-form $l \in \Lambda^1(\mathbb{T}^n)$ and a new deformation $\hat{\varphi}A(\sigma)$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ via the expressions

$$(\check{\varphi}l)(\sigma) = l(\varphi(\sigma)), \quad (\hat{\varphi}A)(\sigma) = A(\varphi(\sigma)) \tag{1.1}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$.

Definition 1.2. The deformation $(\check{\varphi}l)(\sigma) \in \Lambda^1(\mathbb{T}^n)$ is called induced from the deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ under the mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^m\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$. Similarly, the deformation $(\hat{\varphi}A)(\sigma) \in \Gamma(T(\mathbb{T}^n))$ is called induced from the deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ under the mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.

1.2. **Versal deformations.**

Definition 1.3. A vector field deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ is called [2] *versal,* if it generates every other deformation $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$, that is there exists such a mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ and a deformation $g(\tau) \in Diff(\mathbb{T}^n)$ of the identity $Id \in Diff(\mathbb{T}^n)$ that it is equivalent to the deformation obtained from the induced deformation $A(\varphi(\tau)) \in \Gamma(T(\mathbb{T}^n))$:

$$B(\sigma) = Ad_{g(\sigma)}(\hat{\varphi}A)(\sigma) \tag{1.2}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$. Similarly, a 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ is called *versal,* if it generates every other 1-form deformation $p(\tau) \in \Lambda^1(\mathbb{T}^n)$ of the 1-form $l \in \Lambda^1(\mathbb{T}^n)$, that is there exists such a mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ and a deformation $g(\sigma) \in Diff(\mathbb{T}^n)$

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of the identity $I \in Diff(\mathbb{T}^n)$ that it is equivalent to the deformation obtained from the induced deformation $p(\varphi(\sigma)) \in \Lambda^1(\mathbb{T}^n)$:

$$p(\sigma) = Ad^*_{g(\sigma)}(\check{\varphi} l)(\sigma) \tag{1.3}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$.

2. Versality and transversality

2.1. **Transversality.** Let $N \subset M$ - a smooth submanifold of a manifold M. Consider a smooth mapping $A : \Upsilon \to M$, and let a point $\tau \in \Upsilon$ for which $A(\tau) \in N$.

Definition 2.1. A mapping $A : \Upsilon \to M$ is called transvesal [2] to the submanifold $N \subset M$, if

$$T_{A(\tau)}(M) = T_{A(\tau)}(N) + A_* T_\tau(\Upsilon). \tag{2.1}$$

As the diffeomorphism group $Diff(\mathbb{T}^n)$ naturally acts on a fixed vector field $A \in \Gamma(T(\mathbb{T}^n))$, its orbit $Or(A; Diff(\mathbb{T}^n)) = Ad_{Diff(\mathbb{T}^n)}A \subset \Gamma(T(\mathbb{T}^n))$. Thus, a deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ can be considered as a mapping $A : \Upsilon \to \Gamma(T(\mathbb{T}^n))$ of the deformation base $\Upsilon\{\sigma \in \mathbb{C}^m\}$ into the space of vector fields $\Gamma(T(\mathbb{T}^n))$ on the torus $\mathbb{T}^n$. The following lemma [2] holds.

Lemma 2.2. *A deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ is versal iff the mapping $A : \{\tau \in \mathbb{C}^m\} \to \Gamma(T(\mathbb{T}^n))$ is transversal to the orbit of the corresponding element $A \in \Gamma(T(\mathbb{T}^n))$, that is any deformation $B(\sigma) = Ad^*_{g(\sigma)}(\hat{\varphi} A)(\sigma)$ on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.*

