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Existence of Periodic Solution on a Class of Discrete System

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Abstract: Using trigonometric series theory and the contraction mapping principle, we studied the linear inhomogeneous difference systems Eq.(1) and the quasi-linear delay difference systems Eq.(10) in this paper. Some sufficient and necessary conditions on the existence of periodic solutions of Eq.(1) and a new sufficient condition on the existence of unique periodic solution of Eq.(10) are obtained. An example is provided to illustrate the theoretical development.

Keywords: Difference systems; Periodic solution; Trigonometric series; Contraction mapping principle

1. INTRODUCTION

In the last two decades, the existence of periodic solutions of differential and difference equations has been extensively studied (see, for example, [1-4]). Hatvani and Krisztin in [5] studied the existence of periodic solution of linear inhomogeneous differential equation

$$\dot{x}(t) = \int_{-\infty}^{+\infty} [dE(s)]x(t+s) + f(t) \tag{A}$$

where $E : R \to C^{n\times n}$ is left continuous and of bounded total variation on R, i.e., $\gamma = \int_{-\infty}^{\infty} |dE(s)| < \infty, f \in C_T$, and $x \in BC(R; C^n) := \{\psi \in C(R; C^n) : \psi \text{ is bounded on } R\}$. They [5] also gave sufficient and necessary conditions on the existence of periodic solutions for Eq. (A).

Ma et al in [6] studied the uniqueness of periodic solution of quasi-linear functional differential equation

$$\dot{x}(t) = \int_R [dE(s)]x(t+s) + G(t, x(t+\cdot)) \tag{B}$$

where $x(t) \in R^n$, $E : R \to R^{n^2}$ is left continuous and of bounded total variation on R, i.e., $\gamma = \int_{-\infty}^{\infty} |dE(s)| < \infty$, $G : R \times BC(R; R^n) \to R^n$ is continuous, $T > 0$, G is T-periodic with respect to its first variable t, and G maps bounded set to bounded set. Sufficient and necessary conditions on the uniqueness of periodic solutions for Eq. (B) were given in [6]. Although many papers have studied the uniqueness of periodic solution of quasi-linear functional differential equation, only a few papers focused on the uniqueness of periodic solution of certain discrete system. However, in numerical simulations and practical implementations of continuous-time systems, discretization is necessary. We note that in practice, the dynamics of difference systems may be quite different from those of differential systems. Therefore, the dynamics of discrete systems are of both theoretical and practical importance.

In this paper, using trigonometric series theory and the contraction mapping principle, we'll study the linear inhomogeneous difference systems and quasi-linear delay difference systems, establish some sufficient and

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necessary conditions on the existence of periodic solutions for the linear inhomogeneous difference systems, at same time, give a new sufficient condition on the existence of periodic solutions for the quasi-linear delay difference systems. Furthermore, the main results obtained in [5, 6] are extended and improved to difference systems.

2. MAIN RESULTS

In this section, at first, we'll consider the linear inhomogeneous difference periodic systems

$$x(n+1)=\sum_{j=-\infty}^{+\infty} A(j)x(n-j)+f(n) \tag{1}$$

where $A(j) \in C^{n\times n}$, $x(n) \in C^n$, $f \in l_N = \{\{\phi(n)\} \mid \phi(n+N)=\phi(n)\}$, and $N \geq 1$ is a positive integer.

Lemma 1: Suppose that $f(n) \in l_N$, then $f(n)$ can be uniquely expressed as $f(n)=\sum_{k=0}^{N-1}\hat{f}(k)e^{\mu_k n}$, where $\hat{f}(k)=\frac{1}{N}\sum_{n=0}^{N-1} f(n)e^{-\mu_k n}$, $\mu_k=\frac{2k\pi i}{N}$, and $k \in \omega := \{0, 1, 2, \dots, N-1\}$.

Proof: Assume that

$$f(n)=\sum_{k=0}^{N-1} a(k)e^{\mu_k n}, \tag{2}$$

multiplying Eq.(2) by $e^{-\mu_i \eta}$ and summing from 0 to $N-1$ gives

$$\begin{aligned}\sum_{n=0}^{N-1} f(n)e^{-\mu_j n} &= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a(k)e^{\mu_k n}e^{-\mu_j n}\\ &= \sum_{k=0}^{N-1} a(k)\sum_{n=0}^{N-1} e^{(\mu_k-\mu_j)n}.\end{aligned}$$

Since

$$\sum_{n=0}^{N-1} e^{(\mu_k-\mu_j)n}=\begin{cases} N & k=j,\\ 0 & k\neq j,\end{cases}$$

we have

$$\sum_{n=0}^{N-1} f(n)e^{-\mu_k n}=Na(k).$$

Hence

$$a(k)=\frac{1}{N}\sum_{n=0}^{N-1} f(n)e^{-\mu_k n}=\hat{f}(k).$$

This implies that $f(n)$ can be uniquely expressed as $f(n) = \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}$.

