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A GENERALIZATION OF THE ZEROS OF THE MELLIN TRANSFORM OF HERMITE FUNCTIONS

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Abstract. A great deal of work has already been done on the subject of Riemann zeta function. In this paper, we shall study a special class of local ζ-functions, wherein the main result states that the functions have all zeros on the line $Re(\mathfrak{s}) = 1/2$ and prove a generalization of the result of D. Bump and Ng. Eugene which asserts that the zeros of the Mellin transform of Hermite functions have $Re(\mathfrak{s}) = 1/2$.

1. Introduction

We shall begin with the definition of Local zeta function. J. T Tate [5, 6] in the study of Hecke $\mathcal{L}$-functions, defined local ζ-functions as

$$\zeta(\mathfrak{s}, \mathfrak{c}, \mathfrak{f}) = \int_{\mathcal{F}^\times} \mathfrak{f}(\mathfrak{z})\mathfrak{c}(\mathfrak{z})|\mathfrak{z}|^{\mathfrak{s}} d^\times \mathfrak{z},$$

where $\mathcal{F}$ is a local field, $\mathfrak{f}$ is a Schwartz function of $\mathcal{F}$, $\mathfrak{c}$ is a character of $\mathcal{F}^\times$ and integration is taken with respect to Haar measure on $\mathcal{F}^\times$. Weil [1] introduced a representation $\vartheta = \vartheta_\psi$ of the metaplectic group $\widetilde{SL}(2, \mathcal{F})$,for each nontrivial additive character ψ of $\mathcal{F}$. *The Local Riemann Hypothesis (LRH)*, is the assertion that if $\mathfrak{f}$ is taken from some irreducible invariant subspace of the restriction of this representation to a certain compact subgroup G of $\widetilde{SL}(2, \mathcal{F})$, then in fact all zeros of $\zeta(\mathfrak{s}, \mathfrak{c}, \mathfrak{f})$ lie on the line $Re(\mathfrak{s}) = 1/2$. The phenomenon was first observed by Bump and Eugene and they proved that the zeros of the Mellin transform of Hermite functions lie on the line, this corresponds to LRH for $\mathcal{F} = \mathbb{R}$[2]. LRH has also been proved for $\mathcal{F}$ having odd characteristics by Kurlberg [8] and disproved for $\mathcal{F} = \mathbb{C}$ by Kurlberg [8]. In all cases above G is the unique maximal compact subgroup of $SO(2, \mathcal{F})$, for $\mathcal{F} = \mathbb{R}$ and for $\mathcal{F}$ with characteristic congruent to 3 modulo 4, G is nothing but $SO(2, \mathcal{F})$, since this already is compact. In [3] Bump, Choi, Kurlberg and Vaaler offer generalizations of LRH to higher dimensions along with two different proofs of the case $\mathcal{F} = \mathbb{R}$ and $G = SO(2, \mathcal{F})$. In this paper we shall prove:

Theorem 1.1. *Let $\mathfrak{f}$ be an irreducible invariant subspace of the Weil representation restricted to $SU(2, \mathbb{C})$ and $\zeta(\mathfrak{s}, \mathfrak{c}, \mathfrak{f}) \neq 0$, where $\mathfrak{f} \in W$, then all zeros of $\zeta(\mathfrak{s}, \mathfrak{c}, \mathfrak{f})$ lie on the line $Re(\mathfrak{s}) = 1/2$.*

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In other words, we prove that a slightly modified version of LRH (namely taking $G = SU(2,\mathbb{C})$ rather than a compact subgroup of $SO(2,\mathbb{C})$) holds for $F = \mathbb{C}$.

Remark. From now on we will restrict ourselves to the case where the local field is $\mathbb{C}$.

