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Generalized integral guiding functions and periodic solutions for inclusions with causal multioperators

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Abstract

In the present paper the method of generalized integral guiding functions is applied to study the periodic problem for a differential inclusion with a causal multioperator.

Key words: differential inclusion; causal multivalued map; periodic solution; guiding function; integral guiding function
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1 Introduction

The study of systems governed by differential and functional equations with causal operators, which is due to Tonelli [29] and Tychonov [28], attracts the attention of many researchers. The term causal arises from the engineering and the notion of a causal operator turns out to be a powerful tool for unifying problems in ordinary differential equations, integro-differential equations, functional differential equations with finite or infinite delay, Volterra integral equations, neutral functional equations et al. (see monograph [2]). Various problems for functional differential equations with causal operators were considered in recent papers [4, 5, 8, 23, 26]. In particular, boundary and periodic problems were studied in [5] and [23]. In the present paper we apply the method of generalized integral guiding functions to the investigation of the periodic problem for a differential inclusion with a multivalued causal operator.

The main ideas of the method of guiding functions were formulated by Krasnoselskii and Perov in the fifties (see [18, 19]). Being geometrically clear, this method was originally applied to the study of periodic and bounded solutions of ordinary differential equations (see, e.g., [20, 24, 25]). Thereafter the method was extended to differential inclusions (see, e.g., [1, 7]), functional differential equations and inclusions (see, e.g., [6, 10, 13, 14, 16]) and other objects. The sphere of applications was extended to the study of qualitative behavior and bifurcations of solutions (see, e.g. [15, 21, 22]) and asymptotics of solutions (see, e.g., [11, 12, 17]). These and other aspects of the method of guiding functions and its applications, as well as the additional bibliography, may be found in the recent monograph [27].

The paper is organized in the following way. After preliminaries (Section 2), we give the notion of a multivalued causal operator (Section 3.1) and formulate the periodic problem for a differential inclusion with a causal multioperator (Section 3.2). Our main existence result (Theorem 2) is presented for the case when the right-hand side of the inclusion is convex-valued and closed.

2 Preliminaries

In what follows we will use some known notions and notation from the theory of multivalued maps (multimaps) (see, e.g., [1, 3, 7, 9]). Recall some of them.

Let (X, d_X) and (Y, d_Y) be metric spaces. By the symbols $P(Y)$ and $K(Y)$ we denote the collections of all nonempty and, respectively, nonempty and compact subsets of the space Y. If Y is a normed space, $Cv(K)$ and $Kv(Y)$ denote the collections of all nonempty convex closed [and, respectively, compact] subsets of Y.

Definition 1 *A multimap $F : X \to P(Y)$ is called upper semicontinuous (u.s.c.) at a point $x \in X$ if for each open set $V \subset Y$ such that $F(x) \subset Y$ there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies $F(x') \subset V$. A multimap $F : X \to P(Y)$ is called u.s.c. if it is u.s.c. at each point $x \in X$.*

Definition 2 *A multimap $F : X \to P(Y)$ is called lower semicontinuous (l.s.c.) at a point $x \in X$, if for each open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies $F(x') \cap V \neq \emptyset$. A multimap $F : X \to P(Y)$ is called l.s.c. if it is l.s.c. at each point $x \in X$.*

Definition 3 *A multimap $F : X \to P(Y)$ is called continuous if it is both u.s.c. and l.s.c.*

Definition 4 *A multimap $F : X \to P(Y)$ is called closed if its graph*

$$\Gamma_F = \{(x, y) \mid \ (x, y) \in X \times Y, \quad y \in F(x)\}$$

is a closed subset of the space $X \times Y$.

Definition 5 *A multimap $F : X \to P(Y)$ is called compact if its range $F(X)$ is relatively compact in Y.*

Remark 1 *If multimap $F : X \to P(Y)$ is closed and compact, it is u.s.c.*

Let I be a closed subset of $\mathbb{R}$ endowed with the Lebesgue measure.