Lemma 2. Assume that $f(n) \in l_N$ and $f(n)$ can be expressed as $f(n) = \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}$, then

$$\sum_{k=0}^{N-1} \left|\hat{f}(k)\right|^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left|f(n)\right|^2.$$

Proof: Since $f(n) = \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}$, we have

$$\begin{aligned}
|f(n)|^2 &= < f(n), \overline{f(n)} > \\
&= < \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}, \sum_{l=0}^{N-1} \overline{\hat{f}(l)}e^{-\mu_l n} > \\
&= \sum_{k,l=0}^{N-1} < \hat{f}(k), \overline{\hat{f}(l)} > e^{(\mu_k - \mu_l)n},
\end{aligned}$$

then

$$\begin{aligned}
\sum_{n=0}^{N-1}\left|f(n)\right|^2 &= \sum_{k,l=0}^{N-1} < \hat{f}(k), \overline{\hat{f}(l)} > \sum_{n=0}^{N-1} e^{(\mu_k - \mu_l)n} \\
&= \sum_{k=0}^{N-1}\left|\hat{f}(k)\right|^2 N.
\end{aligned}$$

So

$$\sum_{k=0}^{N-1} \left|\hat{f}(k)\right|^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left|f(n)\right|^2.$$

The proof of Lemma 2 is completed.

Theorem 1: Eq.(1) has an unique N-periodic solution if and only if e^{μ_k} are not roots of the characteristic equation

$$\det \Delta(\mu) = 0$$

where $\mu_k = \frac{2k\pi}{N} i, k \in \omega = \{0,1,2,\ldots,N-1\}$, and $\Delta(\mu) = \mu I - \sum_{j=-\infty}^{+\infty} A(j)\mu^{-j}$.

Proof: Assume that Eq.(1) has an unique N-periodic solution $x(n)$. Since $x(n+N) = x(n), f(n+N) = f(n)$, by Lemma 1, $x(n)$ and $f(n)$ can be uniquely expressed as $x(n) = \sum_{k=0}^{N-1} \hat{x}(k)e^{\mu_k n}$, and $f(n) = \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}$, respectively.

Multiplying Eq.(1) by $e^{-\mu_k n}$ and summing from 0 to $N-1$, we obtain

$$\sum_{n=0}^{N-1} x(n+1)e^{-\mu_k n} = \sum_{n=0}^{N-1}\sum_{j=-\infty}^{+\infty} A(j)x(n-j)e^{-\mu_k n} + \sum_{n=0}^{N-1} f(n)e^{-\mu_k n},$$

i.e.,

$$\sum_{n=0}^{N-1} x(n+1)e^{-\mu_k (n+1)}e^{\mu_k} = \sum_{j=-\infty}^{+\infty} A(j)\sum_{n=0}^{N-1} x(n-j)e^{-\mu_k (n-j)}e^{-\mu_k j} + N\hat{f}(k).$$

Hence

$$\left(e^{\mu_k} I - \sum_{j=-\infty}^{+\infty} A(j)e^{-\mu_k j}\right) N\hat{x}(k) = N\hat{f}(k),$$

that is

$$\Delta(e^{\mu_k})\hat{x}(k) = \hat{f}(k). \tag{3}$$

Since the linear equation

$$\Delta(e^{\mu_k})y = \hat{f}(k) \tag{4}$$

has solution, and if we assume that Eq.(4) has an unique solution, then

$$\det \Delta(e^{\mu_k}) \neq 0. \tag{5}$$

In fact, if Eq.(4) has another solution $\hat{y}(k)$, it is immediate that $y(n) = \sum_{k=0}^{N-1} \hat{y}(k)e^{\mu_k n}$ satisfies

$$y(n+1) = \sum_{j=-\infty}^{+\infty} A(j)y(n-j) + \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}. \tag{6}$$

In addition, $x(n) = \sum_{k=0}^{N-1} \hat{x}(k)e^{\mu_k n}$ also satisfies the following equation

$$x(n+1) = \sum_{j=-\infty}^{+\infty} A(j)x(n-j) + \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}. \tag{7}$$

Subtracting Eq.(6) from Eq.(7) gives

$$g(n+1) = \sum_{j=-\infty}^{+\infty} A(j)g(n-j) \tag{8}$$

where $g(n) = x(n) - y(n)$. Since Eq.(1) has an unique solution, then the corresponding homogeneous linear equation Eq. (8) only has a null solution. Therefore, $\det \Delta(e^{\mu_k}) \neq 0$, that is, e^{μ_k} are not roots of the characteristic equation $\det \Delta(\mu) = 0$.

