Stochastic Modeling and Applications
Vol. 25 No. 1 (January-June, 2021)

Received: 27th December 2020 *Revised: 19th January 2021* *Selected: 23rd January 2021*

SOME APPLICATIONS OF UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY FRACTIONAL CALCULUS OPERATOR

SANTOSH V. NAKADE, RAJKUMAR. N. INGLE, AND P.THIRUPATHI REDDY

Abstract. In this paper, we introduce a new subclass of uniformly convex functions with negative coefficients defined by fractional calculus operator. We obtain the coefficient bounds, growth distortion properties, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $TS(\upsilon, \varrho, \mu)$. Furthermore, we obtained modified Hadamard product, convolution and integral operators for this class.

1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_\eta z^\eta \tag{1.1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in E. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{1.2}$$

We denote this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{1.3}$$

We denote this class with $K(\upsilon)$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta,\ (a_\eta \geq 0,\ z \in E) \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary 30C45.

Key words and phrases. analytic, coefficient bounds, fractional calclus,integral operator, hadamard product.

and let $T^*(v) = T \cap S^*(v),\ C(v) = T \cap K(v)$. The class $T^*(v)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [13] and others.

A function $u \in A$ is said to be in the class of uniformly convex functions of order γ and type ϱ, denoted by $UCV(\varrho, \gamma)$, if

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)} - \gamma\right\} > \varrho \left|\frac{zu''(z)}{u'(z)}\right|, \tag{1.5}$$

where $\varrho \geq 0, \gamma \in [-1, 1)$ and $\varrho + \gamma \geq 0$ and it is said to be in the class corresponding class denoted by $SP(\varrho, \gamma)$, if

$$\Re\left\{\frac{zu'(z)}{u(z)} - \gamma\right\} > \varrho \left|\frac{zu'(z)}{u(z)} - 1\right|, \tag{1.6}$$

where $\varrho \geq 0, \gamma \in [-1, 1)$ and $\varrho + \gamma \geq 0$. Indeed it follows from (1.5) and (1.6) that

$$u \in UCV(\gamma, \varrho) \Leftrightarrow zu' \in SP(\gamma, \varrho). \tag{1.7}$$

For $\varrho = 0$, we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$. The function of the class $UCV(1, 0) \equiv UCV$ are called uniformly convex functions, were introduced and studied by Goodman with geometric interpretation in [2]. The class $SP(1, 0) \equiv SP$ is defined by Ronning [9]. The classes $UCV(1, \gamma) \equiv UCV(\gamma)$ and $SP(1, \gamma) \equiv SP(\gamma)$ are investigated by Ronning in [8]. For $\gamma = 0$, the classes $UCV(\varrho, 0) \equiv \varrho - UCV$ and $SP(\varrho, 0) \equiv \varrho - SP$ are defined respectively, by Kanas and Wisniowska in [3, 4].

Further, Murugusundarmoorthy and Magesh [6], Santosh et al. [11], and Thirupathi Reddy and Venkateswarlu [16] have studied and investigated interesting properties for the classes $UCV(\varrho, \gamma)$ and $SP(\varrho, \gamma)$.

For $u \in A$ given by (1.4) and $g(z)$ given by

$$g(z) = z + \sum_{\eta=2}^{\infty} b_\eta z^\eta \tag{1.8}$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{\eta=2}^{\infty} a_\eta b_\eta z^\eta = (g * u)(z)\ (z \in E). \tag{1.9}$$

Note that $u * g \in A$.

Many essentially equivalent definitions of fractional calculus have been given in the literature ((cf.)e.g.,[12] and ([15], p. 45)). We state the following definitions due to Owa and Srivastava [7] which have been used rather frequently in the theory of analytic functions (see also [5]).