Definition 6 *A multifunction $F : I \to K(Y)$ is called measurable if, for each open subset $W \subset Y$, its pre-image*

$$F^{-1}(W) = \{t \in I : F(t) \subset W\}$$

is a measurable subset of I.

Remark 2 *Each measurable multifunction $F : I \to K(Y)$ has a measurable selection, i.e., there exists such measurable function $f : I \to Y$, that $f(t) \in F(t)$ for a.e. $t \in I$.*

In the sequel we will use some standard properties of the topological degree theory of single-valued and multivalued vector fields (see, e.g., [3, 7, 9, 18]).

3 Periodic problem for inclusions with causal multioperators

3.1 Causal multioperators

Let $T > 0$ and $\sigma \geq 0$ be given numbers. By the symbols $C([kT-\sigma, (k+1)T]; \mathbb{R}^n)$ and $L^1((kT, (k+1)T); \mathbb{R}^n)$, where $k \in \mathbb{Z}$, we will denote the corresponding spaces of continuous and integrable functions with usual norms.

For any subset $\mathcal{N} \subset L^1((kT, (k+1)T); \mathbb{R}^n)$ and $\tau \in (kT, (k+1)T)$ we define the restriction of $\mathcal{N}$ on (kT, τ) as

$$\mathcal{N} \mid_{(kT,\tau)} = \{f \mid_{(kT,\tau)} : f \in \mathcal{N}\}.$$

Definition 7 *We will say that $\mathcal{Q}$ is a causal multioperator if for each $k \in \mathbb{Z}$ a multimap*

$$\mathcal{Q} : C([kT - \sigma, (k+1)T]; \mathbb{R}^n) \multimap L^1((kT, (k+1)T); \mathbb{R}^n)$$

is defined in such a way that for each $\tau \in (kT, (k+1)T)$ and for all

$$u(\cdot), v(\cdot) \in C([kT - \sigma, (k+1)T]; \mathbb{R}^n)$$

the condition $u \mid_{[kT-\sigma,\tau]} = v \mid_{[kT-\sigma,\tau]}$ implies $\mathcal{Q}(u) \mid_{(kT,\tau)} = \mathcal{Q}(v) \mid_{(kT,\tau)}$.

Let us consider some examples of causal multioperators. Denote by $\mathcal{C}$ the Banach space $C([-\sigma, 0]; \mathbb{R}^n)$.

Example 1 *Suppose that a multimap* $F : \mathbb{R} \times \mathcal{C} \to Kv(\mathbb{R}^n)$ *satisfies the following conditions:*

$(F1)$ *the multifunction* $F(\cdot, c) : \mathbb{R} \to Kv(\mathbb{R}^n)$ *admits a measurable selection for every* $c \in \mathcal{C}$;

$(F2)$ *the multimap* $F(t, \cdot) : \mathcal{C} \to Kv(\mathbb{R}^n)$ *is u.s.c. for a.e.* $t \in \mathbb{R}$;

$(F3)$ *for every* $r > 0$ *there exists a locally integrable nonnegative function* $\eta_r(\cdot) \in L^1_{loc}(\mathbb{R})$ *such that*

$$\|F(t, c)\| := \sup\{\|y\| : y \in F(t, c)\} \le \eta_r(t) \quad a.e.\ t \in \mathbb{R}\ ,$$

for all $c \in \mathcal{C}$, $\|c\| \le r$.