2. The Weil representation

The Weil (or the metaplectic) representation is an action on $\mathcal{S}(\mathbb{C}) = \{\mathfrak{f}(\mathfrak{u}); \mathfrak{f}(x+iy) = \mathfrak{g}(x,y) \in \mathfrak{F}(\mathbb{R}^2)\}$, where $\mathfrak{F}(\mathbb{R}^2)$ is the Schwartz space. We will often think of the elements of $\mathcal{S}(\mathbb{C})$, not as functions of the complex variable u, but rather as functions of the two real variables x, y satisfying $\mathfrak{u} = x+iy$. In agreement with that we write $d\mathfrak{u}$ and this is nothing but $dxdy$, the Lebesgue measure of $\mathbb{R}^2$. Sometimes we will also use the notation $\langle \mathfrak{f}, \mathfrak{g}\rangle = \int_{\mathbb{C}} \mathfrak{f}(\mathfrak{u})\mathfrak{g}(\mathfrak{u})d\mathfrak{u}$. Let the additive character on $\mathbb{C}$ be $\psi(\mathfrak{u}) = e^{i\pi Re(\mathfrak{u})}$ and introduce the Fourier transform

$$\hat{\mathfrak{f}}(\mathfrak{u}) = \int_{\mathbb{C}} \mathfrak{f}(\mathfrak{u}')\psi(2\mathfrak{u}\mathfrak{u}')d\mathfrak{u}'.$$

It can be easily verified that $\hat{\hat{\mathfrak{f}}}(\mathfrak{u}) = \mathfrak{f}(-\mathfrak{u})$, using this normalization.

We assume, without loss of generality, that the additive character is $\psi(\mathfrak{u}) = e^{i\pi Re(\mathfrak{u})}$ if the objective only is to prove LRH. Changing character does not preserve the irreducible subspaces, but the zeros of the "corresponding ζ- functions" are preserved.

$\widetilde{SL}(2,\mathbb{C})$, the metaplectic double cover of $SL(2,\mathbb{C})$, splits and we have

$$\widetilde{SL}(2,\mathbb{C}) \cong SL(2,\mathbb{C}) \times C_2.$$

Using this identification we write

$$\begin{bmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{bmatrix} = \left(\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix}, 1 \right).$$

The restriction of the metaplectic representation to $SU(2,\mathbb{C})$ is generally given by

$$\left(\vartheta \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \mathfrak{f} \right)(u) = \frac{1}{|\beta|} \int_{\mathbb{C}} \psi\left(\frac{1}{\beta}(\alpha\mathfrak{u}^2 - 2\mathfrak{u}\mathfrak{u}' + \bar{\alpha}\mathfrak{u}^2) \right) \mathfrak{f}(\mathfrak{u}')d\mathfrak{u}'.$$

However, it is much more convenient to see how ϑ acts on the generators of $SL(2,\mathbb{C})$. This is given by

$$\left(\vartheta \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} f \right)(\mathfrak{u}) = \psi(t\mathfrak{u}^2)f(\mathfrak{u}),$$

$$\left(\vartheta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathfrak{f} \right)(\mathfrak{u}) = \hat{\mathfrak{f}}(\mathfrak{u}),$$

and

$$\left(\vartheta \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \mathfrak{f}\right)(\mathfrak{u}) = |\alpha| f(\alpha \mathfrak{u}).$$

In order to find the invariant subspaces of the action of $SU(2,\mathbb{C})$ could of course just as well study the restriction to $SU(2,\mathbb{C})$ the corresponding Lie algebra representation $d\vartheta : SL(2,\mathbb{C}) \to End(\mathcal{S}(\mathbb{C}))$defined by

$$((d\vartheta X)f)(\mathfrak{u}) = \frac{d}{dt}(\vartheta e^{(tX)})\mathfrak{f})(\mathfrak{u})|_{t=0}$$

where the exponential map $SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ lifted to a map $SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$. Since a natural basis for $SL(2,\mathbb{C})$ is

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\},$$

our first objective is to calculate how $d\vartheta$ acts on $\mathcal{S}(\mathbb{C})$ for these vectors. From the definitions we immediately get

$$\begin{aligned}\left(d\vartheta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathfrak{f}\right) &= \frac{d}{dt}\left(\vartheta \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathfrak{f}\right)\bigg|_{t=0} = \frac{d}{dt}\psi(t(x+iy)^2)\mathfrak{f}|_{t=0} \\ &= \frac{d}{dt} e^{i\pi t(x^2-y^2)}\mathfrak{f}|_{t=0} = i\pi(x^2-y^2)\mathfrak{f}\end{aligned}$$

and

$$\begin{aligned}\left(d\vartheta \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \mathfrak{f}\right) &= \frac{d}{dt}\left(\vartheta \begin{bmatrix} 1 & it \\ 0 & 1 \end{bmatrix} \mathfrak{f}\right)\bigg|_{t=0} = \frac{d}{dt}\psi(it(x+iy)^2)\mathfrak{f}|_{t=0} \\ &= \frac{d}{dt} e^{-i2\pi txy}\mathfrak{f}|_{t=0} = i2\pi xy\mathfrak{f}\end{aligned}$$