Definition 1.1. The fractional integral of order μ is defined, for a function $u(z)$, by

$$D_z^{-\mu} u(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta, \quad (\mu > 0) \tag{1.10}$$

and the fractional derivative of order μ is defined, for a function $u(z)$, by

$$D_z^{\mu} u(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta, \quad (0 \leq \mu < 1) \tag{1.11}$$

where $u(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ involved in (1.10) (and that of $(z-\zeta)^{-\mu}$ involved in (1.11) is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Definition 1.2. Under the hypotheses of Definition **??**, the fractional derivative of order $n+\mu$ is defined by

$$D_z^{n+\mu}u(z)=\frac{d^n}{dz^n}D_z^{\mu}u(z), \quad (0\leq\mu<1; n\in \mathrm{N}_0=\mathrm{N}\cup\{0\}). \tag{1.12}$$

With the aid of the above definitions, Owa and Srivastava [7] defined the fractional operator $\mathcal{J}_z^{\mu}$ by

$$\mathcal{J}_z^{\mu}u(z)=\Gamma(2-\mu)z^{\mu}D_z^{\mu}u(z), \quad (\mu\neq 2,3,4,\cdots) \tag{1.13}$$

for functions (1.1) belonging to the class A.

$$\mathcal{J}_z^{\mu}u(z)=z+\sum_{\eta=2}^{\infty}\phi(\mu,\eta)a_\eta z^\eta \tag{1.14}$$

$$\text{where } \phi(\mu,\eta)=\frac{\Gamma(\eta+1)\Gamma(2-\mu)}{\Gamma(\eta-\mu+1)} \tag{1.15}$$

$$\text{and } \phi(\mu,2)=\frac{2}{(2-\mu)}.$$

Now, by making use of the linear operator $\mathcal{J}_z^{\mu}u$, we define a new subclass of functions belonging to the class A.

Definition 1.3. For $-1\leq\upsilon<1$ and $\varrho\geq 0$, we let $TS(\upsilon,\varrho,\mu)$ be the subclass of A consisting of functions of the form (1.4) and satisfying the analytic criterion

$$\Re\left\{\frac{z(\mathcal{J}_z^{\mu}u(z))'}{\mathcal{J}_z^{\mu}u(z)}-\upsilon\right\}\geq\varrho\left|\frac{z(\mathcal{J}_z^{\mu}u(z))'}{\mathcal{J}_z^{\mu}u(z)}-1\right|, \tag{1.16}$$

for $z\in E$.

By suitably specializing the values of μ and s, the class $TS(\upsilon,\varrho,\mu)$ can be reduces to the class studied earlier by Ronning [8, 9]. The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, distortion properties, extreme points, radii of starlikness and convexity, Hadamard product and convolution and integral operators for the class.

2. Coefficient bounds

In this section, we obtain a necessary and sufficient condition for function $u(z)$ is in the class $TS(\upsilon,\varrho,\mu)$.

We employ the technique adopted by Aqlan et al. [1] to find the coefficient estimates for our class.

Theorem 2.1. *The function u defined by* (1.4) *is in the class $TS(\upsilon, \varrho, \mu)$ if*

$$\sum_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)|a_\eta| \leq 1-\upsilon, \tag{2.1}$$

where $-1 \leq \upsilon < 1, \varrho \geq 0$. The result is sharp.

Proof. We have $f \in TS(\upsilon, \varrho, \mu)$ if and only if the condition (1.16) satisfied. Upon the fact that

$$\Re(w) > \varrho|w-1| + \upsilon \Leftrightarrow \Re\{w(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\} > \upsilon, \ -\pi \leq \theta \leq \pi.$$

Equation (1.16) may be written as

$$\Re\left\{\frac{z(\mathcal{J}_z^\mu u(z))'}{\mathcal{J}_z^\mu u(z)}(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\right\}$$
$$= \Re\left\{\frac{z(\mathcal{J}_z^\mu u(z))'1+\varrho e^{i\theta}) - \varrho e^{i\theta}\mathcal{J}_\mu u(z)}{\mathcal{J}_z^\mu u(z)}\right\} > \ \upsilon. \tag{2.2}$$

Now, we let

$$A(z) = z(\mathcal{J}_z^\mu u(z))'1+\varrho e^{i\theta}) - \varrho e^{i\theta}\mathcal{J}_z^\mu u(z)$$
$$B(z) = \mathcal{J}_z^\mu u(z).$$

Then (2.2) is equivalent to

$$|A(z) + (1-\upsilon)B(z)| > |A(z) - (1+\upsilon)B(z)|, \text{ for } 0 \leq \upsilon < 1.$$