It is known (see, e.g., [3, 9]) that under conditions $(F1) - (F3)$ *for each* $k \in \mathbb{Z}$, *the superposition multioperator*

$$\mathcal{P}_F : C([kT - \sigma, (k+1)T]; \mathbb{R}^n) \multimap L^1((kT, (k+1)T); \mathbb{R}^n),$$

$$\mathcal{P}_F(u) = \left\{ f \in L^1((kT, (k+1)T]; \mathbb{R}^n) : \ f(t) \in F(t, u_t) \quad a.e.\ t \in (kT, (k+1)T) \right\} \tag{1}$$

is well defined. Here $u_t \in \mathcal{C}$ *is defined as* $u_t(\theta) = u(t + \theta)$, $\theta \in [-\sigma, 0]$. *It is easy to see that the multioperator* $\mathcal{P}_F$ *is causal.*

Remark 3 *We will say that a multimap* $F : \mathbb{R} \times \mathcal{C} \to K(\mathbb{R}^n)$ *obeying* $(F1)$-$(F2)$ *satisfies the upper Carathéodory conditions. If* $(F2)$ *may be replaced with*

$(F2')$ *the multimap* $F(t, \cdot) : \mathcal{C} \to K(\mathbb{R}^n)$ *is continuous for a.e.* $t \in \mathbb{R}$

we say that F *satisfies the Carathéodory conditions.*

Example 2 *Let* $F : \mathbb{R} \times \mathcal{C} \to Kv(\mathbb{R}^n)$ *be a multimap satisfying conditions* $(F1) - (F3)$ *of Example 1. Suppose that* $\{K(t, s) : -\infty < s \le t < +\infty\}$ *is a continuous (with respect to the norm) family of linear operators in* $\mathbb{R}^n$ *and* $m \in L^1_{loc}(\mathbb{R}; \mathbb{R}^n)$ *is a given locally integrable function. Consider, for each* $k \in \mathbb{Z}$, *the Volterra type integral multioperator* $\mathcal{G} : C([kT - \sigma, (k+1)T]; \mathbb{R}^n) \multimap L^1((kT, (k+1)T); \mathbb{R}^n)$ *defined as*

$$\mathcal{G}(u)(t) = m(t) + \int_{kT}^{t} K(t, s) F(s, u_s) ds,$$

i.e.,

$$\mathcal{G}(u) = \{y \in L^1((kT, (k+1)T); \mathbb{R}^n) : y(t) = m(t) + \int_{kT}^{t} K(t, s) f(s) ds : f \in \mathcal{P}_F(u)\}. \tag{2}$$

It is also obvious that the multioperator $\mathcal{G}$ *is causal.*

Example 3 *Suppose that a multimap* $F : \mathbb{R} \times \mathcal{C} \to K(\mathbb{R}^n)$ *satisfies the following condition of almost lower semicontinuity:*

(F_L) *there exists a sequence of disjoint closed sets* $\{J_n\}$, $J_n \subseteq \mathbb{R}$ $n = 1, 2, ...$ *such that:* (*i*) *meas* $(\mathbb{R} \setminus \bigcup_n J_n) = 0$; (*ii*) *the restriction of* F *on each set* $J_n \times \mathcal{C}$ *is l.s.c.*

Then (see, e.g., [3, 9]) under conditions (F_L), ($F3$), *for each* $k \in \mathbb{Z}$, *the superposition multioperator*

$$\mathcal{P}_F : C\left([kT - \sigma, (k+1)T]; \mathbb{R}^n\right) \multimap L^1\left((kT, (k+1)T); \mathbb{R}^n\right)$$

is also well-defined and causal.

3.2 Periodic problem

Denote by C_T the space of continuous T-periodic functions $x : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|x\|_C = \sup\limits_{t \in [0,T]} \|x(t)\|$. By $\|x\|_2$ we denote the norm of function x in the space L^2,

$$\|x\|_2 = \left(\int\limits_0^T \|x(s)\|^2 \, ds \right)^{\frac{1}{2}}.$$

To define the notion of a periodic causal multioperator, introduce, for $k \in \mathbb{Z}$, the following shift operator $j_k : L^1\left((kT, (k+1)T); \mathbb{R}^n\right) \to L^1\left((0, T); \mathbb{R}^n\right)$:

$$j_k(f)(t) = f(t + kT).$$

Definition 8 *A causal multioperator* $\mathcal{Q}$ *will be called* T*-periodic if, for each* $x \in C_T$ *and* $k \in \mathbb{Z}$,

$$j_k(\mathcal{Q}(x \mid_{[kT-\tau,(k+1)T]})) = \mathcal{Q}(x \mid_{[-\tau,T]}).$$

