Journal of Mathematical Control Science and Applications (JMCSA)
Vol. 3 No. 1 (January-June, 2017), ISSN : 0974-0570

Existence and Uniqueness Theorem of a Coupled System of Linear Schrödinger Equations

Lazhar Bougoffa and Mohammad A Al. Khadhi *

Abstract: In this paper, we study a boundary value problem for of a coupled system of linear Schrödinger equations. Using Lax-Milgram theorem, we prove the existence and uniqueness of the strong solutions.

Keywords: Coupled System of Schrödinger Equations, Strong Solutions.

1. INTRODUCTION

Coupled linear equations of second-order are needed in the formulation of various physical situations. As an example of such type of equations, is the the following coupled system of Schrödinger equations [1,2,3,4,5,6]

$$\begin{cases} -\left(p_1(x)u'\right)' + q_1(x)u = r_1(x)v + f(x), \\ -\left(p_2(x)v'\right)' + q_2(x)v = r_2(x)u + g(x), \end{cases}$$

in the bounded domain $\Omega = (0, 1)$ where $f, g \in L_2(0, 1)$ and $p_i, q_i, r_i \in C^1(0, 1)$, $i = 1, 2$. To this system we attach the following boundary conditions

$$\begin{cases} u(0) = u(1) = 0, \\ v(0) = v(1) = 0. \end{cases}$$

We shall assume: there exist some positive constants $p_{ik}, q_{ik}, r_{ik}, \overline{q}_{ik}, \overline{r}_{ik}, \lambda_{ik}$ and γ_{ik}, $k = 0, 1$ such that $\forall x \in [0, 1]$,

$$(H_1) \qquad \begin{cases} p_{i1} \le p'_i(x) \le p_{i0}, \\ q_{i1} \le q_i(x) \le q_{i0}, \\ r_{i1} \le -r_i(x) \le r_{i0}, \\ \overline{q}_{i1} \le q'_i(x) \le \overline{q}_{i0}, \\ \overline{r}_{i1} \le r'_i(x) \le \overline{r}_{i0}, \end{cases}$$

and

$$(H_2) \qquad \begin{cases} \lambda_{i1} \le -\dfrac{1}{2}\left(p_i(x)q'_i(x)\right)' \le \lambda_{i0}, \\ \gamma_{i1} \le \dfrac{1}{2}\left(p_i(x)r'_i(x)\right)' \le \gamma_{i0}, \end{cases}$$

* Department of Mathematics, Faculty of Science, Al-Imam Muhammad Ibn Saud Islamic University, P. O. Box 84880, Riyadh 11681, Kingdom of Saudi Arabia, *E-mail: bougoffa@hotmail.com*

where $i = 1, 2$. Here and in all that follows: the notation (y . z′) means $\frac{d}{dx}\left(y\frac{dz}{dx}\right)$.

In [1], the author has proved a new theorem concerning the conditions for solvability of this system. Other results on the separation of this system and the afflication of Adomian decomposition method were investigated in [6].

Here, our aim is to prove the existence and uniqueness of the strong solution for the given boundary value problem associated to a coupled system of Schrödinger. The proof is based on Lax-Milgram theorem.

2. PRELIMINARIES

We reformulate the given system as the problem of solving the operator equation

$$LU = F$$

where U, LU and F are respectively the pairs:

$$U = (u, v),$$

$$LU = (\ell_1(u, v), \ell_2(u, v)),$$

and

$$F = (f, g),$$

where

$$\ell_1(u, v) = -(p_1(x)u')' + q_1(x)u - r_1(x)v$$

and

$$\ell_2(u, v) = -(p_2(x)v')' + q_2(x)u - r_2(x)v.$$

The operator L is considered from a space E into the space $L_2(0, 1) \times L_2(0, 1)$,

$$E = \{(u, v) \in (L_2(0, 1))^2 \,/\, u', v', (p_1u')', (p_2v')' \in L_2(0, 1)\},$$

where $u(0) = u(1) = 0$ and $v(0) = v(1) = 0$, with respect to the norm

$$\|U\|_E^2 = \int_0^1 \left[u^2 + v^2 + u'^2 + v'^2 + (p_1u')'^2 + (p_2v')'^2\right]dx.$$

Note that E is Hilbert space with the scalar product

$$(U,W)_E = \int_0^1 \left[u.w_1 + u.w_2 + v.w_1 + v.w_2\right]dx + \int_0^1 \left[u'.w_1' + u'.w_2' + v'.w_1' + v'.w_2'\right]dx$$
$$+\int_0^1 \left[(p_1u')'.(p_1w_1')' + (p_2v')'.(p_2w_2')'\right]dx.$$