Introducing the notation $\mathfrak{F}$ for the operator taking $\mathfrak{f}$ to its Fourier transform $\hat{\mathfrak{f}}$ we see that

$$\begin{aligned} d\vartheta \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} &= \left(\vartheta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^{-1} \left(d\vartheta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\left(\vartheta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \\ &= \mathfrak{F}^{-1} i\pi(x^2-y^2)\mathfrak{F} = -\frac{i}{4\pi}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\end{aligned}$$

and

$$\begin{aligned} d\vartheta \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} &= \left(\vartheta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^{-1} \left(d\vartheta \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}\right)\left(\vartheta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \\ &= \mathfrak{F}^{-1}(i2\pi xy)\mathfrak{F} = -\frac{i}{2\pi}\frac{\partial^2}{\partial x \partial y}.\end{aligned}$$

Hence we have that

$$d\vartheta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = d\vartheta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d\vartheta \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = i\pi(x^2-y^2) - \frac{i}{4\pi}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)$$

and

$$d\vartheta \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = d\vartheta \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - d\vartheta \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} = -i2\pi xy + \frac{i}{2\pi}\frac{\partial^2}{\partial x \partial y}.$$

Finally we get that

$$\left(d\vartheta\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\mathfrak{f}\right) = \frac{d}{dt}\left(\vartheta\begin{bmatrix} e^{it} & 0\\ 0 & e^{-it}\end{bmatrix}\mathfrak{f}\right)(x+iy)\Big|_{t=0} = \frac{d}{dt}\mathfrak{f}(e^{it}(x+iy))\mathfrak{f}|_{t=0}$$
$$= \frac{d}{dt}\mathfrak{f}(x\cos t - y\sin t + i(y\cos t + x\sin t))|_{t=0}$$
$$= -y\frac{\partial\mathfrak{f}}{\partial x} + x\frac{\partial\mathfrak{f}}{\partial y}$$

Definition 2.1. *Let* $\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}(x+iy) = H_{\mathfrak{m}}(\sqrt{2\pi}x)H_n(\sqrt{2\pi}y)e^{-\pi(x^2+y^2)}$, *where* $H_n = (-1)^{\mathfrak{n}}e^{x^2}\dfrac{d^{\mathfrak{n}}}{dx^{\mathfrak{n}}}e^{-x^2}$ *are the Hermite polynomials.*

Proposition 2.2. $W_{\mathfrak{m}} = \bigoplus_{j=0}^{m}\mathbb{C}\mathfrak{f}_{j,\mathfrak{m}-j}$ *are invariant subspaces of the Weil representation restricted to* $SU(2,\mathbb{C})$.

Proof We can write $\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}(x+iy) = h_{\mathfrak{m}}(x)h_n(y)$, where $h_{\mathfrak{m}}$ satisfy

$$\left(x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}\right)h_{\mathfrak{m}} = \frac{2\mathfrak{m}+1}{2\pi}h_{\mathfrak{m}}.$$

Hence we have

$$d\vartheta\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\mathfrak{f}_{\mathfrak{m},\mathfrak{n}} = \left(i\pi(x^2-y^2) - \frac{i}{4\pi}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\right)\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}$$
$$= i\pi\left(\frac{2\mathfrak{m}+1}{2\pi} - \frac{2\mathfrak{n}+1}{2\pi}\right)\mathfrak{f}_{\mathfrak{m},\mathfrak{n}} = i_(\mathfrak{m} - \mathfrak{n}\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}.$$