It is clear that, to provide the periodicity of the causal multioperators in the above examples, it is sufficient to assume that the multimaps F are T-periodic in the first argument:

$$F(t + T, c) = F(t, c)$$

for all $(t, c) \in \mathbb{R} \times \mathcal{C}$ and in Example 2, additionally, that function $m(t)$ and family $K(t, s)$ are also T-periodic:

$$m(t + T) = m(t), \quad \forall t \in \mathbb{R};$$

$$K(t + T, s + T) = K(t, s), \quad \forall -\infty < s \le t < +\infty.$$

It is clear that the condition of periodicity of the causal multioperator allows to consider it only on the space $C([-\tau, T]; \mathbb{R}^n)$.

Given a T-periodic causal multioperator $\mathcal{Q}$, we will consider the existence of solutions to the following problem:

$$x' \in \mathcal{Q}(x), \tag{3}$$

where $x \in C_T$ is an absolutely continuous function.

Denote by L^1_T the space of integrable T-periodic functions $f : \mathbb{R} \to \mathbb{R}^n$.

In this section we will assume that the T-periodic causal multioperator $\mathcal{Q} : C_T \to Cv(L^1_T)$ satisfies the following conditions:

($\mathcal{Q}1$) for each bounded linear operator $A : L^1_T \to E$, where E is a Banach space, the composition $A \circ \mathcal{Q} : C_T \to Cv(E)$ is closed;

($\mathcal{Q}2$) there exists a non-negative T-periodic integrable function $\alpha(t)$ such that

$$\|\mathcal{Q}(x)(t)\| \leq \alpha(t)(1 + \|x(t)\|) \text{ for a.e. } t \in \mathbb{R}$$

for each $x \in C_T$.

To provide condition ($\mathcal{Q}1$) in Examples 1 and 2, it is sufficient to assume, besides the above mentioned periodicity conditions, that the multimap F satisfies conditions $(F1)$ – $(F3)$ (see, e.g. [1], Theorem 1.5.30) and to fulfil condition ($\mathcal{Q}2$), we can suppose, in Example 1, the following sublinear growth condition: for each $x \in C_T$ we have, for some non-negative integrable function $\beta(t)$:

$$\|F(t, x_t)\| \leq \beta(t)(1 + \|x(t)\|) \text{ a.e.} t \in [0, T], \tag{4}$$

and, in Example 2, the global boundedness condition

$$\|F(t, c)\| \leq \gamma(t) \tag{5}$$

for some non-negative integrable function $\gamma(t)$.

To study periodic problem (3) we will need a coincidence point result for a multivalued perturbation of a linear Fredholm operator. Let us give necessary definitions.

Let E_1, E_2 be Banach spaces, $U \subset E_1$ an open bounded set; $l : \text{Dom } l \subseteq E_1 \to E_2$ a linear Fredholm operator of zero index such that $\text{Im } l \subset E_2$ is closed.

Consider continuous linear projection operators $p : E_1 \to E_1$ and $q : E_2 \to E_2$ such that $\text{Im } p = \text{Ker } l$, $\text{Im } l = \text{Ker } q$. By the symbol l_p denote the restriction of the operator l to $\text{Dom } l \cap \text{Ker } p$.

Further, let the continuous operator $k_{p,q} : E_2 \to \text{Dom } l \cap \text{Ker } p$ is defined by the relation $k_{p,q}(y) = l_p^{-1}(y - q(y)),\ y \in E_2$; the canonical projection operator $\pi : E_2 \to E_2/\text{Im } l$ has the form $\pi(y) = y + \text{Im } l,\ y \in E_2$; and $\phi : \text{Coker } l \to \text{Ker } l$ a continuous linear isomorphism.

