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Application of the Taylor Series Method for the Flow of non-Newtonian Fluids between Parallel Plates

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Abstract: This paper applies the Taylor series method to solve the one-dimensional steady laminar flow of a third grade fluid and an Oldroyd six constant fluid between two parallel plates. The fluid flow is produced by an external pressure gradient *dp/dx*. In each case the governing nonlinear boundary value problem is solved and analytical expressions for the fluid velocity, resistance to flow, volume flow rate and the average fluid velocity are obtained. Figures and tables are presented to illustrate the variation of these quantities with the relevant physical parameters. It is shown that in case of a third grade fluid the fluid velocity and other flow variables increase on decreasing the pressure gradient *dp/dx* or by increasing the non-Newtonian parameter β. For an Oldroyd six constant fluid the velocity magnitude increases on decreasing the pressure gradient or on increasing the constant α_1 when $\alpha_2(< \alpha_1)$ and *dp/dx* are fixed. Also the fluid velocity increases on increasing the constant α_2 when $\alpha_1(< \alpha_2)$ and dp / dx are fixed. Similarly, under the same condition, the resistance to the fluid flow and the volume flow rate increase on increasing α_1 or α_2.

Keywords: Third grade fluids, Oldroyd six constant fluids, Parallel plates, Taylor series method.

1. INTRODUCTION

The importance of non-Newtonian fluids is their wide ranging industrial and technological use. Gels, paints, solid-liquid mixtures, foods such as tomato sauce, biological fluids such as blood and synovial fluid found in natural healthy joints are some common examples of such fluids. In this paper we consider one-dimensional steady laminar flow of two non-Newtonian fluid models, namely third grade and Oldroyd six constant fluids, between two stationary parallel plates. The motion is produced by the presence of a constant pressure gradient *dp/dx*.

This problem arises in many practical applications and in particular we quote the underfill flow between parallel plates and solder bumps in flip-chip interconnect system. The flip-chip technology is attractive in electronic packaging because of its high electrical performance, high interconnect density and small size. Underfill is used to improve the reliability of the flip-chip interconnect system and the polymer is filler in the gap between the chip and substrate around the solder joints by the capillary flow. The underfill flow is driven by the pressure difference due to the capillary force. Therefore, the flow of the underfill material can be modelled as steady, laminar flow between parallel plates driven by a pressure gradient. For further details the reader is referred to [13, 17] and the references therein.

The resulting ordinary differential equations governing such a flow are nonlinear and their closed form solution is not obtainable. Consequently, one must resort to alternative forms to express the solution. One

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possible way of representing the solution is by infinite series. The methods using this approach have been receiving great attention over the past few years for studying such non Newtonian fluid flow problems. Amongst these are the methods based on the concept of homotopy as used in topology [4, 5, 6, 7, 10, 15, 16] and the Adomian decomposition method [2, 8, 12]. In addition to these techniques there are traditional perturbation methods [3, 11], which depend upon the existence of a small or large parameter in the problem. These assumptions of a small or large parameter seriously restrict the validity of the ordinary perturbation solutions.

Here we apply the Taylor series method for solving the nonlinear boundary value problems that arise in the considered non-Newtonian fluid flow between two parallel plates. This method [1, 9, 14] expands the solution variable about a point and does not assume the existence of any small or large parameter in the problem. To the best of our knowledge this method has not been previously used in the study of non-Newtonian fluid flows. Therefore, the aim of this communication is to demonstrate the applicability of the Taylor series method for solving the nonlinear boundary value problems that may arise in non-Newtonian fluid mechanics.

With this motivation, we start in section 2 by listing the basic equations, and the constitutive equations for third grade and Oldroyd six constant fluids. In section 3 we give a mathematical formulation of the boundary value problem for both the fluid models. In section 4 we solve these problems to find the fluid velocity, resistance to the flow, volume flow rate and the average fluid velocity. Section 5 contains results and discussion. Finally section 6 consists of concluding remarks.

2. BASIC EQUATIONS

The basic equations governing the flow of an incompressible fluid are the continuity and the momentum equations:

$$\nabla . \, \mathbf{u} = 0 \tag{1}$$

$$\nabla . \, \mathbf{T} = \rho \frac{d\mathbf{u}}{dt} + \rho \mathbf{f} \tag{2}$$

where $\boldsymbol{u}$ is the fluid velocity at time t, ρ the constant fluid density, $\boldsymbol{f}$ the body force per unit mass, d/dt the material derivative and $\boldsymbol{T}$ the Cauchy stress tensor given by a constitutive equation.

A number of constitutive equations relating the stress tensor to the rate of strain tensor have been proposed for explaining the flow behavior of non-Newtonian fluids. In view of the problems to be considered, we mention the constitutive equations for third grade fluids and Oldroyd 6-constant fluids.

2.1 Third Grade Fluid

If p denotes the fluid pressure, η the coefficient of viscosity and α, α^*, β_1, β_2, β_3 the material constants of the fluid. Then the constitutive equation for a third grade fluid is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \; \boldsymbol{S} = \sum_{i=1}^{3} S_i \tag{3}$$

where $\boldsymbol{S}_1 = \eta A_1$, $\boldsymbol{S}_2 = \alpha \boldsymbol{A}_2 + \alpha^* \boldsymbol{A}_1^2$, $\boldsymbol{S}_3 = \beta_1 \boldsymbol{A}_3 + \beta_2(\boldsymbol{A}_2\boldsymbol{A}_1 + \boldsymbol{A}_1\mathbf{A}_2) + \beta_3 tr(\boldsymbol{A}_2)\boldsymbol{A}_1$, $A_1 = \boldsymbol{L}^t + \boldsymbol{L}$, $\boldsymbol{A}_2 = d\boldsymbol{A}_1/dt + \boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^t\boldsymbol{A}_1$, $\boldsymbol{A}_3 = d\boldsymbol{A}_2/dt + \boldsymbol{A}_2 L + \boldsymbol{L}^t\boldsymbol{A}_2$, $\boldsymbol{L} = \text{grad}\, \boldsymbol{u}$ and $\boldsymbol{L}^t$ is transpose of $\boldsymbol{L}$.