For $A(z)$ and $B(z)$ as above, we have

$$|A(z) + (1-\upsilon)B(z)| \geq (2-\upsilon)|z| - \sum_{\eta=2}^{\infty}[\eta+1-\upsilon+\varrho(\eta-1)]\phi(\mu,\eta)|a_\eta||z^\eta|$$

and similarly

$$|A(z) - (1+\upsilon)B(z)| \leq \upsilon|z| - \sum_{\eta=2}^{\infty}[\eta-1-\upsilon+\varrho(\eta-1)]\phi(\mu,\eta)|a_\eta||z^\eta|.$$

Therefore

$$|A(z) + (1-\upsilon)B(z)| - |A(z) - (1+\upsilon)B(z)|$$
$$\geq \ 2(1-\upsilon) - 2\sum_{\eta=2}^{\infty}[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,\eta)|a_\eta|$$

or $\displaystyle\sum_{\eta=2}^{\infty}[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,\eta)|a_\eta| \leq (1-\upsilon),$

which yields (2.1).
On the other hand, we must have

$$\Re\left\{\frac{z(\mathcal{J}_z^\mu u(z))'}{\mathcal{J}_z^\mu u(z)}(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\right\} \geq \upsilon.$$

Upon choosing the values of z on the positive real axis where $0 = |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum_{\eta=2}^{\infty}[\eta-\upsilon+\varrho e^{i\theta}(\eta-1)]\phi(\mu,\eta)|a_\eta|\ r^\eta}{z-\sum_{\eta=2}^{\infty}\phi(\mu,\eta)|a_\eta|\ r^\eta}\right\} \geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum_{\eta=2}^{\infty}[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,\eta)|a_\eta|\ r^\eta}{z-\sum_{\eta=2}^{\infty}\phi(\mu,\eta)|a_\eta|\ r^\eta}\right\} \geq 0.$$

Letting $r \to 1^-$, we get the desired result. Finally the result is sharp with the extremal function u given by

$$u(z) = z - \frac{1-\upsilon}{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}z^\eta \tag{2.3}$$

□

3. Growth and Distortion Theorems

Theorem 3.1. *Let the function* u *defined by* (1.4) *be in the class* $TS(\upsilon,\varrho,\mu)$. *Then for* $|z| = r$

$$r - \frac{1-\upsilon}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)}r^2 \leq |u(z)| \leq r + \frac{1-\upsilon}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)}r^2. \tag{3.1}$$

Equality holds for the function

$$u(z) = z - \frac{1-\upsilon}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)}z^2. \tag{3.2}$$

Proof. We only prove the right hand side inequality in (3.1) ince the other inequality can be justified using similar arguments. In view of Theorem 2.1, we have

$$\sum_{\eta=2}^{\infty}|a_\eta| \leq \frac{1-\upsilon}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)}.$$

Since ,

$$\begin{aligned} u(z) &= z - \sum_{\eta=2}^{\infty} a_\eta z^\eta \\ |u(z)| &= \left| z - \sum_{\eta=2}^{\infty} a_\eta z^\eta \right| \\ &\leq r + \sum_{\eta=2}^{\infty} |a_\eta| r^\eta \\ &\leq r + r^2 \sum_{\eta=2}^{\infty} |a_\eta| \\ &\leq r + \sum_{\eta=2}^{\infty} \frac{1-\upsilon}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)} r^2 \end{aligned}$$

which yields the right hand side inequality of (3.1). □

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. *Let the function* u *defined by* (1.4) *be in the class* $TS(\upsilon, \varrho, \mu)$. *Then for* $|z| = r$

$$1 - \frac{2(1-\upsilon)}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)} r \leq |u'(z)| \leq 1 + \frac{2(1-\upsilon)}{(\eta+1)(2-\upsilon+\varrho)\phi(\mu,2)} r.$$

Equality holds for the function given by (3.2).

Theorem 3.3. *If* $u \in TS(\upsilon, \varrho, \mu)$ *then* $u \in TS(\gamma)$, *where*

$$\gamma = 1 - \frac{(\eta-1)(1-\upsilon)}{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)-(1-\upsilon)}$$

Equality holds for the function given by (3.2).