Let $\mathcal{G} : \overline{U} \to Kv(E_2)$ be a closed multimap such that

(a) $\mathcal{G}(U)$ is a bounded subset of E_2;

(b) $k_{p,q} \circ \mathcal{G} : \overline{U} \to Kv(E_1)$ is compact and u.s.c.

The following assertion holds true (see [3], Lemma 13.1).

Lemma 1 *Suppose that:*

(i) $l(x) \notin \lambda \mathcal{G}(x)$ *for all* $\lambda \in (0,1]$, $x \in \mathrm{Dom}\ l \cap \partial U$;

(ii) $0 \notin \pi \mathcal{G}(x)$ *for all* $x \in \mathrm{Ker}\ l \cap \partial U$;

(iii) $deg_{\mathrm{Ker}\ l}(\phi \pi \mathcal{G}|_{\overline{U}_{\mathrm{Ker}\ l}}, \overline{U}_{\mathrm{Ker}\ l}) \neq 0$, *where the symbol* $deg_{\mathrm{Ker}\ l}$ *denotes the topological degree of a multivalued vector field evaluating in the space* $\mathrm{Ker}\ l$, *and* $\overline{U}_{\mathrm{Ker}\ l} = \overline{U} \cap \mathrm{Ker}\ l$.

Then l *and* $\mathcal{G}$ *has a coincidence point in* U, *i.e., there exists* $x \in U$ *such that* $l(x) \in G(x)$.

Developing notions introduced in [6, 10, 18], let us give the following definition.

Definition 9 *A continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$ *is called the integral guiding function for inclusion (3) if there exists* $N > 0$ *such that*

$$\int_0^T \langle \nabla V(x(s)), f(s) \rangle\, ds > 0 \quad \textit{for all } f \in \mathcal{Q}(x), \tag{6}$$

for each absolutely continuous function $x \in C_T$ *such that* $\|x\|_2 \geq N$ *and* $\|x'(t)\| \leq \|\mathcal{Q}(x)(t)\|$ *a.e.* $t \in [0,T]$.

From the definition it immediately follows that the integral guiding function V is a non-degenerate potential in the sense that

$$\nabla V(x) \neq 0,$$

for all $x \in \mathbb{R}^n$, $\|x\| \geq K = \frac{N}{\sqrt{T}}$. Therefore, on each closed ball $B_{\tilde{K}} \subset \mathbb{R}^n$ centered at the origin of the radius $\tilde{K} \geq K$, the topological degree of the gradient $\deg(\nabla V; B_{\tilde{K}})$ is well defined and, moreover, it does not depend on the radius $\tilde{K}$ (see, e.g., [18, 20]). This generic value of the degree will be called the index $\mathrm{Ind}\ V$ of an integral guiding function V.

Definition 10 *A non-degenerate potential* $V : \mathbb{R}^n \to \mathbb{R}$ *is called the generalized integral guiding function for inclusion (3) if there exists* $N > 0$ *such that*

$$\int_0^T \langle \nabla V(x(s)), f(s) \rangle\, ds \geq 0 \quad \textit{for some } f \in \mathcal{Q}(x), \tag{7}$$

for each absolutely continuous function $x \in C_T$ *such that* $\|x\|_2 \geq N$ *and* $\|x'(t)\| \leq \|\mathcal{Q}(x)(t)\|$ *a.e.* $t \in [0,T]$.

Now we are in position to formulate the main result of this section.

Theorem 1 *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be an generalized integral guiding function for problem (3) such that*

$$\mathrm{Ind}\ V \neq 0.$$

Then problem (3) has a solution.