2.2 Oldroyd Six Constant Fluid

If the unit tensor, the indeterminate part of the stress and the extra stress tensor are denoted by I, p and $\boldsymbol{S}$, respectively, then the constitutive equation for an Oldroyd six constant fluid is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S} \tag{4}$$

The extra stress tensor $\boldsymbol{S}$ satisfies the equation

$$\boldsymbol{S} + \lambda_1 \frac{D\boldsymbol{S}}{Dt} + \frac{\lambda_3}{2}(\boldsymbol{S}\boldsymbol{A}_1 + \boldsymbol{A}_1\boldsymbol{S}) + \frac{\lambda_5}{2}[tr(\boldsymbol{S})]\boldsymbol{A}_1 = \eta(\boldsymbol{A}_1 + \lambda_2 \frac{D\boldsymbol{A}_1}{Dt} + \lambda_4 \boldsymbol{A}_1^2), \tag{5}$$

where $D/Dt = d/dt - \boldsymbol{L} - \boldsymbol{L}^T$, $\boldsymbol{A}_1$ is defined in subsection 2.1 and λ_1, λ_2, λ_3, λ_4 and λ_5 are the material constants.

3. STATEMENT OF PROBLEM

3.1 Third Grade Fluid

We consider the steady laminar flow of a third grade fluid between two stationary parallel plates produced by a constant pressure gradient. The origin of the Cartesian coordinate system is taken to be on the plane of symmetry with the x-axis being in the direction of motion, see figure 1. If the distance between the plates is assumed to be $2h$ then the location of the upper and lower plates are $y = +h$ and $y = -h$, respectively, and the boundary conditions are given by

$$\begin{aligned} S_{xy} &= 0 \quad \text{at} \quad y = 0 \quad \text{(symmetry)} \\ u &= 0 \quad \text{at} \quad y = h \quad \text{(no slip)} \end{aligned} \tag{6}$$

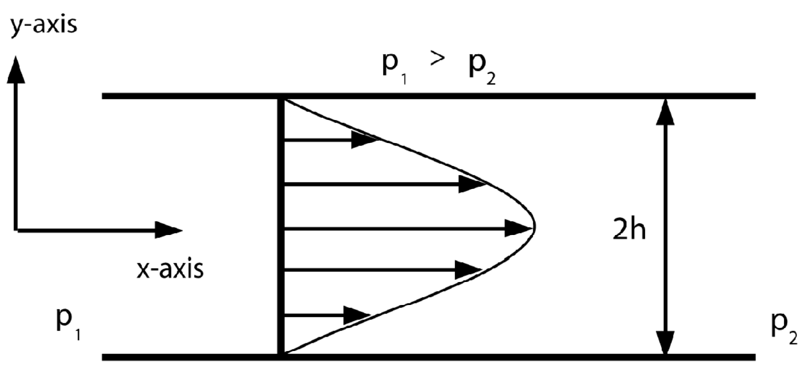

Figure 1: Pressure Driven Flow Between Parallel Plates

Moreover, we assume that $\boldsymbol{u} = [u, 0, 0]$, $u = u(y)$ and $\boldsymbol{S} = \boldsymbol{S}(y)$. Thus the continuity equation is satisfied identically and, in the absence of body forces, the momentum equation (2) reduces to a second-order nonlinear ordinary differential equation, which after integrating and applying the condition of symmetry becomes

$$\frac{du}{dy} + \frac{2\beta}{\eta}\left(\frac{du}{dy}\right)^3 - \frac{y}{\eta}\frac{dp}{dx} = 0 \tag{7}$$

where $\beta = \beta_2 + \beta_3$. Therefore, we need to solve equation (7) subject to the (no-slip) condition $u(h) = 0$. The problem is non-dimensionlized by introducing the following dimensionless variables

$$y^* = \frac{y}{h},\ x^* = \frac{x}{h},\ u^* = \frac{u}{U}, \tag{8}$$

$$\beta^* = \frac{\beta}{\eta(h/U)^2}, \; p^* = \frac{p}{\eta U/h} \tag{9}$$

Thus, dropping the asterisks for convenience, we obtain the following dimensionless boundary value problem

$$\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 - y\left(\frac{dp}{dx}\right) = 0, \qquad u(1) = 0 \tag{10}$$

3.2 Oldroyd Six Constant Fluid

In this problem we consider the steady laminar flow of an Oldroyd six constant fluid between two stationary parallel plates produced by a constant pressure gradient. All other conditions and assumptions as adopted in the previous problem remaining unchanged. Therefore, in the absence of body forces, the momentum equations (2) along with the no-slip and the symmetry conditions (6) generate the following boundary value problem:

$$\frac{du}{dy} + \alpha_1\left(\frac{du}{dy}\right)^3 - \frac{1}{\eta}\left\{\alpha_2\left(\frac{du}{dy}\right)^2 + 1\right\}\left(\frac{dp}{dx}\right)y = 0, \qquad u(h) = 0 \tag{11}$$

where the constants α_1 and α_2 are given by the following expressions

$$\alpha_1 = \lambda_1\lambda_4 - (\lambda_4 - \lambda_2)(\lambda_3 + \lambda_5) \tag{12}$$

$$\alpha_2 = \lambda_1\lambda_3 - (\lambda_3 - \lambda_1)(\lambda_3 + \lambda_5) \tag{13}$$

We now introduce the non-dimensional variables defined in equations (8) and (9), along with $\alpha_1^* = \frac{\alpha_1}{(h/U)^2}$ and $\alpha_2^* = \frac{\alpha_2}{(h/U)^2}$, into equation (11). Thus the non-dimensional form of the boundary value problem (11), after dropping the asterisk, is given by

$$\frac{du}{dy} + \alpha_1\left(\frac{du}{dy}\right)^3 - \left\{\alpha_2\left(\frac{du}{dy}\right)^2 + 1\right\}\left(\frac{dp}{dx}\right)y = 0, \qquad u(1) = 0 \tag{14}$$