Proof. It is sufficient to show that (2.1) implies

$$\sum_{\eta=2}^{\infty} (\eta-\gamma)|a_\eta| \leq 1-\gamma,$$

that is

$$\frac{\eta-\gamma}{1-\gamma} \leq \frac{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}{(1-\upsilon)},$$

then

$$\gamma \leq 1 - \frac{(\eta-1)(1-\upsilon)}{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)-(1-\upsilon)}.$$

The above inequality holds true for $\eta \in \mathbb{N}_0, \eta \geq 2, \varrho \geq 0$ and $0 \leq \upsilon < 1$. □

4. Extreme points

Theorem 4.1. *Let* $u_1(z) = z$ *and*

$$u_\eta(z) = z - \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} z^\eta, \tag{4.1}$$

for $\eta = 2, 3, \cdots$ *. Then* $u(z) \in TS(\upsilon, \varrho, \mu)$ *if and only if* $u(z)$ *can be expressed in the form* $u(z) = \sum\limits_{\eta=1}^{\infty} \zeta_\eta u_\eta(z)$*, where* $\zeta_\eta \geq 0$ *and* $\sum\limits_{\eta=1}^{\infty} \zeta_\eta = 1$.

Proof. Suppose $u(z)$ can be expressed as in (4.1). Then

$$\begin{aligned} u(z) &= \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z) = \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) \\ &= \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ z - \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} z^\eta \right\} \\ &= \zeta_1 z + \sum_{\eta=2}^{\infty} \zeta_\eta z - \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} z^\eta \right\} \\ &= z - \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} z^\eta \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\eta=2}^{\infty} \zeta_\eta \left(\frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} \right) \left(\frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)}{1-\upsilon} \right) \\ &= \sum_{\eta=2}^{\infty} \zeta_\eta = \sum_{\eta=1}^{\infty} \zeta_\eta - \zeta_1 = 1 - \zeta_1 \leq 1. \end{aligned}$$

So, by Theorem 2.1, $u \in TS(\upsilon, \varrho, \mu)$.

Conversely, we suppose $u \in TS(\upsilon, \varrho, \mu)$. Since

$$|a_\eta| \leq \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)}, \ \eta \geq 2.$$

We may set

$$\zeta_\eta = \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)}{1-\upsilon} |a_\eta|, \ \eta \geq 2$$

and $\zeta_1 = 1 - \sum\limits_{\eta=2}^{\infty} \zeta_\eta$. Then □

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta = z - \sum_{\eta=2}^{\infty} \zeta_\eta \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} z^\eta$$
$$= z - \sum_{\eta=2}^{\infty} \zeta_\eta [z - u_\eta(z)] = z - \sum_{\eta=2}^{\infty} \zeta_\eta z + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z)$$
$$= \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) = \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z).$$

Corollary 4.2. *The extreme points of $TS(\upsilon,\varrho,\mu)$ are the functions $u_1(z) = z$ and*

$$u_\eta(z) = z - \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)} z^\eta, \ \eta \geq 2.$$

5. Radii of Close-to-convexity, Starlikeness and Convexity

A function $u \in TS(\upsilon,\varrho,\mu)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{u'(z)\} > \delta, \ (0 \leq \delta < 1; \ z \in E).$$

Also A function $u \in TS(\upsilon,\varrho,\mu)$ is said to be starlike of order δ if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta, \ (0 \leq \delta < 1; \ z \in E).$$

Further a function $u \in TS(\upsilon,\varrho,\mu)$ is said to be convex of order δ if and only if $zu'(z)$ is starlike of order δ that is if

$$\Re\left\{1 + \frac{zu'(z)}{u(z)}\right\} > \delta, \ (0 \leq \delta < 1; \ z \in E).$$

Theorem 5.1. *Let $u \in TS(\upsilon,\varrho,\mu)$. Then u is close-to-convex of order δ in $|z| < R_1$, where*

$$R_1 = \inf_{k\geq 2}\left[\frac{(1-\delta)[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}{\eta(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (2.3).