Remark 4 *Notice that conditions of the theorem are fulfilled if, for example, the function V is even or satisfies the coercivity condition:* $\lim_{\|x\|\to+\infty} V(x) = \pm\infty$.

Proof Step 1. Let us consider the case of the strict integral guiding function for inclusion (3). Let us justify the solvability of the following operator inclusion

$$lx \in \mathcal{Q}(x), \tag{8}$$

where $l \,:\, \mathrm{Dom}\, l := \{x \in C_T \,:\, x \text{ is absolutely continuous}\} \subset C_T \to L^1_T$ is the linear Fredholm operator of zero index. It is easy to see that $\mathrm{Ker}\, l = \mathbb{R}^n$, projection $\pi : L^1_T \to \mathbb{R}^n$ may be given by the formula $\pi f = \frac{1}{T}\int\limits_0^T f(s)\,ds$ and the multioperators $\pi\mathcal{Q}$ and $k_{p,q}\mathcal{Q}$ are convex-valued and compact on bounded subsets.

Now, let, for some $\lambda \in (0,1]$ a function $x \in \mathrm{Dom}\, l$ is the solution of the inclusion

$$lx \in \lambda\mathcal{Q}(x.)$$

It means that $x(\cdot)$ is an absolutely continuous function such that $x'(t) = \lambda f(t)$ a.e. $t \in [0,T]$, for some $f \in \mathcal{Q}(x)$.

Then

$$\int_0^T \langle \nabla V(x(s)), f(s)\rangle\, ds = \frac{1}{\lambda}\int_0^T \langle \nabla V(x(s)), x'(s)\rangle\, ds =$$

$$= \frac{1}{\lambda}\int_0^T V'(x(s))\, ds = \frac{1}{\lambda}(V(x(T)) - V(x(0))) = 0,$$

yielding

$$\|x\|_2 < N.$$

From condition ($\mathcal{Q}2$) it follows that $\|x'\|_2 < M'$, where $M' > 0$. But then there exists also $M > 0$ such that

$$\|x\|_C < M.$$

Now, take as U the ball $B_r \subset C_T$ of the radius $r = \max\{M, NT^{-1/2}\}$. Then we have

$$lx \notin \lambda\mathcal{Q}(x)$$

for all $x \in \partial U$.

Take an arbitrary $u \in \partial U \cap \mathrm{Ker}\, l$. We have $\|u\| \geq NT^{-1/2}$ and considering u as a constant function, from the definition of the strict integral guiding function we obtain

$$\int_0^T \langle \nabla V(u), f(s)\rangle\, ds > 0$$

for each $f \in \mathcal{Q}(u)$. But

$$\int_0^T \langle \nabla V(u), f(s)\rangle\, ds = \langle \nabla V(u), \int_0^T f(s)\, ds\rangle = T\langle \nabla V(u), \pi f\rangle > 0,$$

and, therefore

$$\langle \nabla V(u), y \rangle > 0$$

for each $y \in \pi\mathcal{Q}(u)$.

It means that $0 \notin \pi\mathcal{Q}(u)$ and, moreover, the multifield $\pi\mathcal{Q}(u)$ and the field $\nabla V(u)$ do not admit opposite directions for $u \in \partial U \cap \text{Ker } l$. It means that they are homotopic and, hence,

$$\deg(\pi\mathcal{Q}\big|_{\overline{U}_{\text{Ker } l}}, \overline{U}_{\text{Ker } l}) = \deg(\nabla V, \overline{U}_{\text{Ker } l}) \neq 0,$$

where $\overline{U}_{\text{Ker } l} = \overline{U} \cap \text{Ker } l$. Therefore, all conditions of Lemma 1 are fulfilled and problem (8), and, hence (3) have a solution.

