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WRAPPED GENERALIZED SKEW NORMAL DISTRIBUTION

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Abstract. In this paper we develop a wrapped form of the generalized skew normal distribution of Kumar and Anusree (*Statist.Prob.Lett.,* 2011) and investigate some of its important properties through deriving expressions for its characteristic function, trigonometric moments, mean direction, mean resultant length, circular variance, measures of skewness and kurtosis, etc.

1. Introduction

The subject "Directional Statistics" can be viewed as a branch of statistics that deals with directions. There are several practical situations where observations are represented in the form of directions. For example, measuring daily wind directions, departure directions of animals, ocean current directions, orientation of bird flights, readings of a compass etc. These observations are also known in the literature as "circular data", since they can be viewed as points on a circle. A point on the circle can be represented as an angle with a chosen initial direction and an orientation. Since the magnitude has no importance we consider the observations as points on the circumference of a unit circle. To describe the circular data, the usual procedures adopted in linear statistics are not appropriate and as such specialized statistical techniques under the title "circular statistics" are quite relevant. In this context, each point can be represented either by an angle θ or by a unit vector (x, y) or by a complex number $z = x + iy$.

The angle θ is measured in terms of radians so that the range can be $[0, 2\pi)$ or $[-\pi, \pi)$. Considering zero as the initial direction, it can be seen that the angle varies from 0 to 2π. Thus θ is said to be a circular variable, which is periodic. That is, θ repeats its value at $\theta \pm 2\pi$, $\theta \pm 4\pi$, $\theta \pm 6\pi$, $\cdots$. The probability distribution of such a random variable, whose total probability is concentrated on the circumference of a unit circle is called "circular distribution". Wrapping a linear distribution around a circle of unit radius is a method of generating circular model of a known distribution. If X is a continuous random variable with probability density function (p.d.f.) $f_X(x)$ and θ is the corresponding circular variable, then we define $\theta = \mathrm{X}(\text{mod } 2\pi)$ and the p.d.f. is given by

$$f_W(\theta) = \sum_{k=-\infty}^{\infty} f_X(\theta + 2\pi k) \tag{1.1}$$

for $\theta \in [0, 2\pi)$.

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The idea of wrapping a model for linear data around the unit circle so as to produce a model for circular data has a long history; the wrapped normal distribution dates back to eighteenth century. See [2] and [4], to know more about wrapped normal distribution. The wrapped skew normal distribution on the circle was introduced as a wrapped form of the skew normal distribution through the following p.d.f.

$$f_W(\theta;\mu,\sigma,\lambda)=\frac{2}{\sigma}\sum_{k=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\Phi\left(\lambda\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right), \qquad (1.2)$$

for $\theta\in[0,2\pi)$, $\mu\in[0,2\pi)$, $\sigma>0$ and $\lambda\in\mathbb{R}$, which reduces to wrapped normal distribution when $\lambda=0$. Here $\phi(.)$ is the p.d.f. and $\Phi(.)$ is the cumulative distribution function (c.d.f.) of standard normal distribution. Refer [1] and [5].

Normal and skew normal models are better for modelling a good symmetry and moderate skewness. But when there is a data with plurimodality, both the normal and skew normal are not appropriate. As in [3], a generalized skew normal distribution through the following p.d.f. $f_X(x)$ deal such situations of plurimodality in the assymetric data set, for $x\in\mathbb{R}=(-\infty,\infty),\mu\in\mathbb{R},\sigma>0,\lambda\in\mathbb{R}$ and $\alpha\geq-1$.

$$f_X(x)=\frac{2}{\sigma(\alpha+2)}\phi\left(\frac{x-\mu}{\sigma}\right)\left[1+\alpha\Phi\left(\lambda\left(\frac{x-\mu}{\sigma}\right)\right)\right]. \qquad (1.3)$$

In this paper our intention is to develop a wrapped version of the generalized skew normal distribution. In Section 2 we present the definition and discuss some important properties of the proposed class of distribution.

2. Main results

Here we present the definition of wrapped generalized skew normal distribution and discuss its important properties.

Definition 2.1. An angular random variable θ is said to follow wrapped generalized skew normal distribution with parameters μ, σ, λ and α if its probability density function is of the following form.

$$f_W(\theta;\mu,\sigma,\lambda,\alpha)=\frac{2}{\sigma(\alpha+2)}\sum_{k=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\left[1+\alpha\Phi\left(\lambda\left(\frac{\theta+2\pi k-\mu}{\sigma}\right)\right)\right], \qquad (2.1)$$

for $\theta\in[0,2\pi)$, $\mu\in[0,2\pi)$, $\sigma>0$, $\lambda\in\mathbb{R}$ and $\alpha\geq-1$.