Proof. It is sufficient to show that $|u'(z) - 1| \leq 1 - \delta$, for $|z| < R_1$. We have

$$|u'(z) - 1| = \left|-\sum_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}\right| \leq \sum_{\eta=2}^{\infty} \eta a_\eta |z|^{\eta-1}.$$

Thus $|u'(z) - 1| \leq 1 - \delta$ if

$$\sum_{\eta=2}^{\infty} \frac{\eta}{1-\delta} |a_\eta||z|^{\eta-1} \leq 1. \tag{5.1}$$

But Theorem 2.1 confirms that

$$\sum_{\eta=2}^{\infty} \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)}{1-\upsilon} |a_\eta| \leq 1. \tag{5.2}$$

Hence (5.1) will be true if

$$\frac{\eta|z|^{\eta-1}}{1-\delta} \leq \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)}{1-\upsilon}.$$

We obtain

$$|z| \leq \left[\frac{(1-\delta)[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}{\eta(1-\upsilon)}\right]^{\frac{1}{\eta-1}}, \eta \geq 2$$

as required. □

Theorem 5.2. *Let $u \in TS(\upsilon,\varrho,\mu)$. Then u is starlike of order δ in $|z| < R_2$, where*

$$R_2 = \inf_{k\geq 2}\left[\frac{(1-\delta)[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}{(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (2.3).

Proof. We must show that $\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\delta$, for $|z| < R_2$. We have

$$\begin{aligned}\left|\frac{zu'(z)}{u(z)}-1\right| &= \left|\frac{-\sum\limits_{\eta=2}^{\infty}(\eta-1)a_\eta z^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}a_\eta z^{\eta-1}}\right| \\ &\leq \frac{\sum\limits_{\eta=2}^{\infty}(\eta-1)|a_\eta||z|^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}|a_\eta||z|^{\eta-1}} \\ &\leq 1-\delta. \end{aligned} \tag{5.3}$$

Hence (5.3) holds true if

$$\sum_{\eta=2}^{\infty}(\eta-1)|a_\eta||z|^{\eta-1} \leq (1-\delta)\left(1-\sum_{\eta=2}^{\infty}|a_\eta||z|^{\eta-1}\right)$$

or equivalently,

$$\sum_{\eta=2}^{\infty}\frac{\eta-\delta}{1-\delta}|a_\eta||z|^{\eta-1} \leq 1. \tag{5.4}$$

Hence, by using (5.2) and (5.4) will be true if

$$\frac{\eta-\delta}{1-\delta}|z|^{\eta-1} \leq \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,\eta)}{1-\upsilon}$$

$$\Rightarrow |z| \leq \left[\frac{(1-\delta)[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}{(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}, \eta \geq 2$$

which completes the proof. □

By using the same technique in the proof of Theorem 5.2, we can show that $\left|\frac{zu''(z)}{u'(z)} - 1\right| \leq 1 - \delta$, for $|z| < R_3$, with the aid of Theorem 2.1. Thus we have the assertion of the following Theorem 5.3.

Theorem 5.3. *Let $u \in TS(\upsilon, \varrho, \mu)$. Then u is convex of order δ in $|z| < R_3$, where*

$$R_3 = \inf_{k \geq 2} \left[\frac{(1-\delta)[\eta(1+\varrho) - (\upsilon + \varrho)]\phi(\mu, \eta)}{\eta(\eta - \delta)(1 - \upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (2.3).

6. Inclusion theorem involving modified Hadamard products

For functions

$$u_j(z) = z - \sum_{\eta=2}^{\infty} |a_{\eta,j}| z^{\eta}, \ j = 1, 2 \tag{6.1}$$

in the class A, we define the modified Hadamard product $u_1 * u_2(z)$ of $u_1(z)$ and $u_2(z)$ given by

$$u_1 * u_2(z) = z - \sum_{\eta=2}^{\infty} |a_{\eta,1}||a_{\eta,2}| z^{\eta}.$$

We can prove the following.