Step 2. Now we consider the case of the generalized integral guiding function for inclusion (3). Consider a multimap $B : C_T \to P(L^1_T)$ defined as

$$B(x) = \left\{ \varphi : \ |\varphi(t)| \leq \alpha(t)(1 + \|x_t\|) \text{ and } \gamma(x) \int_0^T \left\langle \nabla V\big(x(s)\big), \varphi(s) \right\rangle ds \geq 0 \right\},$$

where the first relation holds true for a.e. $t \in [0, T]$, $\alpha(\cdot)$ is a function from the condition $(\mathcal{Q}2)$, and

$$\gamma(x) = \begin{cases} 0, & \text{if } \|x\|_2 \leq N, \\ 1, & \text{if } \|x\|_2 > N. \end{cases}$$

It is easy to verify that B is a closed multimap.

Let us consider a multimap $Q^B : C_T \to P(L^1_T)$ given as

$$Q^B(x) = Q(x) \cap B(x).$$

Obviously, the multimap Q^B is closed and the condition (7) is satisfied for all $f \in Q^B(x)$.

For the non-degenerate potential V we define a map $Y_V : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$Y_V(x) = \begin{cases} \nabla V(x), & \text{if } \|\nabla V(x)\| \leq 1, \\ \frac{\nabla V(x)}{\|\nabla V(x)\|}, & \text{if } \|\nabla V(x)\| > 1. \end{cases}$$

It is easy to see that the map Y is continuous.

For any $\varepsilon_m > 0$ we define a multimap $Q_m : C_T \to P(L^1_T)$ as following

$$Q_m(x) = Q^B(x) + \varepsilon_m Y_V(x).$$

The multimap Q_m is closed and for each $\varepsilon_m > 0$ the condition (6) is fulfilled. By applying results of Step 1 we can prove the solvability of the following operator inclusion

$$lx \in Q_m(x),$$

for each $\varepsilon_m > 0$. From which follows the existence of a solution for problem (3).

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References

[1] Borisovich, Yu. G.; Gel'man, B. D.; Myshkis, A. D.; Obukhovskii, V. V. *Introduction to the Theory of Multivalued Maps and Differential Inclusions.* (in Russian). Second edition. Knizhnyi Dom "LIBROKOM", Moscow, 2011.

[2] Corduneanu, C. *Functional Equations with Causal Operators.* Stability and Control: Theory, Methods and Applications, 16. Taylor and Francis, London, 2002.

[3] Deimling, K. *Multivalued Differential Equations.* de Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter, Berlin, 1992.

[4] Drici, Z.; McRae, F.A.; Vasundhara Devi, J. *Differential equations with causal operators in a Banach space.* Nonlinear Anal. 62 (2005), no. 2, 301–313.

[5] Drici, Z.; McRae, F.A.; Vasundhara Devi, J. *Monotone iterative technique for periodic boundary value problems with causal operators.* Nonlinear Anal. Vol. 64 (2006), no. 6, 1271–1277.

[6] Fonda, A. *Guiding functions and periodic solutions to functional differential equations.* Proc. Amer. Math. Soc. 99 (1987), no. 1, 79–85.

[7] Górniewicz, L. *Topological Fixed Point Theory of Multivalued Mappings.* Second edition. Topological Fixed Point Theory and Its Applications, 4. Springer, Dordrecht, 2006.

[8] Jankowski, T. *Boundary value problems with causal operators.* Nonlinear Anal. 68 (2008), no. 12, 3625–3632.

[9] Kamenskii, M.; Obukhovskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces.* de Gruyter Series in Nonlinear Analysis and Applications, 7. Walter de Gruyter, Berlin - New York, 2001.

[10] Kornev, S.; Obukhovskii, V. *On some developments of the method of integral guiding functions.* Funct. Differ. Equ. 12 (2005), no. 3-4, 303–310.

[11] Kornev, S.; Obukhovskii, V.; Yao, J-C. *On asymptotics of solutions for a class of functional differential inclusions.* Discuss. Math. Differ. Incl. Control Optim. 34 (2014), no. 2, 219–227.