4. SOLUTION BY THE TAYLOR SERIES METHOD

For solving problems (10) and (14) using the Taylor series method we assume a series solution of the form

$$u(y) = \sum_{n=0}^{\infty} a_n y^n \tag{15}$$

4.1 Third Grade Fluid

4.1.1 Fluid Velocity

Substituting the expression (15) into the differential equation (10) yields

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}y^k + 2\beta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{k-j}\left\{(j+1)(m+1)(k-m-j+1)a_{j+1}a_{m+1}a_{k-m-j+1}\right\}y^k = y\left(\frac{dp}{dx}\right)$$

Now equating the coefficients of y^0 and y^1 we obtain $a_1 = 0$, $a_2 = (1/2)(dp/dx)$ and the following recursive relationship

$$a_{k+1} = -\frac{2\beta}{k+1}\sum_{j=1}^{k-1}\sum_{m=0}^{k-j}(j+1)(m+1)(k-j-m+1)a_{j+1}a_{m+1}a_{k-j-m+1} \tag{16}$$

where $k = 2, 3, 4, \dots$. The first few coefficients determined from (16) are given by

$$a_3 = a_5 = a_7 = a_9 = \dots = 0 \tag{17}$$

$$a_4 = -\frac{2\beta}{4}\left(\frac{dp}{dx}\right)^3, a_6 = \frac{3(2\beta)^2}{6}\left(\frac{dp}{dx}\right)^5 \tag{18}$$

$$a_8 = -\frac{12(2\beta)^3}{8}\left(\frac{dp}{dx}\right)^7, a_{10} = \frac{55(2\beta)^4}{10}\left(\frac{dp}{dx}\right)^9 \tag{19}$$

Substituting the expression (17)-(19) into equation (15) and using $u(1) = 0$, we obtain

$$\begin{aligned} u(y) = {} & \left(\frac{dp}{dx}\right)\left\{\frac{y^2-1}{2}\right\} - 2\beta\left(\frac{dp}{dx}\right)^3\left\{\frac{y^4-1}{4}\right\} + 3(2\beta)^2\left(\frac{dp}{dx}\right)^5\left\{\frac{y^6-1}{6}\right\} \\ & - 12\,(2\beta)^3\left(\frac{dp}{dx}\right)^7\left\{\frac{y^8-1}{8}\right\} + 55(2\beta)^4\left(\frac{dp}{dx}\right)^9\left\{\frac{y^{10}-1}{10}\right\} + \cdots \end{aligned} \tag{20}$$

We remark that for $\beta = 0$ this solution coincides with the corresponding solution for the flow a Newtonian fluid [13].

4.1.2 Resistance to the fluid flow

An expression for the resistance to the flow of a third grade fluid can be obtained from the following relationship between the shear stress S_{xy} and du/dy:

$$S_{xy} = \eta\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 \tag{21}$$

We introduce $S^*_{xy} = \dfrac{S_{xy}}{\eta(U/h)}$ and make use of the non-dimensional variables defined in (8) and (9) to obtain the following non-dimensional form of the equation (21):

$$S_{xy} = \frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 \tag{22}$$

Substituting for u from equation (20), we obtain

$$S_{xy} = \left(\frac{dp}{dx}\right)y\left\{1 - 2\beta\left(\frac{dp}{dx}\right)^2 y^2 + 3(2\beta)^2\left(\frac{dp}{dx}\right)^4 y^4 - 12(2\beta)^3\left(\frac{dp}{dx}\right)^6 y^6\right.$$

$$+55(2\beta)^4\left(\frac{dp}{dx}\right)^8 y^8+\cdots\Bigg\}+2\beta\left(\frac{dp}{dx}\right)^3 y^3\Bigg\{1-2\beta\left(\frac{dp}{dx}\right)^2 y^2+3 \tag{23}$$

$$\times(2\beta)^2\left(\frac{dp}{dx}\right)^4 y^4-12(2\beta)^3\left(\frac{dp}{dx}\right)^6 y^6+55(2\beta)^4\left(\frac{dp}{dx}\right)^8 y^8+\cdots\Bigg\}^3$$

The force per unit area on the plate $y = 1$ is the skin friction $\tau = S_{xy}$ at $y = 1$ and this is given by

$$\tau = \left(\frac{dp}{dx}\right)\left\{1-2\beta\left(\frac{dp}{dx}\right)^2+3(2\beta)^2\left(\frac{dp}{dx}\right)^4-12(2\beta)^3\left(\frac{dp}{dx}\right)^6+55(2\beta)^4\left(\frac{dp}{dx}\right)^8+\ldots\right\}$$

$$+2\beta\left(\frac{dp}{dx}\right)^3\Bigg\{1-2\beta\left(\frac{dp}{dx}\right)^2+3(2\beta)^2\left(\frac{dp}{dx}\right)^4-12(2\beta)^3\left(\frac{dp}{dx}\right)^6 \tag{24}$$

$$+55(2\beta)^4\left(\frac{dp}{dx}\right)^8+\cdots\Bigg\}^3$$

The corresponding force on the plate $y = -1$ is $-\tau$. This means that there is an equal and opposite drag $\pm\tau$ per unit area on the plates $y = \pm 1$, which resists the fluid motion.

4.1.3 Volume Flow Rate and Average Velocity

The volume flow rate for the flow of a third grade fluid is given by

$$Q(y) = 2Uh\int_0^1 u(y)\mathrm{d}y = Uh\Bigg\{-\frac{1}{3}\frac{dp}{dx}+\frac{1}{5}(2\beta)\left(\frac{dp}{dx}\right)^3-\frac{3}{7}(2\beta)^2\left(\frac{dp}{dx}\right)^5$$

$$+\frac{12}{9}(2\beta)^3\left(\frac{dp}{dx}\right)^7-\frac{55}{11}(2\beta)^4\left(\frac{dp}{dx}\right)^9+\cdots\Bigg\} \tag{25}$$

Knowing the volume flow rate from equation (25), the average fluid velocity $\bar{u}(y)$ is obtained in the form

$$\bar{u}(y)=\frac{Q}{2h}=U\Bigg\{-\frac{1}{3}\frac{dp}{dx}+\frac{1}{5}(2\beta)\left(\frac{dp}{dx}\right)^3-\frac{3}{7}(2\beta)^2\left(\frac{dp}{dx}\right)^5$$

$$+\frac{12}{9}(2\beta)^3\left(\frac{dp}{dx}\right)^7-\frac{55}{11}(2\beta)^4\left(\frac{dp}{dx}\right)^9+\cdots\Bigg\} \tag{26}$$

