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A CERTAIN SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY LAMBDA OPERATOR

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Abstract. In this paper, we introduce a new subclasses of uniformly convex functions with negative coefficients defined by lambda operator. We obtain the coefficient bounds, extreme points and radii of starlikeness and convexity for functions belonging to the class $TS(\upsilon, \varrho, \mu, s)$. Furthermore, partial sums are considered and sharp lower bounds for the ratios of real part of $u(z)$ to $u_k(z)$ and $u'(z)$ to $u'_k(z)$ are determined and also discussed neighbourhood results for this class.

1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_\eta z^\eta \tag{1.1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in E. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{1.2}$$

We denote this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{1.3}$$

We denote this class with $K(\upsilon)$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta, \ (a_\eta \geq 0, \ z \in E) \tag{1.4}$$

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and let $T^*(v) = T \cap S^*(v),\ C(v) = T \cap K(v)$. The class $T^*(v)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [11] and others.

A function $u \in A$ is said to be in the class of uniformly convex functions of order γ and type ϱ, denoted by $UCV(\varrho, \gamma)$, if

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)} - \gamma\right\} > \varrho\left|\frac{zu''(z)}{u'(z)}\right|, \tag{1.5}$$

where $\varrho \geq 0, \gamma \in [-1, 1)$ and $\varrho + \gamma \geq 0$ and it is said to be in the class corresponding class denoted by $SP(\varrho, \gamma)$, if

$$\Re\left\{\frac{zu'(z)}{u(z)} - \gamma\right\} > \varrho\left|\frac{zu'(z)}{u(z)} - 1\right|, \tag{1.6}$$

where $\varrho \geq 0, \gamma \in [-1, 1)$ and $\varrho + \gamma \geq 0$. Indeed it follows from (1.5) and (1.6) that

$$u \in UCV(\gamma, \varrho) \Leftrightarrow zu' \in SP(\gamma, \varrho). \tag{1.7}$$

For $\varrho = 0$, we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$. The function of the class $UCV(1, 0) \equiv UCV$ are called uniformly convex functions, were introduced and studied by Goodman with geometric interpretation in [4]. The class $SP(1, 0) \equiv SP$ is defined by Ronning [9]. The classes $UCV(1, \gamma) \equiv UCV(\gamma)$ and $SP(1, \gamma) \equiv SP(\gamma)$ are investigated by Ronning in [8]. For $\gamma = 0$, the classes $UCV(\varrho, 0) \equiv \varrho - UCV$ and $SP(\varrho, 0) \equiv \varrho - SP$ are defined respectively, by Kanas and Wisniowska in [5, 6].

Further Ahuja et al. [1], Bharathi et al. [2], Murugusundarmoorthy and Magesh [7] and Thirupathi Reddy and Venkateswarlu [15] have studied and investigated interesting properties for the classes $UCV(\varrho, \gamma)$ and $SP(\varrho, \gamma)$.

For $u \in A$ given by (1.1) and $y(z)$ given by $y(z) = z + \sum\limits_{\eta=2}^{\infty} b_\eta z^\eta$ their convolution (or Hadamard product), denoted by $(u * y)$, is defined as

$$(u * y)(z) = z + \sum_{\eta=2}^{\infty} a_\eta b_\eta z^\eta = (y * u)(z),\ (z \in E). \tag{1.8}$$

Note that $u * y \in A$.

Let us recall lambda function [14] defined by

$$\lambda(z, s) = \sum_{\eta=2}^{\infty} \frac{z^\eta}{(2\eta + 1)^\eta}$$

where $z \in E, s \in \mathbb{C}$, when $|z| < 1, \Re(s) > 1$, when $|z| = 1$
and let $\lambda^{(-1)}(z, s)$ be defined such that

$$\lambda(z, s) * \lambda^{(-1)}(z, s) = \frac{1}{(1 - z)^{\mu+1}},\ \mu > -1.$$

We now define $\left(z\lambda^{(-1)}(z, s)\right)$ as the following

$$(z\lambda(z, s)) * \left(z\lambda^{(-1)}(z, s)\right) = \frac{z}{(1 - z)^{\mu+1}} = z + \sum_{\eta=2}^{\infty} \frac{(\mu + 1)_{\eta-1}}{(\eta - 1)!} z^\eta, \mu > -1$$

and obtain the following lambda operator

$$\mathcal{I}_{\mu,s}u(z) = \left(z\lambda^{(-1)}(z,s)\right) * u(z)$$

where $u \in A, z \in E$ and

$$\left(z\lambda^{(-1)}(z,s)\right) = z + \sum_{\eta=2}^{\infty} \frac{(\mu+1)_{\eta-1}(2\eta-1)^s}{(\eta-1)!} z^\eta.$$