We denote the distribution as WGSND$(\mu,\sigma,\lambda,\alpha)$. It is clear that (2.1) satisfies the properties of a probability density function, given by

1. $f_W(\theta)\geq 0$
2. $\int_0^{2\pi} f_W(\theta)d\theta=1$
3. $f_W(\theta)=f_W(\theta+2\pi k)$, for any integer k.

When $\alpha = -1$, then WGSND$(\mu, \sigma, \lambda, \alpha)$ becomes wrapped skew normal distribution with parameters μ, σ, and $-\lambda$. WGSND$(\mu, \sigma, 0, 0)$ is wrapped normal distribution and wrapped half normal distribution if $\lambda = -\infty$ and $\alpha = -1$. The cumulative distribution function is

$$\begin{aligned} F_W(\theta) &= \int_0^\theta \frac{2}{\sigma(\alpha+2)} \sum_{k=-\infty}^{\infty} \phi\left(\frac{\epsilon + 2\pi k - \mu}{\sigma}\right)\left[1 + \alpha\Phi\left(\lambda\left(\frac{\epsilon + 2\pi k - \mu}{\sigma}\right)\right)\right] d\epsilon \\ &= \frac{2}{\alpha+2}\int_0^\theta \sum_{k=-\infty}^{\infty} \frac{1}{\sigma}\phi\left(\frac{\epsilon + 2\pi k - \mu}{\sigma}\right) d\epsilon \\ &\quad + \frac{\alpha}{\alpha+2}\int_0^\theta \sum_{k=-\infty}^{\infty} \frac{2}{\sigma}\phi\left(\frac{\epsilon + 2\pi k - \mu}{\sigma}\right)\Phi\left(\lambda\left(\frac{\epsilon + 2\pi k - \mu}{\sigma}\right)\right) d\epsilon \\ &= \frac{1}{2+\alpha}\left(2F_1(\theta) + \alpha F_2(\theta)\right), \end{aligned}$$

where $F_1(\theta)$ and $F_2(\theta)$ are the cumulative distribution functions of wrapped normal and wrapped skew normal distributions respectively.

In order to obtain Result 2.1, we need the following lemma.

Lemma 2.2. *For any $d \in \mathbb{R}$, $G(d) = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{d^{2n+1}}{2^n n!(2n+1)}$ is convergent.*

Proof. Let A_n be the n^{th} term of the series. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{A_{n+1}}{A_n} &= \lim_{n\to\infty} \frac{d^{2n+3}}{2^{n+1}(n+1)!(2n+3)} \frac{2^n n!(2n+1)}{d^{2n+1}} \\ &= \lim_{n\to\infty} \frac{d^2(2n+1)}{2(n+1)(2n+3)} \\ &= d^2 \lim_{n\to\infty} \frac{(2+\frac{1}{n})}{2(n+1)(2+\frac{3}{n})} \\ &= 0. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{A_{n+1}}{A_n} < 1$, by ratio test $G(d)$ is convergent. □

Now we obtain the characteristic function of WGSND$(\mu, \sigma, \lambda, \alpha)$ through the following result.

Result 2.1. *For any integer p and $i = \sqrt{-1}$, the characteristic function of WGSND$(\mu, \sigma, \lambda, \alpha)$ is given by*

$$\psi(p) = e^{i\mu p - \frac{p^2\sigma^2}{2}}\left(1 + i\frac{\alpha}{\alpha+2}G(\delta\sigma p)\right), \tag{2.2}$$

where $\delta = \lambda\left(1+\lambda^2\right)^{\frac{1}{2}}$.

Proof. The characteristic function of the generalized skew normal distribution with p.d.f. as given in (1.3) is the following, for $t \in \mathbb{R}$ and $i = \sqrt{-1}$.

$$h_X(t) = \frac{2}{\alpha+2} e^{i\mu t - \frac{t^2\sigma^2}{2}}\left(1 + \alpha\Phi(i\delta\sigma t)\right).$$

Now by definition of the characteristic function of circular distributions, we obtain the following for any integer p.

$$\begin{aligned}\psi(p) &= \frac{2}{\alpha+2}e^{i\mu p-\frac{p^2\sigma^2}{2}}\left(1+\alpha\Phi(i\delta\sigma p)\right)\\ &= \frac{2}{\alpha+2}e^{i\mu p-\frac{p^2\sigma^2}{2}}\left(1+\alpha\int_{-\infty}^{i\delta\sigma p}\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}dt\right)\\ &= \frac{2}{\alpha+2}e^{i\mu p-\frac{p^2\sigma^2}{2}}\left(1+\alpha\left(\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}dt+\int_{0}^{i\delta\sigma p}\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}dt\right)\right).\end{aligned}$$