Proof. Really, owing to the versality condition (1.2), for any deformation $B(\tau) \in \Gamma(T(\mathbb{T}^n))$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ one has

$$B(\tau) = Ad_{g(\tau)}(\hat{\varphi} A)(\tau) \tag{2.2}$$

on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$. Then, upon differentiating (2.2) with respect to $\tau \in \Upsilon$ one obtains that

$$B_*(0)\xi = A_*(0)\hat{\varphi}(0)\xi + [C_*(0)\xi, A] \tag{2.3}$$

for any $\xi \in T(\Upsilon)$,. where $[\cdot,\cdot]$ is the usual commutator of vector fields on $\mathbb{T}^n$ and $\nabla_\tau \varphi(\tau)|_{\tau=0} := \xi \in T_0(\Upsilon)$, $\nabla_\tau g(\tau)|_{\tau=0} := C_*(0) \in \Gamma(T(\mathbb{T}^n))$. Now it is easy to see that (2.3) is equivalent to the transversality condition (2.1), if to put $M := \Gamma(T(\mathbb{T}^n)), N := Or(A; Diff(\mathbb{T}^n) \subset \Gamma(T(\mathbb{T}^n))$.

Consider now a smooth mapping $\alpha : Diff(\mathbb{T}^n) \to \Gamma(T(\mathbb{T}^n))$, where

$$\alpha(g) := Ad_g A, \tag{2.4}$$

which induces the tangent mapping $\alpha_* : diff(\mathbb{T}^n) \to T_A(\ \Gamma(T(\mathbb{T}^n)))$, where $diff(\mathbb{T}^n) := T_{Id}(Diff(\mathbb{T}^n)$ is the Lie algebra of vector fields on the torus $\mathbb{T}^n$ and acts as

$$\alpha_* C = [C, A]. \tag{2.5}$$

The kernel $Ker\alpha_*$ is a Lie subalgebra of vector fields commuting with the vector field $A \in \Gamma(T(\mathbb{T}^n))$ and is called its *centralizer.* It is also interesting to observe that the codimension $co\dim Or(A; Diff(\mathbb{T}^n) = \dim Ker\alpha_*$. As a result from reasoning in [2] for small enough $\tau \in \Upsilon$ there exists an invertible mapping $\beta : V \times \Upsilon\{\tau \in \mathbb{C}^m\} \to \Gamma(T(\mathbb{T}^n))$ for V to be a submanifold of $Diff(\mathbb{T}^n)$, transversal to the centralizer $\dim Ker\alpha_*$ and of maximal dimension $\dim V = \dim Or(A; Diff(\mathbb{T}^n)$, allowing the representation

$$\beta(g, \tau) = Ad_g A(\tau) \tag{2.6}$$

on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$ for some $g \in V$. Let now $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ be an arbitrary transversal deformation. Then it can be represented as $B(\sigma) = \beta(v, \tau)$, giving rise to the following expression:

$$B(\sigma) = Ad_{g(\sigma)}A(\varphi(\sigma)) \ , \tag{2.7}$$

where $\varphi(\sigma) := \pi_2 \beta^{-1}(B(\sigma)), g(\sigma) := \pi_1 \beta^{-1}(B(\sigma))$ and π_1 and π_2 are projections of $V \times \Upsilon\{\tau \in \mathbb{C}^k\}$ on the first and the second factor, respectively. The obtained expression (2.7) exactly means that this arbitrary deformation $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ is versal, thus proving the lemma. □

Consider now a 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$. The same way as above one can prove the following dual to Lemma (2.2) proposition.

Proposition 2.3. *A 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ is versal iff the mapping $l : \Upsilon\{\tau \in \mathbb{C}^m\} \to \Lambda^1(\mathbb{T}^n)$ is transversal to the orbit of the corresponding element $l \in \Lambda^1(\mathbb{T}^n)$, that is any deformation $p(\sigma) = Ad^*_{g(\sigma)}(\check{\varphi} l)(\sigma)$ on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.*

Being interested in describing versal deformations of pencils of differential forms, analytically depending on the *"spectral"* parameter $\lambda \in \mathbb{C}$, we will proceed below first to studing their orbits from the Marsden-Weinstein reduction theory point of view.

