Local Feedback Dissipativity through Energy Invariance and Feedback Losslessness Properties in Nonlinear Discrete-time Systems

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Abstract: Sufficient conditions under which a class of nonlinear discrete-time systems which are non-affine in the control input and the states can be rendered locally dissipative are derived. Two methodologies in order to render nonlinear single-input single-output discrete-time systems locally dissipative are given. Dissipativity is seen as a "perturbation" of the storage energy invariance or the feedback losslessness situations, in the sense that the control which makes the system dissipative is based on the control that makes the storage energy function V invariant or on the control that renders the system lossless. An example is used in order to illustrate the feedback dissipativity methodologies.

Keywords: Discrete-time Systems; Nonlinear Systems; Passivity; Feedback dissipativity.

1. INTRODUCTION

The study of dissipativity-related concepts in the nonlinear discrete-time setting is an interesting field for which many poblems remain unsolved. A main problem to consider is the establishment of conditions for a nonlinear discrete-time system to be rendered dissipative or passive via state feedback.

The energy concept is very useful in the analysis of physical systems. Many systems can be studied from its sources and losses of energy. A dissipative system is a system which can not store all the energy given to it from a external source, that is, it dissipates energy in some way. This concept can be formalized by means of the use of different energy-type functions [1], mainly, the storage function (representing the energy stored by the system), the supply function (the energy injected to the system from an external source, which restricts the manner in which the system absorbs energy) and the dissipation funtion (total energy dissipated by the system). The idea of stored energy can be connected to the system stability, considering the stored energy function as a Lyapunov-like function. In fact, dissipative systems benefit from stability properties (see [2] and references therein). The supply function is interpreted as an input power, denomination inherited from the circuit theory. Depending upon the form of the supply function, different kinds of dissipativity are obtained; passivity is the one which has attracted more attention in dynamical systems analysis.

Dissipative and passive systems have interesting properties which may simplify the system analysis and control design [1, 3]. This fact impels to transform a system which is not dissipative (passive) into a dissipative (passive) one. The action of rendering a system dissipative (passive) by means of a static state feedback is known as feedback dissipativity (feedback passivity). Systems which can be rendered dissipative (passive) are referred to as feedback dissipative (feedback passive) systems.

The establishment of conditions for a nonlinear discrete-time system to be rendered dissipative (passive) via a state feedback has not been solved yet in a general manner. This problem has been studied for the losslessness

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[4] and passivity [5] cases for systems with are affine in the control input. In [4, 5], lossless and passive systems with storage functions V such that $V(x(k+1))$ is quadratic in u are considered. Recently [6], the feedback passivity characterization has been extended to general systems without requiring $V(x(k+1))$ to be quadratic in u. The approaches given in [4, 5, 6] use the properties of the relative degree and zero dynamics of the non-passive system, as in the continuous-time case [2]. A non-general solution to the problem of local feedback dissipativity for single-input single-output (SISO) systems of general form using different methods is presented in [7, 8]. They are based on the proposal of a control that fulfills the basic dissipativity inequality. This idea is inherited from the continuous-time counterpart [9]. This paper proposes two new approaches in order to handle with the feedback dissipativity problem in a local way for nonlinear discrete-time systems. One of the approaches is based on the preliminary results given in [8].

Two procedures for dealing with the local feedback dissipativity problem are presented in this work. They are based on the establishment of the input u which satisfies the basic dissipativity inequality. The underlying idea is that dissipativity is seen as a "perturbation" of the storage-energy invariance or the systemlosslessness situations, in the sense that the control which makes the system dissipative (u) is based on the control that makes the storage energy function V invariant or on the control that renders the system lossless (u^*), and u is locally valid in a neighbourhood of u^*. The methods presented are of approximate type, due to the fact that they both are based upon the first-order Taylor series expansion at u of $V(f(x, u))$ and $s(h(x, u), v)$, with V and s the storage and the supply energy functions, respectively. The errors of the approximations are bounded, and sufficient conditions are given under which the approximations made are valid. The approaches are local since the (V, s)-dissipativity of the feedback transformed system is assured in a compact subset of $\mathcal{X} \times \mathcal{U}$ containing the fixed point of the system. The orbits of the feedback transformed system are assured not to leave this compact by means of the stability properties of the class of dissipativity treated.

The second method which renders a system dissipative by means of rendering it lossless can be also seen as an alternative way of treating the feedback losslessness problem to the one proposed in [4]. The local feedback losslessness methodology proposed is derived from the feedback dissipativity one presented in [7].

In this paper, the stabilization problem is not treated. Some stability properties of the feedback transformed system are only just mentioned. The results presented could be adapted for stabilization purposes together with techniques, such as, the ones presented in [10, 11] or the *Energy Shaping and Damping Injection*, which was proposed for the discrete-time general case in [7].