4.2 Oldroyd Six Constant Fluid

4.2.1 Fluid Velocity

Next, introducing equation (15) into equation (14) yields

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}y^k+\alpha_1\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{k-j}\left\{(j+1)(m+1)(k-m-j+1)a_{j+1}a_{m+1}a_{k-m-j+1}\right\}y^k$$
$$-\alpha_2\left(\frac{dp}{dx}\right)\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\left\{(j+1)(k-j)a_{j+1}a_{k-j}\right\}y^k=y\left(\frac{dp}{dx}\right) \tag{27}$$

As before, equating coefficients of y^0 and y^1, we obtain $a_1 = 0$ and $a_2 = \frac{1}{2}\frac{dp}{dx}$ and the recursive relation

$$a_{k+1} = -\frac{\alpha_1}{k+1}\sum_{j=1}^{k-1}\sum_{m=0}^{k-j}(j+1)(m+1)(k-j-m+1)a_{j+1}a_{m+1}a_{k-j-m+1}$$
$$+\frac{\alpha_2}{k+1}\frac{dp}{dx}\sum_{j=0}^{k-1}(j+1)\,(k-j)\,a_{j+1}\,a_{k-j} \tag{28}$$

where $k = 2, 3, \ldots$ and the first few coefficients determined from this relation are given by

$$a_3 = a_5 = a_7 = a_9 = \ldots = 0 \tag{29}$$

$$a_4 = -\frac{\alpha_1-\alpha_2}{4}\left(\frac{dp}{dx}\right)^3, a_6 = \frac{(\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)}{6}\left(\frac{dp}{dx}\right)^5 \tag{30}$$

$$a_8 = -\frac{(\alpha_1-\alpha_2)(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)}{8}\left(\frac{dp}{dx}\right)^7, \tag{31}$$

$$a_{10} = \frac{(\alpha_1-\alpha_2)}{10}\left(\frac{dp}{dx}\right)^9\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2)$$
$$+ (\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-2\alpha_2) + \alpha_1(\alpha_1-\alpha_2)^2\} \tag{32}$$

Hence, substituting equations (29)-(32) into equation (15) we obtain

$$u(y) = \left(\frac{dp}{dx}\right)\frac{\left(y^2-1\right)}{2}-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^3\frac{\left(y^4-1\right)}{4}+(\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)$$
$$\times\left(\frac{dp}{dx}\right)^5\frac{\left(y^6-1\right)}{6}-(\alpha_1-\alpha_2)(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^7\frac{\left(y^8-1\right)}{8}$$
$$+ (\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2) + (\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2) \tag{33}$$
$$\times (6\alpha_1-2\alpha_2) + \alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^9\frac{y^{10}-1}{10}+\cdots$$

We notice that for $\alpha_1 = \alpha_2$ this solution coincides with the corresponding solution for the flow of a Newtonian fluid [13].

4.2.2 Resistance to the Flow

In this case the resistance to the flow is found by the dimensionless relationship

$$S_{xy} = \left(\frac{du}{dy}\right)\left\{\frac{1+\alpha_1\left(\frac{du}{dy}\right)^2}{1+\alpha_2\left(\frac{du}{dy}\right)^2}\right\} \tag{34}$$

which in view of equation (33) yields

$$\begin{aligned} S_{xy} = & \Bigg[\left(y\frac{dp}{dx}\right)\Bigg\{1-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^2 y^2 + (\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2) \\ & \times\left(\frac{dp}{dx}\right)^4 y^4 - (\alpha_1-\alpha_2)\times(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^6 y^6 \\ & + (\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2) + (\alpha_1-\alpha_2) \\ & (3\alpha_1-2\alpha_2)\times(6\alpha_1-2\alpha_2) + \alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^8 y^8 + \cdots\Bigg\}\Bigg] \\ & \times\Bigg[1+\alpha_1\left(\frac{dp}{dx}\right)^2 y^2 \Bigg\{1-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^2 y^2 + (\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2) \\ & \times\left(\frac{dp}{dx}\right)^4 y^4 - (\alpha_1-\alpha_2)\times(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^6 y^6 \\ & + (\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2) + (\alpha_1-\alpha_2) \\ & (3\alpha_1-2\alpha_2)\times(6\alpha_1-2\alpha_2) + \alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^8 y^8 + \cdots\Bigg\}^2\Bigg] \\ & \times\Bigg[1+\alpha_2\left(\frac{dp}{dx}\right)^2 y^2 \Bigg\{1-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^2 y^2 + (\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2) \\ & \times\left(\frac{dp}{dx}\right)^4 y^4 - (\alpha_1-\alpha_2)\times(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^6 y^6 \\ & + (\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2) + (\alpha_1-\alpha_2) \\ & (3\alpha_1-2\alpha_2)\times(6\alpha_1-2\alpha_2) + \alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^8 y^8 + \cdots\Bigg\}^2\Bigg]^{-1} \end{aligned} \tag{35}$$

Thus the drag on the plates $y = \pm 1$, which resists the fluid motion, is $\pm\tau = \pm S_{xy}(y = 1)$ and is given by

$$\tau = \Bigg[\left(\frac{dp}{dx}\right)\Bigg\{1-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^2 + (\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)$$

$$\times\left(\frac{dp}{dx}\right)^4-(\alpha_1-\alpha_2)\times(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^6$$
$$+(\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2)+(\alpha_1-\alpha_2)$$
$$\left.\left.(3\alpha_1-2\alpha_2)\times(6\alpha_1-2\alpha_2)+\alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^8+\cdots\right\}\right]$$

$$\times\left[1+\alpha_1\left(\frac{dp}{dx}\right)^2\left\{1-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^2+(\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)\right.\right.$$
$$\times\left(\frac{dp}{dx}\right)^4-(\alpha_1-\alpha_2)\times(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^6 \tag{36}$$
$$+(\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2)+(\alpha_1-\alpha_2)$$
$$\left.\left.(3\alpha_1-2\alpha_2)\times(6\alpha_1-2\alpha_2)+\alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^8+\cdots\right\}^2\right]$$
$$\times\left[1+\alpha_2\left(\frac{dp}{dx}\right)^2\left\{1-(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^2+(\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)\right.\right.$$
$$\times\left(\frac{dp}{dx}\right)^4-(\alpha_1-\alpha_2)\times(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^6$$
$$+(\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2)+(\alpha_1-\alpha_2)$$
$$\left.\left.(3\alpha_1-2\alpha_2)\times(6\alpha_1-2\alpha_2)+\alpha_1(\alpha_1-\alpha_2)^2\left(\frac{dp}{dx}\right)^8+\cdots\right\}^2\right]^{-1}$$