Theorem 6.1. *Let the function u_j, $j = 1, 2$, given by* (6.1) *be in the class $TS(\upsilon, \varrho, \mu)$ respectively. Then $u_1 * u_2(z) \in TS(\upsilon, \varrho, \mu, \xi)$, where*

$$\xi = 1 - \frac{(1-\upsilon)^2}{(\eta+1)(2-\upsilon)(2-\upsilon+\varrho)(1+\lambda) - (1-\upsilon)^2}.$$

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest ξ such that

$$\sum_{\eta=2}^{\infty} \frac{[\eta - \xi + \varrho(\eta - 1)]\phi(\mu, \eta)}{1 - \xi} |a_{\eta,1}||a_{\eta,2}| \leq 1.$$

Since $u_j \in TS(\upsilon, \varrho, \mu), \ j = 1, 2$ then we have

$$\sum_{\eta=2}^{\infty} \frac{[\eta(1+\varrho) - (\upsilon+\varrho)]\phi(\mu, \eta)}{1-\upsilon} |a_{\eta,1}| \leq 1$$

$$\text{and } \sum_{\eta=2}^{\infty} \frac{[\eta(1+\varrho) - (\upsilon+\varrho)]\phi(\mu, \eta)}{1-\upsilon} |a_{\eta,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{\eta=2}^{\infty} \frac{[\eta(1+\varrho) - (\upsilon+\varrho)]\phi(\mu, \eta)}{1-\upsilon} \sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\frac{[\eta-\xi+\varrho(\eta-1)]\phi(\mu,\eta)}{1-\xi}|a_{\eta,1}||a_{\eta,2}|$$
$$\leq \frac{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}{1-\upsilon}\sqrt{|a_{\eta,1}||a_{\eta,2}|},\ \eta\geq 2,$$

that is

$$\sqrt{|a_{\eta,1}||a_{\eta,2}|}\leq \frac{(1-\xi)[\eta(1+\varrho)-(\upsilon+\varrho)]}{1-\upsilon)[\eta-\xi+\varrho(\eta-1)]}.$$

Note that

$$\sqrt{|a_{\eta,1}||a_{\eta,2}|}\leq \frac{(1-\upsilon)}{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}.$$

Consequently, we need only to prove that

$$\frac{(1-\upsilon)}{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)}\leq \frac{(1-\xi)[\eta(1+\varrho)-(\upsilon+\varrho)]}{1-\upsilon)[\eta-\xi+\varrho(\eta-1)]},\ \eta\geq 2,$$

or, equivalently, that

$$\xi\leq 1-\frac{(\eta-1)(1+\varrho)(1-\upsilon)^2}{[\eta(1+\varrho)-(\upsilon+\varrho)]^2\phi(\mu,\eta)-(1-\upsilon)^2},\ \eta\geq 2.$$

Since

$$A(k)=1-\frac{(\eta-1)(1+\varrho)(1-\upsilon)^2}{[\eta(1+\varrho)-(\upsilon+\varrho)]^2\phi(\mu,\eta)-(1-\upsilon)^2},\ \eta\geq 2$$

is an increasing function of $\eta,\eta\geq 2$, letting $\eta=2$ in last equation, we obtain

$$\xi\leq A(2)=1-\frac{(1+\varrho)(1-\upsilon)^2}{[2-\upsilon+\varrho]^2\phi(\mu,\eta)-(1-\upsilon)^2}.$$

Finally, by taking the function given by (3.2), we can see that the result is sharp. □

7. Convolution and Integral Operators

Let $u(z)$ be defined by (1.4) and suppose that $g(z)=z-\sum\limits_{\eta=2}^{\infty}|b_\eta|z^\eta$. Then,the Hadamard product (or convolution) of $u(z)$ and $g(z)$ defined here by

$$u(z)*g(z)=u*g(z)=z-\sum_{\eta=2}^{\infty}|a_\eta||b_\eta|z^\eta.$$

Theorem 7.1. *Let $u\in TS(\upsilon,\varrho,\mu)$ and $g(z)=z-\sum\limits_{\eta=2}^{\infty}|b_\eta|z^\eta, 0\leq|b_\eta|\leq 1$. Then $u*g\in TS(\upsilon,\varrho,\mu)$.*

Proof. In view of Theorem 2.1, we have

$$\begin{aligned}&\sum_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)|a_\eta||b_\eta|\\ \leq&\sum_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,\eta)|a_\eta|\\ \leq&(1-\upsilon).\end{aligned}$$