3. Torus diffeomorphism group and its orbits

Let us now consider the action of the diffeomorphism group $Diff(\mathbb{T}^n)$ on the space $\mathcal{G} := diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$, being the semidirect product $\Gamma(T(\mathbb{T}^n)) \ltimes \Lambda^1(\mathbb{T}^n) \simeq diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$. It is well known [?] that the semidirect sum $\mathcal{G} = diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$ is a metrized Lie algebra with the Lie structure

$$[a_1 \ltimes l_1, a_2 \ltimes l_2] := [a_1, a_2] \ltimes (ad^*_{a_1} l_2 - ad^*_{a_2} l_1), \tag{3.1}$$

allowing to identify it with its adjoint space $\mathcal{G}^* \simeq \mathcal{G}$ via the nondegenerate and symmetric scalar product

$$(a_1 \ltimes l_1, a_2 \ltimes l_2) = (l_1, a_2) + (l_2, a_1) \tag{3.2}$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \mathcal{G}^* \simeq \mathcal{G}$, where $(\cdot,\cdot) : \Lambda^1(\mathbb{T}^n) \times \Gamma(T(\mathbb{T}^n)) \to \mathbb{C}$ is the standard pairing.

Consider now the point product $\check{\mathcal{G}} := \prod_{z \in \mathbb{S}^1} \tilde{\mathcal{G}}$ of Lie algebra $\mathcal{G}$ and endow it wit the central extension generated by a two-cocycle $\omega_2 : \check{\mathcal{G}} \times \check{\mathcal{G}} \to \mathbb{C}$, where

$$\omega_2(a_1 \ltimes l_1, a_2 \ltimes l_2) := \int_{\mathbb{S}^1} [(l_1, \partial a_2/\partial z) - (l_2, \partial a_1/\partial z)] dz \tag{3.3}$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \check{\mathcal{G}}$. Thus, the adjoint space $\check{\mathcal{G}}^*$ is a Poisson manifold [2, 7, 6, 4] endowed with the canonical Lie-Poisson structure

$$\begin{aligned} \{f,h\}_0 &:= (a \ltimes l, [\nabla f(a \ltimes l), \nabla h(a \ltimes l)]) + \\ &+ \int_{\mathbb{S}^1} [< \nabla f_a(a \ltimes l), \frac{\partial}{\partial z} \nabla h_l(a \ltimes l) > - < \nabla h_a(a \ltimes l), \frac{\partial}{\partial z} \nabla f_l(a \ltimes l) >] dz, \end{aligned} \tag{3.4}$$

where $f, h \in \mathcal{D}(\check{\mathcal{G}}^*)$, $\nabla f(a \ltimes l) := \nabla f_l(a \ltimes l) \ltimes \nabla f_a(a \ltimes l) \in \check{\mathcal{G}}$, $\nabla h(a \ltimes l) := \nabla h_l(a \ltimes l) \ltimes \nabla h_a(a \ltimes l) \in \check{\mathcal{G}}$ and $\nabla : \mathcal{D}(\check{\mathcal{G}}^*) \to \check{\mathcal{G}}$ is the usual functional gradient mapping. If to take now a constant vector field $d(a \ltimes l)/ds = J(\alpha) := \sum_{j,k=\overline{1,n}} \alpha_{jk}\ \partial/\partial x_j \ltimes dx_k \in \check{\mathcal{G}}^*$, depending on the constant parameters $\alpha_{kj} \in \mathbb{C}, j,k = \overline{1,n}$, one can construct [5, 3] by means of the Lie differentiation $L_{J(\alpha,\beta)}$ of the bracket (3.4) a new Poisson bracket

$$\begin{aligned} \{f,h\}_1 &:= L_{J(\alpha,\beta)}\{f,h\}_0 - \{L_{J(\alpha,\beta)} f, h\}_0 - \{f, L_{J(\alpha,\beta)} h\}_0 = \\ &= (J(\alpha), [\nabla f(a \ltimes l), \nabla h(a \ltimes l)]), \end{aligned} \tag{3.5}$$

defined for any $f, h \in \mathcal{D}(\check{\mathcal{G}}^*)$ and satisfying the Jacobi condition.