[12] Kornev, S.; Obukhovskii, V. *On asymptotic behavior of solutions of differential inclusions and the method of guiding functions.* (in Russian), Differ. Uravn., 51 (2015), no. 6, 700-705. Translation: Differ. Equ. 51 (2015), no. 6, 711–716.

[13] Kornev, S. *The method of generalized integral guiding function in the periodic problem of differential inclusions.* (in Russian) The Bulletin of Irkutsk State University. Mathematics. 13 (2015), 16–31.

[14] Kornev, S.V.; Obukhovskii V.V.; Zecca, P. *On the method of generalized integral guiding functions in the periodic problem of functional differential inclusions.* (in Russian), Differ. Uravn., 52 (2016), no. 10, 1335-1344. Translation: Differ. Equ. 52 (2016), no. 10, 1282–1292.

[15] Kornev, S.V.; Liou, Y.-C. *Multivalent guiding functions in the bifurcation problem of differential inclusions.* The J. of Nonlin. Sci. and Appl. 9 (2016), Issue 8, 5259–5270.

[16] Kornev, S.; Obukhovskii, V.; Zecca, P. *On the method of generalized integral guiding functions in the periodic problem of functional differential inclusions with causal operator.* Appl. Anal. 96 (2017), Issue 3, 418–428.

[17] Obukhovskii, V.; Kamenskii, M.; Kornev, S.; Liou, Y.-C. *On asymptotics of solutions for a class of differential inclusions with a regular right-hand part.* J. of Nonlin. Conv. Anal. (in print).

[18] Krasnosel'skii, M. A. *The Operator of Translation Along the Trajectories of Differential Equations.* Translations of Mathematical Monographs, Vol. 19. American Mathematical Society, Providence, R.I. 1968

[19] Krasnosel'skii, M. A.; Perov, A. I. *On a certain principle of existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations.* (in Russian) Dokl. Akad. Nauk SSSR 123(1958), 235–238.

[20] Krasnosel'skii, M. A.; Zabreiko, P. P. *Geometrical Methods of Nonlinear Analysis.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 263. Springer-Verlag, Berlin, 1984.

[21] Kryszewski, W. *Homotopy properties of set-valued mappings.* Univ. N. Copernicus Publishing, Torun, 1997.

[22] Loi, N.V.; Obukhovskii, V.; Zecca, P. *On the global bifurcation of periodic solutions of differential inclusions in Hilbert spaces,* Nonlinear Anal. 76(2013), 80-92.

[23] Lupulescu, V. *Causal functional differential equations in Banach spaces.* Nonlinear Anal. 69 (2008), no. 12, 4787–4795.

[24] Mawhin, J. *Topological Degree Methods in Nonlinear Boundary Value Problems.* CBMS Regional Conference Series in Mathematics, 40. American Mathematical Society, Providence, R.I., 1979.

[25] Mawhin, J.; Ward, J.R., Jr. *Guiding-like functions for periodic or bounded solutions of ordinary differential equations.* Discrete Contin. Dyn. Syst. 8 (2002), no. 1, 39–54.

[26] Obukhovskii, V., Zecca, P., *On certain classes of functional inclusions with causal operators in Banach spaces.* Nonlinear Anal. 74 (2011), no. 8, 2765–2777.

[27] Obukhovskii, V.; Zecca, P.; Loi, N.V.; Kornev, S. *Method of Guiding Functions in Problems of Nonlinear Analysis.* Lecture Notes in Mathematics, 2076. Springer, Heidelberg, 2013.

[28] Tikhonov, A.N. *Functional Volterra-type equations and their applications to certain problems of mathematical physics* (in Russian), Bull. Mosk. Gos. Univ. Sekt. A, 1(1938), no. 8, 1–25.

[29] Tonelli, L. *Sulle equazioni funzionali di Volterra,* Bull. Calcutta Math. Soc. 20(1930), 31–48.