Using the recurrence formulas $H_{n+1}(x) = 2xH_n(x) - 2\mathfrak{n}H_{n-1}(x)$ and $H_n'(x) = 2\mathfrak{n}H_{n-1}(x)$ [4] we get

$$d\vartheta\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\mathfrak{f}_{\mathfrak{m},\mathfrak{n}} = -y\frac{\partial\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}}{\partial x} + x\frac{\partial\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}}{\partial y}$$
$$= -y\big(\sqrt{2\pi}2\mathfrak{m}\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}} - 2\pi x\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}\big)$$
$$= +x\big(\sqrt{2\pi}2\mathfrak{n}\mathfrak{f}_{\mathfrak{m},\mathfrak{n}-1} - 2\pi y\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}\big)$$
$$= \sqrt{2\pi}\big(-2\mathfrak{m}y\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}} + 2nx\mathfrak{f}_{\mathfrak{m},\mathfrak{n}-1}\big)$$
$$= -.2\mathfrak{m}\frac{\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}+1} + 2\mathfrak{n}\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}-1}}{2} + 2\mathfrak{n}\frac{\mathfrak{f}_{\mathfrak{m}+1,\mathfrak{n}-1} + 2\mathfrak{m}\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}-1}}{2}$$
$$= n\mathfrak{f}_{\mathfrak{m}+1,\mathfrak{n}-1} - \mathfrak{m}\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}+1}$$

and

$$d\vartheta\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}\mathfrak{f}_{\mathfrak{m},\mathfrak{n}} = \frac{1}{2}d\vartheta\left[\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\right]\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}$$
$$= \frac{1}{2}d\vartheta\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}i_(\mathfrak{m} - \mathfrak{n}\mathfrak{f}_{\mathfrak{m},\mathfrak{n}}$$
$$-\frac{1}{2}d\vartheta\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}(n\mathfrak{f}_{\mathfrak{m}+1,\mathfrak{n}-1} - \mathfrak{m}\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}+1})$$
$$= -i\mathfrak{n}\mathfrak{f}_{\mathfrak{m}+1,\mathfrak{n}-1} - i\mathfrak{m}\mathfrak{f}_{\mathfrak{m}-1,\mathfrak{n}+1}.$$

The proposition follows since $W_{\mathfrak{m}}$ obviously is closed under all three basis operators *Remark.* Using the three basis operators given above it is easy to see that $W_{\mathfrak{m}}$ is irreducible.

Instead of choosing the basis $\{\mathfrak{f}_{\mathfrak{m}-\mathfrak{n},\mathfrak{n}}\}_{n=0}^{m}$ for $W_{\mathfrak{m}}$ it is sometimes more convenient to use the basis of eigenfunctions of $d\vartheta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Because of the symmetry in the commutator relations of the basis elements of $SU(2,\mathbb{C})$, these eigenfunctions have the same set of eigenvalues as $\{\mathfrak{f}_{\mathfrak{m}-\mathfrak{n},\mathfrak{n}}\}_{n=0}^{m}$. Call this new basis $\{b_{\mathfrak{m},\mathfrak{n}}\}$, where $\mathfrak{n} = -\mathfrak{m}, -\mathfrak{m}+2, ..., \mathfrak{m}$ and $b_{\mathfrak{m},\mathfrak{n}}(re^{i\theta}) = e^{i\mathfrak{n}\theta} b_{\mathfrak{m},\mathfrak{n}}(r)$. The elements of the basis is determined by the relations above up to multiplication by a constant, choosing these constants correctly we get:

Proposition 2.3. *Let*

$$L_n^{(\theta)}(y) = \frac{y^{-\theta}e^{y}}{n!}\frac{d^{\mathfrak{n}}}{dy^{\mathfrak{n}}}(y^{n+\theta}e^{-y})$$

be the Laguerre polynomials. We have that

$$b_{\mathfrak{m},\mathfrak{n}}(re^{i\theta}) = e^{i\mathfrak{n}\theta} r^{|\mathfrak{n}|} L^{(|\mathfrak{n}|)}_{(\mathfrak{m}-|\mathfrak{n}|)/2}(2\pi r^2) e^{(-\pi r^2)}.$$

Proof We assume $n \geq 0$, the argument is same as for $n < 0$. Since $b_{\mathfrak{m},\mathfrak{n}} \in W_{\mathfrak{m}}$, we see that $b_{\mathfrak{m},\mathfrak{n}}$ is of the form $\mathfrak{p}(u,\bar{u})e^{-\pi|u|^2}$, where $\mathfrak{p}$ is a polynomial of degree m. That $b_{\mathfrak{m},\mathfrak{n}}(re^{i\theta}) = e^{i\mathfrak{n}\theta} b_{\mathfrak{m},\mathfrak{n}}(r)$ means that $\mathfrak{p}(u,\bar{u})$ only consists of terms of the form $u^a\bar{u}^b$, where $a - b = n$. In particular we must have that $b_{\mathfrak{m},\mathfrak{n}}(re^{i\theta}) = e^{i\mathfrak{n}\theta} r^{\mathfrak{n}} q_{\mathfrak{m},\mathfrak{n}}(2\pi r^2) e^{(-\pi r^2)}$, where $q_{\mathfrak{m},\mathfrak{n}}$ is a polynomial of degree $(\mathfrak{m}-\mathfrak{n}/2$. Since the subspaces $W_{\mathfrak{m}}$ are orthogonal to each other, for $m \neq m'$, we have