For $W = (w_1, w_2) \in E$ define the operator Mw_i, $i = 1, 2$ by

$$Mw_i = w_i - (p_iw_i')'.$$

Consider the scalar products $(\ell_1(u, v), Mw_1)_{L_2}$ and $(\ell_2(u, v), Mw_2)_{L_2}$. Employing integration by parts, and taking into account of the given boundary conditions, we obtain

$$(\ell_1(u,v),w_1)_{L_2} = \int_0^1 \left[q_1 u.w_1 - r_1 v.w_1 + p_1 u'.w_1'\right]dx, \tag{1}$$

$$\left(\ell_2(u,v),w_2\right)_{L_2} = \int_0^1 \left[q_2 u.w_2 - r_2 v.w_2 + p_2 v'.w_1'\right]dx, \tag{2}$$

$$\left(\ell_1(u,v),-\left(p_1 w_1'\right)'\right)_{L_2} = \int_0^1 \left[p_1 q_1 u'.w_1' - p_1 r_1 v'.w_1' + (p_1 u')'.(p_1 w_1')'\right]dx \tag{3}$$

$$+\int_0^1 \left[p_1 q_1' u.w_1' - p_1 r_1' v.w_1'\right]dx,$$

and

$$\left(\ell_2(u,v),-\left(p_2 w_2'\right)'\right)_{L_2} = \int_0^1 \left[p_2 q_2 u'.w_2' - p_2 r_2 v'.w_2' + (p_2 v')'.(p_2 w_2')'\right]dx \tag{4}$$

$$+\int_0^1 \left[p_2 q_2' u.w_2' - p_2 r_2' v.w_2'\right]dx.$$

Adding (1) and (3), we obtain

$$\left(\ell_1(u,v),Mw_1\right)_{L_2} = \int_0^1 \left[q_1 u.w_1 - r_1 v.w_1 + p_1 u'.w_1'\right]dx \tag{5}$$

$$+\int_0^1 \left[p_1 q_1 u'.w_1' - p_1 r_1 v'.w_1' + (p_1 u')'.(p_1 w_1')'\right]dx$$

$$+\int_0^1 \left[p_1 q_1' u.w_1' - p_1 r_1' v.w_1'\right]dx,$$

also, adding (2) and (4), we get

$$\left(\ell_2(u,v),Mw_2\right)_{L_2} = \int_0^1 \left[q_2 u.w_2 - r_2 v.w_2 + p_2 v'.w_2'\right]dx \tag{6}$$

$$+\int_0^1 \left[p_2 q_2 u'.w_2' - p_2 r_2 v'.w_2' + (p_2 v')'.(p_2 w_2')'\right]dx$$

$$+\int_0^1 \left[p_2 q_2' u.w_2' - p_2 r_2' v.w_2'\right]dx\cdot$$

If we assume side to side (5) and (6), we get

$$\left(\ell_1(u,v),Mw_1\right)_{L_2} + \left(\ell_2(u,v),Mw_2\right)_{L_2}\cdot = \int_0^1 \left[q_1 u.w_1 - r_1 v.w_1 + p_1 u'.w_1'\right]dx + \int_0^1 \left[p_1 q_1 u'.w_1' - p_1 r_1 v'.w_1' + (p_1 u')'.(p_1 w_1')'\right]dx$$

$$+\int_0^1 \left[q_2 u.w_2 - r_2 v.w_2 + p_2 v'.w_1'\right]dx + \int_0^1 \left[p_2 q_2 u'.w_2' - p_2 r_2 v'.w_2' + (p_2 v')'.(p_2 w_2')'\right]dx$$

$$+\int_0^1 \left[p_1 q_1 u'.w_1' - p_1 r_1 v'.w_1' + (p_1 u')'.(p_1 w_1')'\right]dx + \int_0^1 \left[p_2 q_2' u.w_2' - p_2 r_2' v.w_2'\right]dx,$$

Now we are in a position to give the following definition of the strong solution as follows

Definition 1: A solution $U = (u, v) \in E$ is called a strong solution of

$$LU = F,$$

if

$$\Phi(U,W) = \Psi(W),\ \forall W = (w_1, w_2) \in E,$$

where the bilinear form $\Phi\ (U, W)$ is defined by

$$\Phi\ (U, W) = \int_0^1 \left[q_1 u.w_1 - r_1 v.w_1 + p_1 u'.w_1'\right]dx + \int_0^1 \left[p_1 q_1 u'.w_1' - p_1 r_1 v'.w_1' + (p_1 u')'.(p_1 w_1')'\right]dx$$

$$+\int_0^1 \left[q_2 u.w_2 - r_2 v.w_2 + p_2 v'.w_1'\right]dx + \int_0^1 \left[p_2 q_2 u'.w_2' - p_2 r_2 v'.w_2' + (p_2 v')'.(p_2 w_2')'\right]dx$$

$$+\int_0^1 \left[p_1 q_1' u.w_1' - p_1 r_1' v.w_1'\right]dx + \int_0^1 \left[p_2 q_2' u.w_2' - p_2 r_2' v.w_2'\right]dx,$$

and

$$\Psi(W) = \left(\ell_1(u,v), Mw_1\right)_{L_2} + \left(\ell_2(u,v), Mw_2\right)_{L_2}$$

is a linear functional.