4.2.3 Volume Flow Rate and Average Velocity

In the case of an Oldroyd six constant fluid, the volume flow rate is given by

$$Q\;=2Uh\int_0^1 u(y)\mathrm{d}y=Uh\left[-\frac{1}{3}\left(\frac{dp}{dx}\right)+\frac{1}{5}(\alpha_1-\alpha_2)\left(\frac{dp}{dx}\right)^3-\frac{1}{7}(\alpha_1-\alpha_2)\right.$$
$$\times(3\alpha_1-2\alpha_2)\left(\frac{dp}{dx}\right)^5+\frac{1}{9}(\alpha_1-\alpha_2)(2\alpha_1-\alpha_2)(6\alpha_1-5\alpha_2)\left(\frac{dp}{dx}\right)^7$$
$$-\frac{1}{11}(\alpha_1-\alpha_2)\{(2\alpha_1-\alpha_2)(3\alpha_1-2\alpha_2)(6\alpha_1-5\alpha_2)+(\alpha_1-\alpha_2) \tag{37}$$
$$\left.\times(3\alpha_1-2\alpha_2)(6\alpha_1-2\alpha_2)+\alpha_1(\alpha_1-\alpha_2)^2\}\left(\frac{dp}{dx}\right)^9+\cdots\right]$$

and the average fluid velocity is given by

$$\bar{u}(y) = \frac{Q}{2h} = \Bigg[-\frac{1}{3}\frac{dp}{dx} + \frac{1}{5}(\alpha_1 - \alpha_2)\left(\frac{dp}{dx}\right)^3 - \frac{1}{7}(\alpha_1 - \alpha_2)(3\alpha_1 - 2\alpha_2)$$

$$\times\left(\frac{dp}{dx}\right)^5 + \frac{1}{9}(\alpha_1 - \alpha_2)(2\alpha_1 - \alpha_2)(6\alpha_1 - 5\alpha_2)\left(\frac{dp}{dx}\right)^7 \tag{38}$$

$$-\frac{1}{11}(\alpha_1 - \alpha_2)\{(3\alpha_1 - 2\alpha_2)(6\alpha_1 - 5\alpha_2) + (\alpha_1 - \alpha_2)$$

$$\times(3\alpha_1 - 2\alpha_2)(6\alpha_1 - 2\alpha_2) + \alpha_1(\alpha_1 - \alpha_2)^2\}\left(\frac{dp}{dx}\right)^9 + \cdots \Bigg]$$

5. RESULTS AND DISCUSSION

In this section we study the effect of the variation of the physical parameters on the fluid velocity, the resistance to the flow and the volume flow rate. For this purpose we present tables and figures of the analytical expressions derived in sections 4 for various values of the governing parameters.

5.1 Third Grade Fluid

Figures 2(a) and (b) show the fluid velocity field $u(y)$ given in equation (20) for various values of the non-Newtonian parameter β and of the pressure gradient dp/dx, respectively, when one of these is held fixed. From these figures we observe that the magnitude of the fluid velocity distribution increases with increasing values of the non-Newtonian parameter or the pressure drop. Therefore, we may conclude that the larger the value of the parameters β or $|dp/dx|$ then the larger is the magnitude of the fluid velocity.

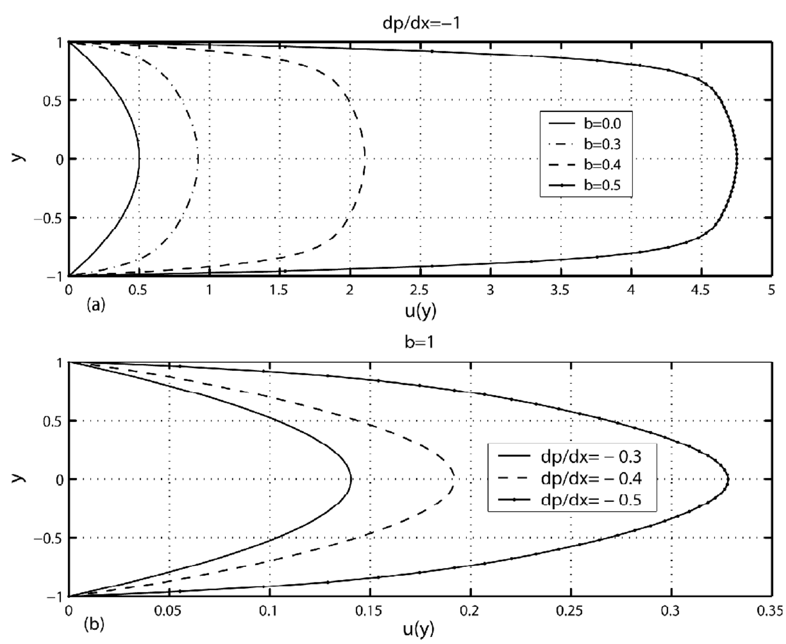

Figure 2: Effect of the Variation of (a) β When *dp/dx* = –1, and (b) *dp/dx* When β = 1, on the Fluid Velocity Given by the Expression (20)

Figures 3(a) and (b) show the volume flow rate (25) as a function of β when dp/dx =–1.0, –1.1, –1.2 and as a function of dp/dx when β = 1.0. 1.4, 1.8, 2.0, respectively. We observe from figure 3(a) that the flow rate increases with decreasing values of *dp/dx* and for a particular value of *dp/dx* the flow rate increases as the value of β increases. A similar observation may be made from the result presented in figure 3(b). Therefore, we conclude that the volume flow rate of a third grade fluid increases with increasing values of β or |*dp/dx*|.