$$\begin{aligned} 0 = \langle \overline{b_{\mathfrak{m},\mathfrak{n}}}, b_{\mathfrak{m}',\mathfrak{n}} \rangle &= 2\pi \int_0^\infty r^{\mathfrak{n}} \overline{q_{\mathfrak{m},\mathfrak{n}}(2\pi r^2)} e^{(-\pi r^2)} r^{\mathfrak{n}} q_{\mathfrak{m}',\mathfrak{n}}(2\pi r^2) e^{(-\pi r^2)} r dr \\ &= \frac{1}{2(2\pi)^{\mathfrak{n}}} \int_0^\infty \overline{q_{\mathfrak{m},\mathfrak{n}}(y)} q_{\mathfrak{m}',\mathfrak{n}}(y) y^{\mathfrak{n}} e^{-y} dy. \end{aligned}$$

This proves that $q_{\mathfrak{m},\mathfrak{n}}(y) = L^{(|\mathfrak{n}|)}_{(\mathfrak{m}-|\mathfrak{n}|)/2}(y)$ if we normalize correctly.

3. Properties of the local Tate ζ-function

Definition 3.1. *We define the local Tate ζ-function*

$$\zeta(\mathfrak{s},\mathfrak{c},\mathfrak{f}) = \int_{\mathbb{C}^\times} \mathfrak{f}(u)\mathfrak{c}(u)|u|^{2s-2} du$$

for all characters $\mathfrak{c}$ of $\mathbb{C}^\times$ and $\mathfrak{f} \in \mathcal{S}(\mathbb{C})$.

All characters of $\mathbb{C}^\times$ can be written using polar coordinates in the form $\mathfrak{c}(r,\theta) = r^{i\theta}e^{ij\theta}$ with $j \in \mathbb{Z}$. Since $\zeta(\mathfrak{s}, r^{i\theta}e^{ij\theta}, \mathfrak{f}) = \zeta(\mathfrak{s} + i\theta/2, e^{ij\theta}, \mathfrak{f})$, the real part of the zeros of ζ does not depend on θ. Hence our attention will be drawn to the following object:

Definition 3.2. *Let $j, \mathfrak{m} \in \mathbb{N}, \mathfrak{c}_j = e^{ij\theta}$ and $g_j = r^{2\mathfrak{s}-2}\mathfrak{c}_j$. We set*

$$\zeta_{\mathfrak{m}}^{(j)}(\mathfrak{s}) = \langle \mathfrak{f}_{\mathfrak{m},0}, g_j \rangle = \zeta(\mathfrak{s}, \mathfrak{c}_j, \mathfrak{f}_{\mathfrak{m},0}).$$

In order for Theorem 1.1.1 to be true it is essential that all elements in the invariant subspaces define the same ζ-function $\zeta_{\mathfrak{m}}^{(j)}$, up to multiplication by a constant. That this really is the case is shown in the next proposition.

Proposition 3.3. *If $\mathfrak{f} \in W_{\mathfrak{m}}$ then $\zeta(\mathfrak{s}, \mathfrak{c}_j, \mathfrak{f}) = \mathfrak{c}_{\mathfrak{f},j}$. $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s})$, where $\mathfrak{c}_{\mathfrak{f},j}$ is constant not depending on $\mathfrak{s}$.*

Proof Let $\mathfrak{f} = \sum_{j=0}^{m} c_{2j-\mathfrak{m}} b_{\mathfrak{m},2j-\mathfrak{m}}$. For $_{(}\mathfrak{m} - j)/2 \in \mathbb{N}$, we see that

$$\begin{aligned} \zeta(\mathfrak{s}, \mathfrak{c}_j, \mathfrak{f}) &= \sum_{j=0}^{m} c_{2j-\mathfrak{m}} \zeta(\mathfrak{s}, \mathfrak{c}_j, b_{\mathfrak{m},2j-\mathfrak{m}}) \\ &= \sum_{j=0}^{m} \mathfrak{c}_{2j-\mathfrak{m}} \int_0^\infty \int_0^{2\pi} e^{i(2j-\mathfrak{m})\theta} b_{\mathfrak{m},2j-\mathfrak{m}}(r) r^{(2\mathfrak{s}-1)} e^{ik\theta} d\theta dr \\ &= c_j \zeta(\mathfrak{s}, \mathfrak{c}_i j b_{\mathfrak{m},j}), \end{aligned}$$

Other$m$ give $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) \equiv 0$.