Consider now the infinitesimal $Diff(\mathbb{T}^n)$-actions on the space $\check{\mathcal{G}}^* \simeq \check{\mathcal{G}}$ subject to the Poisson brackets (3.4) and (3.5):

$$d(a \ltimes l)/d\tau = \{h, a \ltimes l\}_0 = (-[\nabla h_l, a] + \frac{\partial}{\partial z} \nabla h_l) \ltimes (ad^*_{\nabla h_l} l - ad^*_a \nabla h_a - \frac{\partial}{\partial z} \nabla h_a) \tag{3.6}$$

subject to any function $h \in \mathcal{D}(\check{\mathcal{G}}^*)$ and

$$d(a \ltimes l)/d\xi = \{f, a \ltimes l\}_1 = -\sum_{j=\overline{1,n}} \alpha_{jk}\ [\nabla f_l, \partial/\partial x_j] \ltimes ad^*_{\nabla f_l} dx_k \tag{3.7}$$

subject to a Casimir function $f \in \mathcal{D}(\check{\mathcal{G}}^*)$, respectively to the evolution parameters τ and $\xi \in \mathbb{C}$. Making use of the vector fields (3.6) and (3.7), one can construct the following integrable on the space $\check{\mathcal{G}}^*$ distributions:

$$\mathcal{D}_0 = \{(-[\nabla h_l, a] + \frac{\partial}{\partial z}\nabla h_l) \ltimes (ad^*_{\nabla h_l} l - ad^*_a \nabla h_a - \frac{\partial}{\partial z}\nabla h_a) : h \in I_1(\check{\mathcal{G}}^*)\}, \tag{3.8}$$

where $I_1(\check{\mathcal{G}}^*)$ is the space of Casimir functions for the Poisson bracket (3.5), and

$$\mathcal{D}_1 = \{-\sum_{j,k=\overline{1,n}} \alpha_{jk}\ [\nabla f_l, \partial/\partial x_j] \ltimes ad^*_{\nabla f_l} dx_k : f \in \mathcal{D}(\check{\mathcal{G}}^*)\}, \tag{3.9}$$

as $[\mathcal{D}_0, \mathcal{D}_0] \subset \mathcal{D}_0$ and $[\mathcal{D}_1, \mathcal{D}_1] \subset \mathcal{D}_1$. The set of maximal integral submanifolds of (3.9) generates the foliation $\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0$, whose leaves are the intersections of fixed integral submanifolds $\check{\mathcal{G}}^*_J \subset \check{\mathcal{G}}^*$ of the distribution $\mathcal{D}_1$ passing through an element $a \ltimes l \in \mathcal{G}^*$ with the leaves of the distribution $\mathcal{D}_0$. If the foliation $\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0$ is sufficiently smooth, one can define the quotient manifold $\check{\mathcal{G}}^*_{red} := \check{\mathcal{G}}^*_J/(\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0)$ with its associated projection mapping $\check{\mathcal{G}}^*_J \to \check{\mathcal{G}}^*_{red}$. The structure of the reduced manifold $\check{\mathcal{G}}^*_{red}$ is characterized by the following theorem.

Theorem 3.1. *On the manifold $\check{\mathcal{G}}^*_{red}$ the pair of Poisson structures $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ are compatible, that is for any parameter $\lambda \in \mathbb{R}$ the algebraic sum $\{\cdot,\cdot\}_0 + \lambda\{\cdot,\cdot\}_1$ is Poisson too.*

A proof of Theorem 3.1 is strongly based on the classical differential-geometric Marsden-Weinstein reduced space construction.

As a consequence of Theorem 3.1 and reasonings, based on the structure of the distribution (3.8), one can describe its invariants on and a leave $\check{\mathcal{G}}^*_J$ and generate the related coordinates on the reduced manifold $\check{\mathcal{G}}^*_{red} = \check{\mathcal{G}}^*_J/(\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0)$. Thus, the related with (3.6) reduced flow on the manifold $\check{\mathcal{G}}^*_{red}$ will present the canonical representation of the studied versal deformation subject to a metric Lie algebra generated by the semidirect sum $\Gamma(T(\mathbb{T}^n)) \ltimes \Lambda^1(\mathbb{T}^n) \simeq diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$ of the smooth affine vector fields $\Gamma(T(\mathbb{T}^n))$ on the torus $\mathbb{T}^n$ and its adjoint space $\Lambda^1(\mathbb{T}^n)$. Their detailed analytical structure is under preparation and will be presented in other place.

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