On the other hand, assume that $\det \Delta(e^{\mu_k}) \neq 0$ for each $k \in \omega$, then Eq.(4) has an unique solution for each $k \in \omega$. That is, for each $k \in \omega$, Eq.(4) uniquely defines a $c(k)$ such that

$$\Delta(e^{\mu_k})c(k) = \hat{f}(k). \tag{9}$$

It is obvious that $z(n) = \sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ satisfy Eq.(7). Let $\phi(n) = \sum_{k=0}^{N-1} \beta(k)e^{\mu_k n}$ to be another solution of Eq.(7), then $\phi(n) - z(n)$ is a solution of the corresponding homogeneous linear equation Eq.(8). Since $\det \Delta(e^{\mu_k}) \neq 0$, then Eq. (8) has an unique null solution. Therefore, $\phi(n) = z(n)$, that is, $z(n)$ is the unique N-periodic solution of Eq. (8). By Lemma 1, it is easy to know that Eq.(7) is equivalent to Eq.(1). Hence, $z(n) = \sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ is the unique N-periodic solution of Eq.(1). This completes the proof of Theorem 1.

Let C^{n*} be the space of n-dimension row vector. $\omega = \{0, 1, 2, \ldots, N-1\}$, $A(K) = \{a \in C^{n*} \mid a\Delta(e^{\mu_k}) = 0, k \in \omega\}$ and $l_N(E) = \{f \in l_N \mid a\hat{f}(k) = 0$ for all $a \in A(K), k \in \omega\}$. We'll prove a more general result giving a necessary and sufficient condition for the existence of N-periodic solutions in the general case when $det\Delta(e^{\mu_k}) = 0$ for some integers. That is,

Theorem 2: Eq.(1) has a N-periodic solution if and only if $f(n) \in l_N(E)$.

Proof: First, assume that $x(n)$ is a N-periodic solution of Eq.(1). Multiplying Eq.(1) by $e^{-\mu_k n}$ and summing from 0 to $N-1$, we obtain

$$\Delta(e^{\mu_k})\hat{x}(k) = \hat{f}(k),$$

that is, the linear equation Eq. (4) has solutions.

From elementary linear algebra, Eq.(4) has solutions if and only if $a\hat{f}(k) = 0$ for all $a \in A(K)$ such that $a\Delta(\mu_k) = 0$. Thus, the existence of a N-periodic solution of Eq.(1) implies $f(n) \in l_N(E)$.

On the other hand, assume that $f(n) \in l_N(E)$, then Eq.(4) has solutions. Choose $c(k)$ such that

$$\Delta(e^{\mu_k})c(k) = \hat{f}(k),$$

it is obvious that $z(n) = \sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ is the N-periodic solution of

$$x(n+1) = \sum_{j=-\infty}^{+\infty} A(j)x(n-j) + \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}.$$

By Lemma 1, Eq.(1) and Eq.(7) have same solutions, hence $\sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ is N-periodic solution of Eq.(1). Therefore, if $f(n) \in l_N(E)$, then Eq.(1) has at least one N-periodic solution. The proof is completed.

In the rest of this section, we consider the quasi-linear delay difference equation

$$x(n+1) = \sum_{j=-\infty}^{+\infty} A(j)x(n-j) + G(n, x(n+\cdot)) \tag{10}$$

where $A(j) \in C^{n\times n}$, $x(n) \in C^n$, G is N-periodic with respect to its first variable n, and G maps bounded set to bounded set. Let $|\cdot|$ denote any norm of C^n, for any matrix $D \in C^{n\times n}$, $|D|$ denotes operator norm induced by the norm in C^n. From Theorem 1, we know that Eq.(1) has an unique N-periodic solution if and only if $det\Delta(e^{\mu_k}) \neq 0$ for all $k \in \omega$. At the same time, $x(n)$ can be given by

$$x(n) = \sum_{k=0}^{N-1} \Delta^{-1}(e^{\mu_k})\hat{f}(k)e^{\mu_k n}. \tag{11}$$

Therefore, we have the following theorem on the existence of unique N-periodic solution of Eq.(10).

Theorem 3: Assume that $\det \Delta(e^{\mu k}) \neq 0$, for each $k \in \omega$, $G(n, \varphi)$ satisfies Lipschitz condition for $\varphi \in l_N$ with Lipschitz constant L satisfying

$$L^2 \sum_{k=0}^{N-1} \left|\Delta^{-1}(e^{\mu_k})\right|^2 < 1, \tag{12}$$

Then Eq.(10) has an unique N-periodic solution.