A simple computation gives us

$$\mathcal{I}_{\mu,s}u(z) = z + \sum_{\eta=2}^{\infty} \phi(\mu,s,\eta)a_\eta z^\eta \tag{1.9}$$

$$\text{where } \phi(\mu,s,\eta) = \frac{(\mu+1)_{\eta-1}(2\eta-1)^s}{(\eta-1)!}, \tag{1.10}$$

where $(\mu)_\eta$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(\mu)_\eta = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu)} = \begin{cases} 1, & \text{if } \eta = 0; \\ \mu(\mu+1)\cdots(\mu+\eta-1), & \text{if } \eta \in \mathbb{N} \end{cases}.$$

Now, by making use of the lambda operator $\mathcal{I}_{\mu,s}u$, we define a new subclass of functions belonging to the class A.

Definition 1.1. For $-1 \le \upsilon < 1$ and $\varrho \ge 0$, we let $S(\upsilon,\varrho,\mu,s)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)} - \upsilon\right\} \ge \varrho\left|\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)} - 1\right|, \tag{1.11}$$

for $z \in E$.

By suitably specializing the values of μ and s, the class $S(\upsilon,\varrho,\mu,s)$ can be reduces to the class studied earlier by Ronning [8, 9]. The main object of the paper some usual properties of the geometric function theory such as coefficient bounds, extreme points, radii of starlikness and convexity, partial sums for the class and neighbourhood results for the class.

2. Coefficient bounds

In this section, we obtain a necessary and sufficient condition for function $u(z)$ is in the classes $S(\upsilon,\varrho,\mu,s)$ and $TS(\upsilon,\varrho,\mu,s)$.

Theorem 2.1. *The function u defined by* (1.1) *is in the class* $S(\upsilon,\varrho,\mu,s)$ *if*

$$\sum_{\eta=2}^{\infty}[\eta(1+\varrho) - (\upsilon+\varrho)]\phi(\mu,s,\eta)|a_\eta| \le 1-\upsilon, \tag{2.1}$$

where $-1 \le \upsilon < 1, \varrho \ge 0$.

Proof. It suffices to show that

$$\varrho\left|\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)} - 1\right| - \Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)} - 1\right\} \le 1-\upsilon.$$

We have

$$\varrho\left|\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)}-1\right|-\Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)}-1\right\}$$

$$\leq(1+\varrho)\left|\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)}-1\right|\leq\frac{(1+\varrho)\sum\limits_{\eta=2}^{\infty}(\eta-1)\phi(\mu,s,\eta)|a_\eta|}{1-\sum\limits_{\eta=2}^{\infty}\phi(\mu,s,\eta)|a_\eta|}.$$

This last expression is bounded above by $(1-\upsilon)$ by

$$\sum_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,s,\eta)|a_\eta|\leq 1-\upsilon$$

and hence the proof is complete. □

Theorem 2.2. *A necessary and sufficient condition for $u(z)$ of the form (1.4) to be in the class $TS(\upsilon,\varrho,\mu,s)$, $-1\leq\upsilon<1,\varrho\geq 0$ is that*

$$\sum_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,s,\eta)|a_\eta|\leq 1-\upsilon. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity.