Lemma 3.4. *If $_{(}\mathfrak{m} - i)/2 \in \mathbb{N}$ we have that*

$$\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) = \Gamma\left(s + \frac{i}{2}\right) \pi^{(1-s)} p_{\mathfrak{m}}^{(i)}(\mathfrak{s}),$$

where $p_{\mathfrak{m}}^{(i)}(\mathfrak{s})$ is a real polynomial of degree $_{(}\mathfrak{m} - i)/2$. Otherwise $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) \equiv 0$

Proof Since $H_{\mathfrak{m}}$ is odd if m is odd and even if m is even, the trigonometric identities [7]

$$\cos^{2\mathfrak{n}} \theta = \frac{1}{2^{2\mathfrak{n}}} \binom{2\mathfrak{n}}{n} + \frac{1}{2^{2\mathfrak{n}-1}} \sum_{j=1}^{n} \binom{2\mathfrak{n}}{n-j} \cos(2j\theta)$$

and

$$\cos^{2\mathfrak{n}-1} \theta = \frac{1}{2^{2\mathfrak{n}-2}} \sum_{j=1}^{n} \binom{2\mathfrak{n}-1}{n-j} \cos((2j-1)\theta),$$

can be used to write

$$H_{\mathfrak{m}}(\sqrt{2\pi} r \cos\theta) = \sum_{j=0}^{[m/2]} r^{m-2j} a_j(r^2) \cos(_{(}\mathfrak{m} - 2j)\theta)$$

for some real polynomials $a_j(r)$ with deg $a_j = j$. This implies that if $_(\mathfrak{m} - i)/2 \notin \mathbb{N}$ then $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) \equiv 0$ and if $_(\mathfrak{m} - i)/2 \in \mathbb{N}$ we have

$$\begin{aligned}
\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) &= \int_0^\infty \int_0^{2\pi} H_{\mathfrak{m}}(\sqrt{2\pi} r \cos\theta) e^{-\pi r^2} r^{2s-1} e^{ii\theta} d\theta dr \\
&= 2\pi \int_0^\infty r^i a_{\frac{m-i}{2}}(r^2) r^{2s-1} e^{-\pi r^2} dr = \pi \sum_{j=0}^{\frac{m-i}{2}} \int_0^\infty r^{2s-1+i+2j} e^{-\pi r^2} dr \\
&= \pi \sum_{j=0}^{_(\mathfrak{m}-i)/2} b_j \frac{1}{2\pi^{s+j+i/2}} \Gamma\left(s + j + \frac{i}{2}\right) \\
&= \sum_{j=0}^{_(\mathfrak{m}-i)/2} b_j \frac{1}{2\pi^{s+j+i/2-1}} \Gamma\left(s + j + \frac{i}{2} - 1\right) \dots \left(s + \frac{i}{2}\right) \Gamma\left(s + \frac{i}{2}\right) \\
&= \Gamma\left(s + \frac{i}{2}\right) \pi^{1-s} p_{\mathfrak{m}}^{(i)}(\mathfrak{s}),
\end{aligned}$$

where $p_{\mathfrak{m}}^{(i)}(\mathfrak{s})$ a real polynomial of degree $_(\mathfrak{m} - i)/2$.

Remark. Theorem 1.1.1 implies that $p_{\mathfrak{m}}^{(i)}(1 - s) = (-1)^{\frac{m-i}{2}} p_{\mathfrak{m}}^{(i)}(\mathfrak{s})$ so $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s})$ fulfills a functional equation much like the functional equation for the Riemann ζ-function.