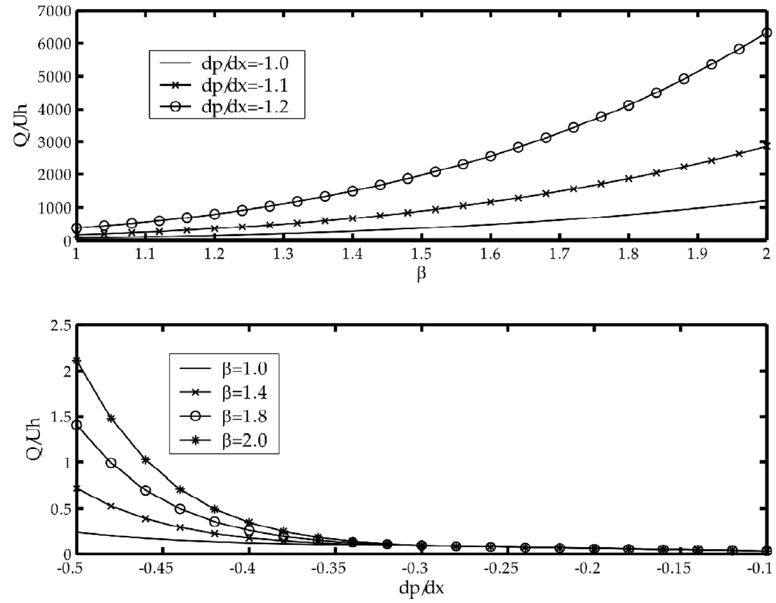

Figure 3: Effect of Variation of β and *dp/dx* on the Volume Flow Rate Given by Equation (25)

Finally, we study the variation of the resistance to the flow of a third grade fluid with the variation of the non-Newtonian parameter β and the pressure gradient. For this purpose we tabulate in table 1 the values of τ given in equation (24) for various values of β ∈ {0.0, 0.5, 1.0, 1.5, 2.0, 2.5} and of the pressure gradient *dp/dx* ∈ {–0.3, –0.4, –0.5}. From this table we observe that for a particular value of β the value of τ increases with increasing values of *dp/dx*. Similarly for any particular value of the parameter *dp/dx*, the magnitude of τ increases with increasing values of the parameter β. Therefore we may conclude that the resistance to the flow of a third grade fluid increases for increasing values of the non-Newtonian parameter β or the pressure drop |*dp/dx*|.

Table 1

Effect of β and $\frac{dp}{dx}$ on the Resistance to the Flow, Equation (24)

β	$\frac{dp}{dx} = -0.3$	$\frac{dp}{dx} = -0.4$	$\frac{dp}{dx} = -0.5$
0.0	–0.3000	–0.4000	–0.5000
0.5	–0.3004	–0.4007	–0.5947
1.0	–0.3115	–0.6737	–9.6901
1.5	–0.3836	–5.3531	-1.0985×10^{3}
2.0	–0.7096	–118.6044	-4.8691×10^{4}
2.5	–2.3801	-2.0348×10^{3}	-9.575×10^{5}

5.2 Oldroyd Six Constant Fluid

Since expression (33) for the fluid velocity involves terms containing factors of the form $\alpha_1 - \alpha_2$, $3\alpha_1 - 2\alpha_2$, etc., we therefore present in figures 4 and 5 results for a wide range of values of α_1 and α_2.

Figure 4 shows the fluid velocity (33) for $dp/dx = -1$, $\alpha_1 = 1$ and for various values of α_2 chosen in the intervals (0, 1), (1, 2), (2, 3) and (3, 4). From this figure we observe that the magnitude of the fluid velocity decreases with increasing values of $\alpha_2 \in (0, 2)$ and it increases with increasing values of α_2 when this choice is made in (1, 2), (2, 3) or (3, 4). Therefore we conclude that the magnitude of the fluid velocity decreases with increasing values of α_2 when it is smaller than the fixed value of α_1. However, this trend in the variation of the velocity magnitude becomes exactly the opposite when the value of α_2 is larger than the fixed value of α_1.

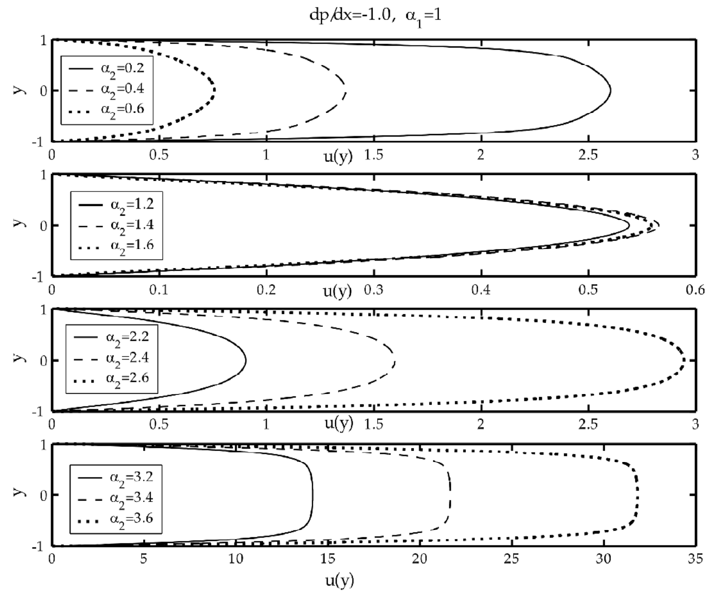

Figure 4: Effect of Variation of α_2 on the Fluid Velocity, given in Equation (33), when α_1 and *dp/dx* are Fixed

Figure 5 shows the fluid velocity magnitude given in equation (33) for $dp/dx = -1$, $\alpha_2 = 1$ and various values of $\alpha_1 \in (0, 1)$, (1, 2), (2, 3) and (3, 4). From this figure we see that the magnitude of the fluid velocity decreases with increasing values of $\alpha_1 \in (0, 1)$ and increases with increasing values of $\alpha_1\ \varepsilon$ (1, 2), (2, 3) or (3, 4). Therefore, we again conclude that the magnitude of the fluid velocity in equation (33) decreases when the value of α_1 is smaller than the fixed value of α_2. Otherwise the velocity magnitude increases with increasing value of α_1.