If $u\in TS(\upsilon,\varrho,\mu,s)$ and z is real then

$$\frac{1-\sum\limits_{\eta=2}^{\infty}\eta\phi(\mu,s,\eta)a_\eta z^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}\phi(\mu,s,\eta)a_\eta z^{\eta-1}}-\upsilon\geq\varrho\left|\frac{\sum\limits_{\eta=2}^{\infty}(\eta-1)\phi(\mu,s,\eta)|a_\eta|}{1-\sum\limits_{\eta=2}^{\infty}\phi(\mu,s,\eta)|a_\eta|}\right|.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality
$\sum\limits_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,s,\eta)|a_\eta|\leq 1-\upsilon.$ □

Theorem 2.3. *Let $u(z)$ defined by (1.4) and $g(z)=z-\sum\limits_{\eta=2}^{\infty}b_\eta z^\eta$ be in the class $TS(\upsilon,\varrho,\mu,s)$. Then the function $h(z)$ defined by*

$$h(z)=(1-\zeta)u(z)+\zeta g(z)=z-\sum_{\eta=2}^{\infty}c_\eta z^\eta,$$

where $c_\eta=(1-\zeta)a_\eta+\zeta b_\eta,\ 0\leq\zeta<1$ is also in the class $TS(\upsilon,\varrho,\mu,s)$.

Proof. Let the function

$$u_j=z-\sum_{\eta=2}^{\infty}a_{\eta,j}z^\eta,\ a_{\eta,j}\geq 0,\ j=1,2, \tag{2.3}$$

be in the class $TS(\upsilon,\varrho,\mu,s)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z)=\zeta u_1(z)+(1-\zeta)u_2(z),\ 0\leq\zeta\leq 1,$$

is in the class $TS(\upsilon, \varrho, \mu, s)$. Since

$$g(z) = z - \sum_{\eta=2}^{\infty}[\zeta a_{\eta,1} + (1-\zeta)a_{\eta,2}]z^{\eta},$$

an easy computation with the aid of Theorem 2.2 gives,

$$\begin{aligned}&\sum_{\eta=2}^{\infty}[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)\zeta a_{\eta,1} + \sum_{\eta=2}^{\infty}[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)(1-\zeta)a_{\eta,2}\\ &\leq \zeta(1-\upsilon) + (1-\zeta)(1-\upsilon)\\ &\leq 1-\upsilon,\end{aligned}$$

which implies that $g \in TS(\upsilon, \varrho, \mu, s)$. Hence $TS(\upsilon, \varrho, \mu, s)$ is convex □

3. Extreme points

The proof of Theorem 3.1, follows on lines similar to the proof of the theorem on extreme points given in Silverman [11]

Theorem 3.1. *Let* $u_1(z) = z$ *and*

$$u_{\eta}(z) = z - \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}z^{\eta}, \tag{3.1}$$

for $\eta = 2, 3, \cdots$. *Then* $u(z) \in TS(\upsilon, \varrho, \mu, s)$ *if and only if* $u(z)$ *can be expressed in the form* $u(z) = \sum\limits_{\eta=2}^{\infty} \zeta_{\eta} u_{\eta}(z)$, *where* $\zeta_{\eta} \geq 0$ *and* $\sum\limits_{\eta=1}^{\infty} \zeta_{\eta} = 1$.

Next we prove the following closure theorem.

4. Closure theorem

Theorem 4.1. *Let the function* $u_j(z), j = 1, 2, \cdots, l$ *defined by* (2.3) *be in the classes* $TS(\upsilon_j, \varrho, \mu, s)$, $j = 1, 2, \cdots, l$ *respectively. Then the function* $h(z)$ *defined by*

$$h(z) = z - \frac{1}{l}\sum_{\eta=2}^{\infty}\left(\sum_{j=1}^{l} a_{\eta,j}\right)z^{\eta}$$

is in the class $TS(\upsilon, \varrho, \mu, s)$, *where* $\upsilon = \min\limits_{1\leq j\leq l}\{\upsilon_j\}$, *where* $-1 \leq \upsilon_j \leq 1$.