3. EXISTENCE AND UNIQUENESS OF SOLUTION

Theorem 1: Let $F = (f(x), g\ (x)) \in L_2\ (0, 1) \times L_2\ (0, 1)$. Then there exists one and only one strong solution $W_0 = (w_{10}, w_{20}) \in E$ of problem

$$LU = F.$$

Proof: Clearly the bilinear form $\Phi(U, W)$ is a bounded bilinear functional and coercive for $U = (u, v) \in E$ and $W = (w_1, w_2) \in E$. Indeed, for $U = (u, v) \in E$ and using conditions (H_1)we get

$$\Phi(U,U) \geq d_1 \int_0^1 u^2 dx + d_2 \int_0^1 v^2 dx + d_3 \int_0^1 u'^2 dx + d_4 \int_0^1 v'^2 dx \tag{7}$$

$$+\int_0^1 \left[(p_1 q_1' + p_2 q_2')u.u' - (p_1 r_1' + p_2 r_2')v.v'\right]dx$$

$$+\int_0^1 \left[(p_1 u')'^2 + (p_2 v')'^2\right]dx,$$

where $d_1 = \min(q_{11}, q_{21})$, $d_2 = \min\ (r_{11}, r_{21})$, $d_3 = \min\ (p_{11}, p_{11}q_{11}, p_{21}q_{21})$ and $d_4 = \min(p_{11}r_{11}, p_{21}, p_{21}r_{21})$.

We observe that the following term in (7) can be expressed as

$$\int_0^1 \left[(p_1 q_1' + p_2 q_2')u.u' - (p_1 r_1' + p_2 r_2')v.v'\right]dx = \tfrac{-1}{2}\int_0^1 \left[(p_1 q_1')' + (p_2 q_2')'\right]u^2 dx$$

$$+\ \tfrac{1}{2}\int_0^1 \left[(p_1 r_1')' + (p_2 r_2')'\right]v^2 dx,$$

using conditions (H_2), we obtain

$$\Phi\ (U, U)\ \geq \beta\ \|U\|_E,$$

where $\beta = \min\ (d_i, \lambda_{11} + r_{21}, \gamma_{21} + \gamma_{21}, 1)$, $i = 1, \dots 4$.

Also, for $(f(x), g(x)) \in L_2\ (0, 1) \times L_2\ (0, 1)$,

$$\begin{aligned}\psi(W) &= \left(\ell_1(u,v), Mw_1\right)_{L_2} + \left(\ell_2(u,v), Mw_2\right)_{L_2} \\ &= \left(f, Mw_1\right)_{L_2} + \left(g, Mw_2\right)_{L_2}\end{aligned}$$

is a bounded linear functional on E. Indeed,

$$\left|\psi(W)\right| \leq \|f\|_{L_2} \|Mw_1\|_{L_2} + \|g\|_{L_2} \|Mw_2\|_{L_2}.$$

Thus

$$\left|\psi(W)\right| \leq \max(\|f\|_{L_2}, \|g\|_{L_2})\left(\|w_1\|_E + \|w_2\|_E\right).$$

So, $\psi(W) \leq \max\ (\|f\|_{L2}, \|g\|_{L2})\ \|W\|_E$. Thus by Lax-Milgram theorem, there exists a unique solution $W_0 \in E$.

The following inequality follows immediately.

Corollary 1.

$$\|W_0\|_E \leq C\, \|F\|_{L_2 \times L_2}, \forall W_0 \in E,$$

where $C > 0$ is independent on W_0

REFERENCES

[1] Cao Xuan Chuan, *Solvability of a coupled system of Schrödinger equations,* C. R. Acad. Sci. Paris, t. 319, Série II, pp. 625–630, 1994.

[2] Cao Xuan Chuan, *Exactly solvable and the concept of shape invariance,* J. Pys. A Math. Gene, 21, pp. L1165–L1174, 1991.

[3] L. Infeld and T. D. Hull, *The factorisation method,* Rev. Mod. Pys., 23, pp. 21–68, 1951.

[4] E. Written, *Dynamical breaking of supersymmetry,* Nucl. Phys. B, 185, pp. 513–554, 1981.

[5] L. E. Gedenshtein, *Derivation of exact spectra of the Schrödinger equation by means of supersymmetry,* JETP Lett., **38**, pp. 356–359, 1983.

[6] L. Bougoffa and S. Bougouffa, *Adomian method for solving some coupled systems of two equations,* Appl. Math. Comput. **177**, pp. 553–560, 2006.