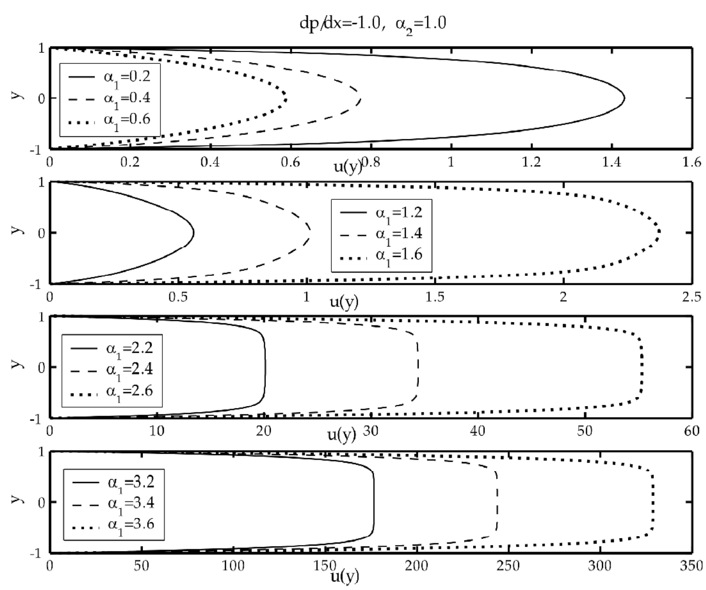

Figure 5: Effect of Variation of α_1 on the Fluid Velocity, given in Equation (33), when α_2 and *dp/dx* are Fixed

Hence, the overall conclusion remains that the larger the value of α_1 or α_2, with one of them being fixed along with *dp/dx*, then the larger the magnitude of the fluid velocity. The only exception to this conclusion occurs when $\alpha_1 < \alpha_2$ or $\alpha_2 < \alpha_1$, in which case the velocity magnitude decreases with increasing values of α_1 or α_2.

Figure 6 shows the fluid velocity (33) as a function of *y* for various values of $dp/dx \in \{-0.2, -0.4, -0.6, -0.8\}$, when α_1 and α_2 are fixed at 1 and 2, respectively. From this figure we observe that the fluid velocity is smallest for $dp/dx = -0.2$ and it increases when the value of *dp/dx* decreases down to –0.8. Therefore, we conclude that when α_1 and α_2 are fixed, the magnitude of the fluid velocity increases with increasing value of the pressure drop.

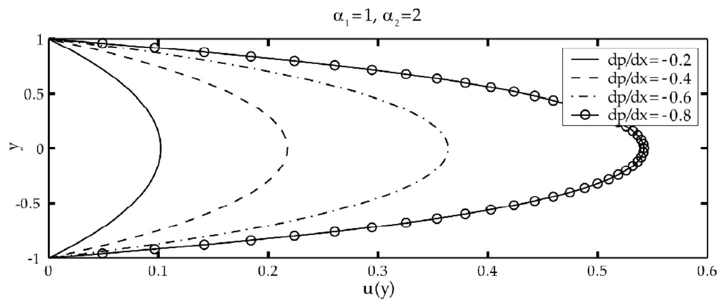

Figure 6: Effect of the Variation of Pressure Gradient *dp/dx* on the Fluid Velocity, given in Equation (33), when α_1 and α_2 are Fixed

In view of the conclusion made in case of the fluid velocity variation we present in figures 7(a) the volume flow rate (37) as a function of the pressure gradient *dp/dx* for $\alpha_1 = 1$ and $\alpha_2 \in \{2.0, 2.2, 2.4, 2.6\}$. Also we show in figure 7(b), the case when $\alpha_2 = 1$ and $\alpha_1 \in \{2.0, 2.2, 2.4, 2.6\}$. From figure 7(a) we observe that the flow rate increases as the value of α_1 increases. Moreover, the flow rate increases with decreasing value of *dp/dx*. A similar behavior of the flow rate is observed in figure 7(b). Thus we conclude that the volume flow rate of an Oldroyd six constant fluid increases with increasing value of α_1 or α_2, one of these being fixed, and decreasing the value of the pressure gradient *dp/dx*.

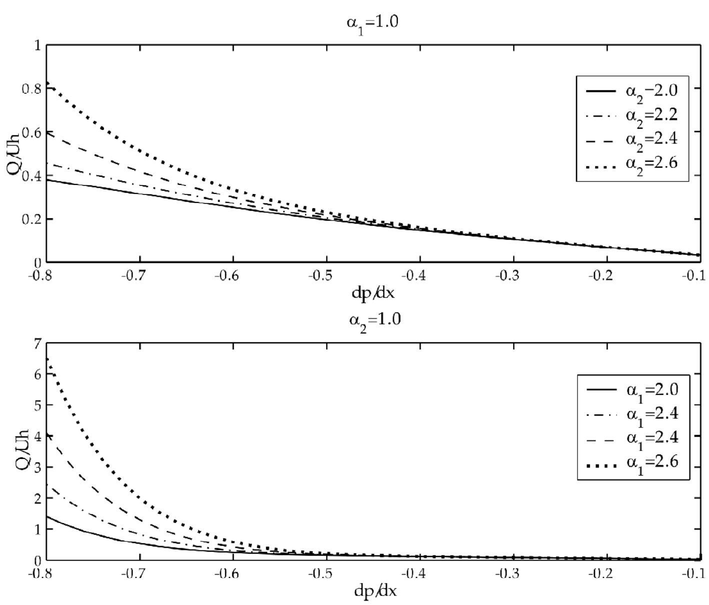

Figure 7: Effect of Variation of (a) *dp/dx* and α_2 when α_1 is Fixed, and (b) *dp/dx* and α1 when α_2 is Fixed, on the Volume Flow Rate Given by Equation (37).