Before the proof of the theorem, we have a previous lemma: If we consider the operator $E : l_N \to l_N$ defined by

$$Ef(n) = \sum_{k=0}^{N-1} \Delta^{-1}(e^{\mu_k})\hat{f}(k)e^{\mu_k n}. \tag{13}$$

Then it is easy to know $Ef(n)$ is the unique N-periodic solution of Eq.(1).

Lemma 3: Assume that $E : l_N \to l_N$ is the operator defined by Eq.(13), then E is a linear operator, and

$$\|E\| \leq \left(\sum_{k=0}^{N-1} \left|\Delta^{-1}(e^{\mu_k})\right|^2 \right)^{\frac{1}{2}},$$

where $\|E\|$ is the norm of the operator E.

Proof: It is obvious that the operator E is a linear operator.

By Lemma 2 and Cauchy inequality, we have

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} \left|Ef(n)\right|^2 &= \sum_{k=0}^{N-1} \left|\Delta^{-1}(e^{\mu_k})\hat{f}(k)\right|^2 \\ &\leq \left(\sum_{k=0}^{N-1} \left|\Delta^{-1}(e^{\mu_k})\right|^2\right)\left(\sum_{k=0}^{N-1} \left|\hat{f}(k)\right|^2\right) \\ &= \left(\frac{1}{N}\sum_{n=0}^{N-1} \left|f(n)\right|^2\right)\left(\sum_{k=0}^{N-1} \left|\Delta^{-1}(e^{\mu_k})\right|^2\right). \end{aligned}$$

Hence

$$\sum_{n=0}^{N-1}\left|Ef(n)\right|^2 \le \left(\sum_{n=0}^{N-1}\left|f(n)\right|^2\right)\left(\sum_{k=0}^{N-1}\left|\Delta^{-1}(e^{\mu_k})\right|^2\right).$$

Since

$$\|E\| = \sup_{f\neq 0}\frac{|Ef|}{\|f\|} = \sup_{\|f\|\neq 0}\frac{\left(\sum_{n=0}^{N-1}\left|Ef(n)\right|^2\right)^{\frac{1}{2}}}{\left(\sum_{n=0}^{N-1}\left|f(n)\right|^2\right)^{\frac{1}{2}}},$$

then $\|E\| \le \left(\sum_{k=0}^{N-1}\left|\Delta^{-1}(e^{\mu_k})\right|^2\right)^{\frac{1}{2}}$.

Here, $|Ef|$ denotes the norm of the function Ef.

The proof of Theorem 3: Define the operator $T: l_N \to l_N$ by

$$(Tf)(n) = G(n, Ef(n+\cdot)). \tag{14}$$

By Theorem 1, $f \in l_N$ is the fixed point of T if and only if $Ef(n)$ is a N-periodic solution of Eq.(1). So, it suffices to prove that T has an unique fixed point in l_N.

For $f_1, f_2 \in l_N$,

$$\begin{aligned}|Tf_1(n) - Tf_2(n)| &= |G(n,Ef_1(n+\cdot)) - G(n, Ef_2(n+\cdot))|\\ &\le L\|Ef_1 - Ef_2\|\\ &\le L\left(\sum_{k=0}^{N-1}\left|\Delta^{-1}(e^{u_k})\right|^2\right)^{\frac{1}{2}}\|f_1 - f_2\|,\end{aligned}$$

then, by Inequality (12), $T: l_N \to l_N$ is a contraction mapping. By contraction mapping principle, T has an unique fixed point f^* in l_N. Hence, Ef^* is the unique N-periodic solution of Eq.(10). This concludes the proof of Theorem 3.

3. EXAMPLE

In this section, we provide an example to illustrate the effectiveness of main result in this paper.

Example 1: Consider the discrete system given in Eq. (1) with

$$A(j) = \begin{bmatrix} 1/2^j & 0 \\ 0 & 1/2^j \end{bmatrix}, \quad f(n) = \sin n\pi.$$

Then

$$x(n+1) = \sum_{j=-\infty}^{+\infty} A(j)x(n-j) + f(n)$$

has at least one 2-periodic solution.

Proof: Since $f(n) = \sin n\pi \in l_N$, $N = 2$, $\mu_0 = 0$, $\mu_1 = \pi i$. So $e^{\mu_0} = 1$, $e^{\mu_1} = -1$. It is straightforward to check that 1 and –1 are not roots of the characteristic equation

$$\det \Delta(\mu) = 0$$

where $\Delta(\mu) = \mu I - \sum_{j=-\infty}^{+\infty} A(j)\mu^{-j}$. Therefore, by Theorem 1, the difference system given in this example has at least one 2-periodic solution.

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