Lemma 3.5. $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s})$ *admits the functional equation*

$$_(\mathfrak{m} + 1)\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) = \pi \zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s} + 1) - \frac{1}{\pi}\left(s + \frac{i}{2} - 1\right)\left(s - \frac{i}{2} - 1\right)\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s} - 1).$$

Proof Since we have that

$$\Delta g_i(\mathfrak{s}) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\partial r^2} + \frac{\partial^2}{\partial \theta^2}\right) r^{2s-2} e^{i\theta i} = ((2s-2)^2 - i^2) g_i(\mathfrak{s} - 1)$$

and

$$\left(-\frac{1}{4\pi}\Delta + \pi(x^2 + y^2)\right)\mathfrak{f}_{\mathfrak{m},0} =_(\mathfrak{m} + 1)\mathfrak{f}_{\mathfrak{m},0},$$

we immediately get

$$\begin{aligned}
_(\mathfrak{m} + 1)\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s}) &= \langle _(\mathfrak{m} + 1)\mathfrak{f}m, 0, g_i(\mathfrak{s})\rangle = \left\langle \left(-\frac{1}{4\pi}\Delta + \pi(x^2 + y^2)\right)\mathfrak{f}m, 0, g_i(\mathfrak{s})\right\rangle \\
&= \left\langle \mathfrak{f}m, 0, \left(-\frac{1}{4\pi}\Delta + \pi(x^2 + y^2)\right) g_i(\mathfrak{s})\right\rangle \\
&= \left\langle \mathfrak{f}m, 0, -\frac{1}{4\pi}((2s-2)^2 - i^2) g_i(\mathfrak{s} - 1) + \pi g_i(\mathfrak{s} + 1)\right\rangle \\
&= -\frac{1}{\pi}\left(s + \frac{i}{2} - 1\right)\left(s - \frac{i}{2} - 1\right)\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s} - 1) + \pi\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s} + 1).
\end{aligned}$$

From [3] we have the following lemma:

Lemma 3.6. *Let $\mathfrak{q}(\mathfrak{s})$ be a polynomial, and assume that the zeros of $\mathfrak{q}(\mathfrak{s})$lie in the closed strip $\{\mathfrak{s}; Re(\mathfrak{s}) \in [-\mathfrak{c}, \mathfrak{c}]\}$ with $\mathfrak{c} > 0$. Then if $\mathfrak{a} > 0, \mathfrak{b} > 0$, the zeros of*

$$r(\mathfrak{s}) = (\mathfrak{s} + \mathfrak{a})\mathfrak{q}(\mathfrak{s} + \mathfrak{b}) - (\mathfrak{s} - \mathfrak{a})\mathfrak{q}(\mathfrak{s} - \mathfrak{b})$$

lie in the open strip $\{\mathfrak{s}; Re(\mathfrak{s}) \in (-\mathfrak{c}, \mathfrak{c})\}$.

Remark. The lemma is proved for $\mathfrak{b} = 2$, but this does not change the proof.

Proof of Theorem 1.1.1. We only need to show that $\zeta_{\mathfrak{m}}^{(i)}(\mathfrak{s})$ has all its zeros on $Re(\mathfrak{s}) = 1/2$. Letting $\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}) = \mathfrak{p}_{\mathfrak{m}}^{(i)}(\mathfrak{s} + 1/2)$and inserting this in Lemma 1.4.3 we get

$$\begin{aligned}(\mathfrak{m} + 1)&\Gamma\left(\mathfrak{s} + \frac{\mathsf{k}+1}{2}\right)\pi^{\frac{1}{2}-\mathfrak{s}}\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}) = \pi\Gamma\left(\mathfrak{s} + 1 + \frac{\mathsf{k}+1}{2}\right)\pi^{-\frac{1}{2}-\mathfrak{s}}\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}+1)\\ &- \frac{1}{\pi}\left(\mathfrak{s} + \frac{\mathsf{k}-1}{2}\right)\left(\mathfrak{s} - \frac{\mathsf{k}+1}{2}\right)\Gamma\left(\mathfrak{s} - 1 + \frac{\mathsf{k}+1}{2}\right)\pi^{\frac{3}{2}-\mathfrak{s}}\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}-1).\end{aligned}$$

Simplifying this gives

$$(\mathfrak{m} + 1)\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}) = \left(\mathfrak{s} + \frac{\mathsf{k}+1}{2}\right)\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}+1) - \left(\mathfrak{s} - \frac{\mathsf{k}+1}{2}\right)\mathfrak{q}_{\mathfrak{m}}^{(i)}(\mathfrak{s}-1)$$