Proof. Since $u_j(z) \in TS(\upsilon_j, \varrho, \mu, s),\ j = 1, 2, \cdots, l$ for applying (2.3) to Theorem 2.2, then we observe that

$$\begin{aligned}
&\sum_{\eta=2}^{\infty}[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)\left(\frac{1}{l}\sum_{j=1}^{l}a_{\eta,j}\right)\\
=&\frac{1}{l}\sum_{j=1}^{l}a_{\eta,j}\left(\sum_{\eta=2}^{\infty}[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)a_{\eta,j}\right)\\
\leq&\frac{1}{l}\sum_{j=1}^{l}(1-\upsilon_j)\\
\leq&1-\upsilon
\end{aligned}$$

which in view of Theorem 2.2, again implies that $h(z) \in TS(\upsilon, \varrho, \mu, s)$ and so the proof is complete. □

Theorem 4.2. *Let $u \in TS(\upsilon, \varrho, \mu, s)$. Then*

(1). *u is starlike of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$ i.e., $\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta$, $|z| < r_1$, where*

$$r_1 = \inf_{\eta\geq 2}\left\{\left(\frac{1-\delta}{\eta-\delta}\right)\frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon}\right\}^{\frac{1}{\eta-1}}.$$

(2). *u is convex of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$ i.e., $\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \delta$, $|z| < r_2$, where*

$$r_2 = \inf_{\eta\geq 2}\left\{\left(\frac{1-\delta}{\eta-\delta}\right)\frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon}\right\}^{\frac{1}{\eta}}.$$

Each of these results are sharp for the extremal function $u(z)$ given by (3.1)*.*

Proof. Given $u \in A$ and u is starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| < 1 - \delta. \tag{4.1}$$

For the left hand side (4.1), we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \leq \frac{\sum\limits_{\eta=2}^{\infty}(\eta-1)a_\eta|z|^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}a_\eta|z|^{\eta-1}}.$$

The last expression is less that $1-\delta$ if $\sum\limits_{\eta=2}^{\infty}\frac{\eta-\delta}{1-\delta}a_\eta|z|^{\eta-1} < 1$.

Using the fact, that $u \in TS(\upsilon, \varrho, \mu, s)$ if and only if

$$\sum_{\eta=2}^{\infty}\frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon}a_\eta < 1.$$

We can say (4.1) is true if

$$\frac{\eta-\delta}{1-\delta}|z|^{\eta-1} < \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon}$$

$$\Rightarrow\ |z|^{\eta-1} < \frac{(1-\delta)[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{(\eta-\delta)(1-\upsilon)}$$

which yields the starlikeness of the family.

(2). Using the fact that u is convex if and only if zu' is starlike, we can prove (2), on lines similar to the proof of (1). □

5. Partial Sums

Following the earlier works by Silverman [12] and Silvia [13] on partial sums of analytic functions. We consider in this section partial sums of functions in this class $S(\upsilon,\varrho,\mu,s)$ and obtain sharp lower bounds for the ratios of real part of $u(z)$ to $u_q(z)$ and $u'(z)$ to $u'_q(z)$.

Theorem 5.1. *Let $u(z)\in S(\upsilon,\varrho,\mu,s)$ be given by (1.1) and define the partial sums $u_1(z)$ and $u_q(z)$ by*

$$u_1(z)=z \text{ and } u_q(z)=z+\sum_{\eta=2}^{q}a_\eta z^\eta,\ (q\in\mathbb{N}\setminus\{1\}). \tag{5.1}$$

Suppose that $\sum_{\eta=2}^{\infty} d_\eta|a_\eta|\le 1$,

$$\text{where } d_\eta=\frac{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon} \tag{5.2}$$

Then $u\in S(\upsilon,\varrho,\mu,s)$.

$$\text{Further more, } \Re\left[\frac{u(z)}{u_q(z)}\right] > 1-\frac{1}{d_{q+1}},\ (z\in E,\ q\in\mathbb{N}) \tag{5.3}$$

$$\text{and } \Re\left[\frac{u_q(z)}{u(z)}\right] > \frac{d_{q+1}}{1+d_{q+1}}. \tag{5.4}$$