Note that the first integral reduces to $\frac{1}{2}$. On substituting $u = it$, the second integral becomes

$$\begin{aligned}\int_{0}^{i\delta\sigma p}\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}dt &= -i\int_{0}^{-\delta\sigma p}\frac{1}{\sqrt{2\pi}}e^{\frac{u^2}{2}}du\\ &= i\int_{-\delta\sigma p}^{0}\frac{1}{\sqrt{2\pi}}e^{\frac{u^2}{2}}du\\ &= \frac{i}{\sqrt{2\pi}}\int_{0}^{\delta\sigma p}e^{\frac{u^2}{2}}du.\end{aligned}$$

Thus $\psi(p)$ reduces to

$$\begin{aligned}\psi(p) &= \frac{2}{\alpha+2}e^{i\mu p-\frac{p^2\sigma^2}{2}}\left(1+\frac{\alpha}{2}+\frac{i\alpha}{\sqrt{2\pi}}\int_{0}^{\delta\sigma p}e^{\frac{u^2}{2}}du\right)\\ &= e^{i\mu p-\frac{p^2\sigma^2}{2}}\left(1+i\frac{\alpha}{\alpha+2}\sqrt{\frac{2}{\pi}}\int_{0}^{\delta\sigma p}e^{\frac{t^2}{2}}dt\right). \qquad (2.3)\end{aligned}$$

On expanding $e^{\frac{t^2}{2}}$ under the integral sign and integrating (2.3), it reduces to (2.2) in the light of Lemma 2.2. □

Expanding (2.2) and expressing in the form $\psi(p) = \xi_p + i\beta_p$, we obtain the p^{th} trigonometric moments ξ_p and β_p as shown in the following result.

Result 2.2. *For any integer p, the p^{th} trigonometric moments of WGSND(μ, σ,λ,α) are expressed as*

$$\xi_p = e^{-\frac{p^2\sigma^2}{2}}\left(\cos\mu p-\frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin\mu p\right) \qquad (2.4)$$

and

$$\beta_p = e^{-\frac{p^2\sigma^2}{2}}\left(\sin\mu p+\frac{\alpha}{\alpha+2}G(\delta\sigma p)\cos\mu p\right). \qquad (2.5)$$

As a consequence of Result 2.2 we have the following result which gives an approximation to the p.d.f. of WGSND($\mu,\sigma,\lambda,\alpha$). The proof follows from the representation of the p.d.f. given by

$$f_W(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty}(\xi_p \cos p\theta + \beta_p \sin p\theta)\right).$$

Result 2.3. *The p.d.f. of WGSND($\mu, \sigma, \lambda, \alpha$) can be expressed in the following form.*

$$f_W(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} e^{-\frac{p^2\sigma^2}{2}}\left[\left(\cos\mu p - \frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin\mu p\right)\cos p\theta \right.\right.$$
$$\left.\left. + \left(\sin\mu p + \frac{\alpha}{\alpha+2}G(\delta\sigma p)\cos\mu p\right)\sin p\theta\right]\right).$$

Using $\mu_p = \arctan\left(\frac{\beta_p}{\xi_p}\right)$, we obtain the mean direction $\nu = \mu_1$ as in Result 2.4.

Result 2.4. *For WGSND($\mu, \sigma, \lambda, \alpha$), the mean direction is*

$$\nu = \arctan\left(\frac{\sin\mu + \frac{\alpha}{\alpha+2}G(\delta\sigma)\cos\mu}{\cos\mu - \frac{\alpha}{\alpha+2}G(\delta\sigma)\sin\mu}\right). \tag{2.6}$$

Now using the expression $\rho_p = \sqrt{\xi_p^2 + \beta_p^2}$, the mean resulatant length $\tau = \rho_1$ is obtained and is expressed in the following result. Results 2.6 and 2.7 are obtained from (2.7) using $V_0 = 1 - \tau$ and $\kappa_0 = \sqrt{-2\log\tau}$.

Result 2.5. *The mean resultant length of wrapped generalized skew normal distribution is*

$$\tau = \eta e^{-\frac{\sigma^2}{2}}, \tag{2.7}$$

where $\eta = \sqrt{1 + \left(\frac{\alpha}{\alpha+2}G(\delta\sigma)\right)^2}$.