We can also prove Theorem 1.1.1 in a different way by using the following well known theorem:

Theorem 3.7. *Let $\{\mathfrak{p}_n\}_{n=0}^{\infty}$ be a sequence of polynomials such that the degree of $\mathfrak{p}_n$ is n and the polynomials are orthogonal with respect to some Borel measure μ on $\mathbb{R}$. Then $\mathfrak{p}_n$ have n distinct real roots.*

Proof The theorem is obviously true for $n = 0$. Assume that $\mathfrak{p}_{\mathsf{k}}$ has k distinct roots for $\mathsf{k} < n$. Without loss of generality we assume that all polynomials have one as their leading coefficient. Then $\mathfrak{p}_{\mathsf{k}}$ is real for $\mathsf{k} < n$ and $\mathfrak{p}_n = \mathfrak{f}_n + ig_n$, where g_n has degree less than n. Moreover,

$$0 = (\mathfrak{p}_n, \mathfrak{p}_{\mathsf{k}}) = (\mathfrak{f}_n, \mathfrak{p}_{\mathsf{k}}) - i(g_n, \mathfrak{p}_{\mathsf{k}})$$

for $\mathsf{k} < 0$, hence $(g_n, \mathfrak{p}_{\mathsf{k}}) = 0$. But the degree of g_n is less than n so we must have $g_n \equiv 0$. Thus $\mathfrak{p}_n$ is real. If $\mathfrak{p}_n$ does not have n distinct real roots then it could be written as $\mathfrak{p}_n(x) = (x - \alpha)(x - \bar{\alpha})q(x) = |x - \alpha|^2 q(x)$ for $x \in \mathbb{R}$. Since the degree of q is less than n we must have $(\mathfrak{p}_n, q) = 0$, but on the other hand we have that $\mathfrak{p}_n(x)q(x) \geq 0$ for all x. This is a contradiction, hence $\mathfrak{p}_n$ has n distinct real roots.

Proposition 3.8. *The polynomials $\mathfrak{p}_{\mathfrak{m}}^{(\mathsf{k})}(1/2 + it)$ are orthogonal with respect to the measure $|\Gamma((\mathsf{k}+1)/2 + it)|^2 dt$, where dt is the Lebesgue measure on $\mathbb{R}$.*

Proof As we have noticed before the functions $b_{\mathfrak{m}}, \mathfrak{n}$ are orthogonal, hence for $m \neq m'$ we have

$$\begin{aligned}0 = \left\langle \overline{b_{\mathfrak{m},\mathfrak{n}}}, b_{\mathfrak{m}',\mathfrak{n}} \right\rangle &= 2\pi \int_0^{\infty} \overline{b_{\mathfrak{m},\mathfrak{n}}(r)} b_{\mathfrak{m}',\mathfrak{n}}(r) r dr\\ &= 2\pi \int_{-\infty}^{\infty} \overline{b_{\mathfrak{m},\mathfrak{n}}(e^u)e^u} b_{\mathfrak{m}',\mathfrak{n}}(e^u) e^u du.\end{aligned}$$

Using Plancherel's formula it follows that $2\pi\mathfrak{F}b_{\mathfrak{m},\mathfrak{n}}(e^u)e^u(-2t)$,is an orthogonal sequence ($\mathfrak{F}$ denotes the ordinary Fourier transform) and this is just

$$\begin{aligned}2\pi\mathfrak{F}b_{\mathfrak{m},k}(e^u)e^u(-2t) &= 2\pi\int_0^\infty b_{\mathfrak{m},-k}(r)r^{i2t}dr \\ &= \int_0^\infty\int_0^{2\pi} b_{\mathfrak{m},-k}(re^{i\theta})r^{i2t}e^{ik\theta}d\theta dr = c_{\mathfrak{m},k}\zeta_{\mathfrak{m}}^{(k)}(1/2+it) \\ &= c_{\mathfrak{m},k}\Gamma\left(\frac{\mathsf{k}+1}{2}+it\right)\pi^{1/2-it}p_{\mathfrak{m}}^{(k)}(1/2+it).\end{aligned}$$

Remark. Theorem 1.1.1 follows immediately if we combine Theorem 1.4.5 with Proposition 1.4.6.

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