Proof. For the coefficients d_η given by (5.2) it is not difficult to verify that

$$d_{\eta+1} > d_\eta > 1. \tag{5.5}$$

$$\text{Therefore we have } \sum_{\eta=2}^{q}|a_\eta| + d_{q+1}\sum_{\eta=q+1}^{\infty}|a_\eta| \le \sum_{\eta=2}^{\infty} d_\eta|a_\eta| \le 1 \tag{5.6}$$

by using the hypothesis (5.2). By setting

$$g_1(z)=d_{q+1}\left[\frac{u(z)}{u_q(z)}-\left(1-\frac{1}{d_{q+1}}\right)\right]$$

$$=1+\frac{d_{q+1}\sum_{\eta=q+1}^{\infty}a_\eta z^{\eta-1}}{1+\sum_{\eta=2}^{q}a_\eta z^{\eta-1}} \tag{5.7}$$

and applying (5.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{q+1}\sum\limits_{\eta=q+1}^{\infty}|a_\eta|}{2-2\sum\limits_{\eta=2}^{q}|a_\eta|-d_{q+1}\sum\limits_{\eta=q+1}^{\infty}|a_\eta|} \leq 1 \tag{5.8}$$

which readily yields the assertion (5.3) of Theorem 5.1. In order to see that

$$u(z)=z+\frac{z^{q+1}}{d_{q+1}} \text{ gives sharp result, we observe that for } z=re^{\frac{i\pi}{q}} \text{ that} \tag{5.9}$$

$$\frac{u(z)}{u_q(z)}=1+\frac{z^q}{d_{q+1}} \;\rightarrow\; 1-\frac{1}{d_{q+1}} \text{ as } z\rightarrow 1^-.$$

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1+d_{q+1})\left(\frac{u_q(z)}{u(z)}-\frac{d_{q+1}}{1+d_{q+1}}\right) \\ &= 1-\frac{(1+d_{\eta+1})\sum\limits_{\eta=q+1}^{\infty}a_\eta z^{\eta-1}}{1+\sum\limits_{\eta=2}^{\infty}a_\eta z^{\eta-1}} \end{aligned} \tag{5.10}$$

and making use of (5.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{q+1})\sum\limits_{\eta=q+1}^{\infty}|a_\eta|}{2-2\sum\limits_{\eta=2}^{q}|a_\eta|-(1-d_{q+1})\sum\limits_{\eta=q+1}^{\infty}|a_\eta|}$$

□

which leads is immediately to the assertion (5.4) of Theorem 5.1.

The bound in (5.4) is sharp for each $q\in\mathbb{N}$ with the external function $u(z)$ given by (5.9). The proof of the Theorem 5.1 is thus complete.

Theorem 5.2. *If $u(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left[\frac{u'(z)}{u_q'(z)}\right] \geq 1-\frac{q+1}{d_{q+1}}. \tag{5.11}$$

Proof. By setting

$$g(z) = d_{q+1}\left[\frac{u'(z)}{u'_q(z)}\right] - \left(1 - \frac{q+1}{d_{q+1}}\right)$$

$$= \frac{1 + \frac{d_{q+1}}{q+1}\sum\limits_{\eta=q+1}^{\infty} \eta a_\eta z^{\eta-1} + \sum\limits_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}}{1 + \sum\limits_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}}$$

$$= 1 + \frac{\frac{d_{q+1}}{q+1}\sum\limits_{\eta=q+1}^{\infty} \eta a_\eta z^{\eta-1}}{1 + \sum\limits_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1}}.$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{q+1}}{q+1}\sum\limits_{\eta=q+1}^{\infty} \eta|a_\eta|}{2 - 2\sum\limits_{\eta=2}^{q} \eta|a_\eta| - \frac{d_{q+1}}{q+1}\sum\limits_{\eta=q+1}^{\infty} \eta|a_\eta|}. \tag{5.12}$$

$$\text{Now } \left|\frac{g(z)-1}{g(z)+1}\right| \leq 1 \text{ if } \sum_{\eta=2}^{q} \eta|a_\eta| + \frac{d_{q+1}}{q+1}\sum_{\eta=q+1}^{\infty} \eta|a_\eta| \leq 1. \tag{5.13}$$

Since the left hand side of(5.13)is bounded above by $\sum\limits_{\eta=2}^{q} d_\eta|a_\eta|$ if

$$\sum_{\eta=2}^{q}(d_\eta - \eta)|a_\eta| + \sum_{\eta=q+1}^{\infty} d_\eta - \frac{d_{q+1}}{q+1}\eta|a_\eta| \geq 0 \tag{5.14}$$

and the proof is complete.
The result is sharp for the extremal function $u(z) = z + \frac{z^{q+1}}{d_{q+1}}$. □