□

Theorem 7.2. *Let $u \in TS(\upsilon,\varrho,\mu)$ and α be real number such that $\alpha > -1$. Then the function $F(z) = \frac{\alpha+1}{z^\alpha}\int_0^z t^{\alpha-1}u(t)dt$ also belongs to the class $TS(\upsilon,\varrho,\mu)$.*

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{\eta=2}^{\infty}|A_\eta|z^\eta, \text{ where } A_\eta = \left(\frac{\alpha+1}{\alpha+\eta}\right)|a_\eta|.$$

Since $\alpha > -1$, than $0 \leq A_\eta \leq |a_\eta|$. Which in view of Theorem 2.1, $F \in TS(\upsilon,\varrho,\mu)$.

□

References

[1] Aqlan, E., Jahangiri, J. M. and Kulkarni, S. R., New classes of k-uniformly convex and starlike functions, Tamkang J. Math., 35(3), 261 - 266, (2004).

[2] Goodman, A. W., On uniformly convex functions, Ann. Pol. Math., 56, 87-92, (1991).

[3] Kanas, S. and Wisniowska, A., Conic regions and k-uniform convexity, Comput. Appl. Math., 105 , 327-336, (1999).

[4] Kanas, S. and Wisniowska, A., Conic domains and starlike functions, Rev. Roum. Math. Pures Appl., 45 , 647-657, (2000).

[5] Kim, Y.C. and Srivastava, H.M., Fractional integral and other linear operators asso ciated with the Gaussian hypergeometric function, Complex Variables Theory Appl. 34, 293 - 312, (1997).

[6] Murugusundarmoorthy, G. and Magesh, N., Certain subclasses of starlike functions of complex order involving generalised hypergeometric functions, Int. J. Math. Sci., 45 , 12 pages, (2010).

[7] Owa, S. and Srivastava, H.M., Univalent and starlike generalized hypergeometric functions, Can. J. Math. 39, 1057-1077, (1987).

[8] Ronning, F., On starlike functions associated with parabolic regions, Ann. Univ. Mariae. Curie-Sklodowska Sect. A, 45 , 117-122, (1991).

[9] Ronning, F., Uniformly convex functions and a corresponing class of starlike functions, Proc. Amer. Math. Soc., 118 , 189-196, (1993).

[10] Schild, A. and Silverman, H., Convolutions of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Skalodowska Sect. A, 29 ,99 - 107 (1975).

[11] Santosh, M. P., Rajkumar, N. I., Thirupathi Reddy, P. and Venkateswarlu, B., A new subclass of analytic functions defined by linear operator, Adv. Math. Sci, Journal, 9(1), 205-217, (2020).

[12] Samko, S.G., Kilbas, A. A. and Marichev, O.I., Fractional Integral and Derivatives, Theory and Applications, Gordon and Breach, New York, Philadelphia, London, Paris, Montreux, Toronto and Melbourne, 1993.

[13] Silverman, H., Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 , 109-116, (1975).

[14] Spanier, J. and Oldham, K. B., The zeta numbers and realted functions, Chapter 3 in An Atlas of functions, Washington, Dc:Hemisphere, 25-33, (1987).
[15] Srivaetava, H.M. and Buschman, R.G., Theory and Applications of Convolution Integral Equations, Kluwer Academic Publishers, dordrecht, Boston and London, 1992.
[16] Thirupathi Reddy, P. and Venkateswarlu, B., On a certain subclass of uniformly convex functions defined by bessel functions, Transylvanian J. of Math. and Mech., 10 (1) , 43 - 49, (2018).

Santosh V. Nakade: Department of Mathematics,Sharda Mahavidyalaya,, Prabhani - 431 401, Maharashtra, India.,
E-mail address: santosh_nakade@yahoo.in

Rajkumar. N. Ingle: Department of Mathematics, Bahirji Smarak Mahavidyalay, Bashmathnagar - 431 512, Hingoli Dist., Maharashtra, India.
E-mail address: ingleraju11@gmail.com

P.Thirupathi Reddy: Department of Mathematics, Kakatiya University, Warangal-506 009, Telangana, India.
E-mail address: reddypt2@gmail.com