Result 2.6. *The circular variance of WGSND($\mu, \sigma, \lambda, \alpha$) is obtained as*

$$V_0 = 1 - \eta e^{-\frac{\sigma^2}{2}}. \tag{2.8}$$

Result 2.7. *For WGSND($\mu, \sigma, \lambda, \alpha$), the circular standard deviation is given by*

$$\kappa_0 = \sqrt{\sigma^2 - 2\log\eta}. \tag{2.9}$$

Now we obtain the p^{th} central trigonometric moments in the following result and thus we obtain the measures of circular skewness and kurtosis in Results 2.9 and 2.10, respectively. The central trigonometric moments $\bar{\xi_p}$ and $\bar{\beta_p}$ are derived using the equations

$$\bar{\xi_p} = \xi_p \cos p\mu + \beta_p \sin p\mu$$

and

$$\bar{\beta_p} = \beta_p \cos p\mu - \xi_p \sin p\mu.$$

Result 2.8. *The p^{th} central trigonometric moments of WGSND($\mu,\sigma,\lambda,\alpha$) are obatined as following, for $p = 0, \pm1, \pm2, \ldots$.*

$$\bar{\xi_p} = e^{-\frac{p^2\sigma^2}{2}}\left[\cos p\omega - \frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos p\omega\right] \tag{2.10}$$

and

$$\bar{\beta_p} = e^{-\frac{p^2\sigma^2}{2}}\left[\sin p(2\mu-\omega) + \frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos p(2\mu-\omega)\right], \tag{2.11}$$

where $\omega = \left(\mu - \arctan\frac{a}{b}\right)$, *in which*

$$a = \sin\mu + \frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos\mu$$

and

$$b = \cos\mu - \frac{\alpha}{\alpha+2}G(p\delta\sigma)\sin\mu.$$

Proof. By definition, we have

$$\begin{aligned}
\bar{\xi_p} &= e^{-\frac{p^2\sigma^2}{2}}\left(\cos\mu p - \frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin\mu p\right)\cos\left(p\arctan\frac{a}{b}\right) \\
&\quad + e^{-\frac{p^2\sigma^2}{2}}\left(\sin\mu p + \frac{\alpha}{\alpha+2}G(\delta\sigma p)\cos\mu p\right)\sin\left(p\arctan\frac{a}{b}\right) \\
&= e^{-\frac{p^2\sigma^2}{2}}\Bigg(\cos(\mu p)\cos\left(p\arctan\frac{a}{b}\right) - \frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin(\mu p)\cos\left(p\arctan\frac{a}{b}\right) \\
&\quad + \sin(\mu p)\sin\left(p\arctan\frac{a}{b}\right) + \frac{\alpha}{\alpha+2}G(\delta\sigma p)\cos(\mu p)\sin\left(p\arctan\frac{a}{b}\right)\Bigg) \\
&= e^{-\frac{p^2\sigma^2}{2}}\left(\cos\left(\mu p - p\arctan\frac{a}{b}\right) - \frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin\left(\mu p - p\arctan\frac{a}{b}\right)\right) \\
&= e^{-\frac{p^2\sigma^2}{2}}\left(\cos p\omega - \frac{\alpha}{\alpha+2}G(\delta\sigma p)\sin p\omega\right),\ \omega = \left(\mu - \arctan\frac{a}{b}\right).
\end{aligned}$$

Similarly we obtain

$$\bar{\beta_p} = e^{-\frac{p^2\sigma^2}{2}}\left(\sin p(2\mu-\omega) + \frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos p(2\mu-\omega)\right).$$

□

Result 2.9. *The measure of circular skewness γ_1 of WGSND($\mu,\sigma,\lambda,\alpha$) is given by*

$$\gamma_1 = \frac{e^{-\frac{p^2\sigma^2}{2}}\left(\sin p(2\mu-\omega) + \frac{\alpha}{\alpha+2}G(p\delta\sigma)\cos p(2\mu-\omega)\right)}{\left(1-\eta e^{-\frac{\sigma^2}{2}}\right)^{3/2}}. \tag{2.12}$$

The result is obvious from the equation of the measure of circular skewness given by

$$\gamma_1 = \frac{\bar{\beta_2}}{V_0^{3/2}}.$$

Now using the equation

$$\gamma_2 = \frac{\bar{\xi}_2 - \tau^4}{V_0^2},$$

the measure of circular kurtosis is obtained as given in the following result.

Result 2.10. *The measure of kurtosis of WGSND($\mu, \sigma, \lambda, \alpha$) is*

$$\gamma_2 = \frac{e^{-\frac{p^2\sigma^2}{2}}\left(\cos p\omega - \dfrac{\alpha}{\alpha+2}G(\delta\sigma p)\sin p\omega\right) - \eta^4 e^{-2\sigma^2}}{\left(1-\eta e^{-\frac{\sigma^2}{2}}\right)^2}. \tag{2.13}$$

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