As before, in order to observe the variation of τ given in equation (36) we tabulate in table 2 the values of τ for $\alpha_1 = 1.0, 1.5, 2.0, 2.5, 3.0$ and $dp/dx = -1.0, -1.2, -1.4, -1.6$ when $\alpha_2 = 1$. From this table we observe that, for a given value of α_1, the value τ increases with increasing value of dp/dx. Also for a specific value of dp/dx the value of τ increases if we increase α_1. Almost a similar trend for the variation of τ was observed when α_1 is fixed and τ is calculated for various values of α_2 and dp/dx. Therefore, we conclude that the resistance to the fluid flow increases if we increase any one of α_1, α_2 and dp/dx.

Table 2

Effect of α_1 and $\frac{dp}{dx}$ on the Resistance to the Flow, Equation (36), when $\alpha_2 = 1$.

$\alpha 1$	$\frac{dp}{dx} = -1.0$	$\frac{dp}{dx} = -1.2$	$\frac{dp}{dx} = -1.4$	$\frac{dp}{dx} = -1.6$
1.0	−1.0000	−1.2000	−1.4000	−1.6000
1.5	−18.4284	−96.3717	−398.191	-1.3590×10^3
2.0	−217.9908	-1.1686×10^3	−4.80423	-1.6269×10^4
2.5	-1.1177×10^3	-5.9512×10^3	-2.4314×10^4	-8.1960×10^4
3.0	-3.795×10^3	-2.0093×10^4	-8.1763×10^4	-2.7486×10^5

6. CONCLUSIONS

The flow of a third grade and an Oldroyd six constant fluid between two stationary parallel plates, generated by a constant pressure gradient, is studied. The analytical solutions for the fluid velocity, the resistance to the fluid flow, the volume flow rate and the average fluid velocity are derived by using the Taylor series method. Figures and tables are presented to illustrate the effect of dp/dx, β, α_1 and α_2 on these analytical solutions. It is found that (i) for a third grade fluid the fluid velocity and other flow variables increase with increasing values of $|dp/dx|$ and of the parameter β. (ii) For the Oldroyd six constant model the fluid velocity increases with increasing value of α_2, when $\alpha_1 (< \alpha_2)$ and dp/dx are fixed. Also the same conclusion holds for the fluid velocity magnitude, when $\alpha_2 (< \alpha_1)$ and dp/dx are fixed. The resistance to flow and the volume flow rate increases on increasing $|dp/dx|$ and α_1 or α_2, one of these being fixed.

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REFERENCES

[1] Abdul-Majid Wazwaz. A comparison between Adomian decomposition method and Taylor series method in the series solution. Applied Mathematics and Computation, Vol. **97**, pp. 37–44, 1998.

[2] Adomian, G. Convergent series solution of nonlinear equations. *J. Comput. Appl. Math.*, Vol. **11**, No. 2, 1984.

[3] A. H. Nayfeh. Introduction to Perturbation Techniques. Wiley-Interscience, New York, 1981.

[4] A. M. Siddiqui, A. Zeb, Q. K. Ghori, A. M. Benharbit. Homotopy perturbation method for heat transfer flow of a third grade fluid. Chaos, Solitons and Fractals, Vol. **36**, pp. 182-192, 2008.

[5] A. M. Siddiqui, M. Ahmed, Q. K. Ghori. Couette and Poiseuille flow for Non-Newtonian Fluids. Int. J. Non linear Sci. Numer. Simulat., Vol. **7**, No.1, pp. 15–26, 2006.

[6] A. M. Siddiqui, M. Ahmed, S. Islam, Q. K. Ghori. Homotopy Analysis of Couette and Poiseuille flows for fouth grade fluids. Acta Mechanica, Vol. **180** No. 1–4, pp. 117–132, 2005.

[7] A. M. Siddiqui, R. Mahmood, Q. K. Ghori. Thin film flow of a third grade fluid on a moving belt by He's Homotopy Perturbation Method. Int. J. non Linear Sci. Numer. Simulat., Vol. **7**, No.1, pp. 07–14, 2006.

[8] Xue-Hui Chen, Lian-Cun Zheng, Xin-Xin Zhang. MHD boundary layer flow of a non-Newtonian fluid on a moving surface with a power-law velocity. Chin. Phys. Lett., Vol. **24**, No.7, 1989–1991, 2007.

[9] C. M. Bender and S. A. Orszag. Advanced Mathematical Methods for Scientists and Engineers. McGraw-Hill, New York, 1978.

[10] J. H. He Homotopy perturbation method a new nonlinear analytical technique. Appl. Math. and Comput., Vol. 1**35**, pp. 73, 2000.

[11] M. Yürüsoy and M. Pakdemmirli. Approximate analytical solutions for the flow of a third grade fluid in a pipe. Int. J. Non-Linear Mech., Vol. **37**, 187–195, 2002.

[12] Mehdi Deghan. The use of Adomian decomposition method for solving the one-dimensional parabolic equation with non-local boundary specifications. Int. J. Comput. Math., Vol. **81** No.1, pp. 25–34, 2004.

[13] M. K. Schwiebert and W. H. Leong. Underfill flow as viscous flow between parallel plates driven by capillary action. IEEE Comp., Packag., Manufact. Technol. C, Vol. **19**, 133–137, 1996.

[14] R.E. O' Malley Jr. Thinking about ordinary differential equations. Cambridge Texts in Applied Mathematics (No. 18). Cambridge University Press, 1997.

[15] T. Hayat, Z. Abbas, M. Sajid. Series solution for the upper-convected Maxwell fluid over a porous stretching plate. Phy. Lett. A, Vol. **358**, 396–403, 2006.

[16] T. Hayat, M. Sajid. Analytic solution for axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet. Int. J. Heat and Mass Transfer, Vol. **50**, pp. 75–84, 2007.

[17] W. B. Young and W. L.Yang. Underfill viscous flow between parallel plates and solder bumps. IEEE Transactions on components and packging technologies, Vol. **25**, No. 4, 695–700, 2002.