Theorem 5.3. *If $u(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left[\frac{u'_q(z)}{u'(z)}\right] \geq \frac{d_{q+1}}{q+1+d_{q+1}}. \tag{5.15}$$

Proof. By setting

$$g(z) = [q+1+d_{q+1}]\left[\frac{u'_q(z)}{u'(z)} - \frac{d_{q+1}}{q+1+d_{q+1}}\right] = 1 - \frac{\left(1 + \frac{d_{q+1}}{q+1}\right)\sum\limits_{\eta=q+1}^{\infty} \eta a_\eta z^{\eta-1}}{1 + \sum\limits_{\eta=2}^{q} \eta a_\eta z^{\eta-1}}$$

and making use of (5.14), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1 + \frac{d_{q+1}}{q+1}\right)\sum\limits_{\eta=q+1}^{\infty} \eta|a_\eta|}{2 - 2\sum\limits_{\eta=2}^{q} \eta|a_\eta| - \left(1 + \frac{d_{q+1}}{q+1}\right)\sum\limits_{\eta=q+1}^{\infty} \eta|a_\eta|} \leq 1$$

which leads us immediately to the assertion of the Theorem 5.3. □

6. Neighbouhood for the class $S^{\xi}(\upsilon, \varrho, \mu, s)$

In this section, we determine the neighbourhoods for the class $S^{\xi}(\upsilon, \varrho, \mu, s)$ which we define as follows:

Definition 6.1. A function $u \in A$ is said to be in the class $S^{\xi}(\upsilon, \varrho, \mu, s)$ if there exist a function $g \in S(\upsilon, \varrho, \mu, s)$ such that

$$\left|\frac{u(z)}{g(z)} - 1\right| < 1 - \upsilon, \;\; (z \in E, 0 \leq \upsilon < 1). \tag{6.1}$$

For any function $u(z) \in A, z \in E$ and $\delta \geq 0$, we define

$$N_{\eta,\delta}(u) = \left\{g \in \Sigma : g(z) = z + \sum_{\eta=2}^{\infty} b_\eta z^\eta \text{ and } \sum_{\eta=2}^{\infty} \eta|a_\eta - b_\eta| \leq \delta\right\} \tag{6.2}$$

which is the $(\eta, \delta)-$neighbourhood of $u(z)$.

The concept of neighbourhoods was first introduced by Goodman [3] and generalized by Ruscheweyh [10].

Theorem 6.2. *If $g \in S(\upsilon, \varrho, \mu, s)$ and*

$$\xi = 1 - \frac{\delta(1-\upsilon)}{2[(1-\upsilon) - (2+\varrho-\upsilon)\phi(\mu, s, 2)]} \tag{6.3}$$

then $N_{\eta,\delta}(g) \subset S^{\xi}(\upsilon, \varrho, \mu, s)$.

Proof. Suppose $u \in N_{\eta,\delta}(g)$. We then find from (6.2) that

$$\sum_{\eta=2}^{\infty} n|a_\eta - b_\eta| \leq \delta \tag{6.4}$$

which yields the coefficient inequality

$$\sum_{\eta=2}^{\infty} |a_\eta - b_\eta| \leq \frac{\delta}{2}, \;\; (\eta \in \mathbb{N}). \tag{6.5}$$

Next, since $g \in S(\upsilon, \varrho, \mu, s)$, we have $\sum\limits_{\eta=2}^{\infty} b_\eta \leq \frac{(2+\varrho-\upsilon)\phi(\mu,s,2)}{1-\upsilon}$. So that

$$\begin{aligned}\left|\frac{u(z)}{g(z)} - 1\right| &< \frac{\sum\limits_{\eta=2}^{\infty} |a_\eta - b_\eta|}{1 - \sum\limits_{\eta=2}^{\infty} b_\eta} = \frac{\delta(1-\upsilon)}{2[(1-\upsilon) - (2+\varrho-\upsilon)\phi(\mu, s, 2)]} \\ &= 1 - \xi\end{aligned}$$

provided ξ is given by (6.3). Thus the proof of the is completed. □

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