The feedback-dissipativity methodologies proposed in this paper are designed for nonlinear discrete-time systems of general form. They both can be applied in order to render dissipative linear systems. In this case, the errors of the approximations used would be smaller and some calculations would be simplified. However, some other methods proposed for linear systems would be more appropriate. In particular, for the linear case, the problem of feedback dissipativity is solved in the framework of the positive real control [12, 13] and in the framework of the robust dissipative control problem [14, 15]. In these works, dissipative systems with a supply function of the form $s(y,u) = y^T Q y + 2y^T S u + u^T R u$ are considered. The feedback passivity linear problem is also treated in [16] by using the properties of the relative degree and the zero dynamics of the non-passive system.

Section 2 gives basic definitions used in the sequel. Section 3 deals with the local feedback dissipativity problem in two steps: first, the storage energy function of the system is rendered invariant, second, the control which makes the system dissipative is obtained from an approximation of the basic dissipativity equality. Section 4 presents another methodology in order to render a system dissipative by means of a static state feedback, it also

consists of two steps: first, the system is rendered lossless, second, the control which makes the system dissipative is proposed in order to satisfy an approximation of the basic dissipativity inequality. The feedback dissipativity methodologies are illustrated by means of an example. For the example, the feedback passivity problem is treated and some local stability properties of the passified system are highlighted. Conclusions are presented in the last section.

2. PRELIMINARY DEFINITIONS

Let a nonlinear SISO discrete-time system of the form,

$$x(k+1) = f(x(k), u(k)), \ x \in \mathcal{X}, \ u \in \mathcal{U} \tag{1a}$$

$$y(k) = h(x(k), u(k)), \ y \in \mathcal{Y} \tag{1b}$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ are smooth maps with $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U}, \mathcal{Y} \subset \mathbb{R}$ open sets. $k \in \mathbb{Z}_+ := \{0, 1, 2, ...\}$. All considerations will be restricted to an open set of $\mathcal{X} \times \mathcal{U}$ containing $(\overline{x}, \overline{u})$, having $\overline{x}$ as an isolated fixed point of $f(x, \overline{u})$, with $\overline{u}$ a constant, i.e., $f(\overline{x}, \overline{u}) = \overline{x}$. A positive definite smooth function $V : \mathcal{X} \to \mathbb{R}$, associated with system (1) and addressed as the *storage function* is considered. Function V is considered to have a strict local minimum in $\overline{x}$. A second smooth function is also taken into account, referred to as the *supply function*, and denoted by $s(y, u)$, with $s : \mathcal{Y} \times \mathcal{U} \to \mathbb{R}$.

Definition 1: The supply function $s(y,u)$ is said to satisfy the zero-input-output (ZIO) property if

$$s(0, u) = 0, \ \forall \ u \in \mathcal{U}, \ \ s(y, 0) = 0, \ \forall \ y \in \mathcal{Y} \tag{2}$$

Definition 2: A smooth function $\phi : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, such that $\phi(\cdot, u)$ is positive (respectively, strictly positive) for each $u \in \mathcal{U}$, with $\phi(0, 0) = 0$ is referred to as a dissipation rate (resp., strict dissipation rate) function in the sense proposed in [3, 7].

The dissipativity definition in the discrete-time nonlinear setting given in [4] will be rewritten in the following way.

Definition 3: [7] System (1) with storage function $V(x)$ and supply function $s(y, u)$ is said to be locally (V, s)-dissipative (resp., locally strictly (V, s)-dissipative) if there exists a dissipation rate (resp., strict dissipation rate) function ϕ such that

$$V(f(x,u)) - V(x) = s(h(x,u),u) - \phi(x,u), \ \forall (x,u) \in \mathcal{X} \times \mathcal{U} \tag{3}$$

The system is said to be locally V-passive if it is locally (V, s)-dissipative with a supply function of the form $s(y,u) = yu$. The system is said to be locally (V, s)-lossless if $\phi(x, u) = 0, \ \forall (x, u)$.

Equality (3) can be also written as

$$V(f(x, u)) - V(x) \le s(h(x, u), u) \tag{4}$$

Let $\alpha : \mathcal{X} \times \mathcal{U} \to \mathcal{U}$ be a $\mathcal{C}^1$ function with $\alpha(0, 0) = 0$. A nonlinear static state feedback control law is denoted by the expression $u = \alpha(x, v)$. The system $x(k+1) = f(x(k), \alpha(x(k), v(k)))$ is referred to as the feedback transformed system, which may be also denoted by $x(k+1) = \overline{f}(x(k), v(k))$. In addition, $\overline{h}(x, v)$ denotes the function $h(x, \alpha(x, v))$.

Definition 4: A feedback control law $u = \alpha(x, v)$ is locally regular if for all $(x, v) \in \mathcal{X} \times \mathcal{U}$, it follows that $\supset \alpha = \supset v \ne 0$.

Definition 5: Consider system (1) and two scalar functions $V(x)$ and $s(y, v)$ as a storage function and a supply function, respectively. The system is said to be locally feedback (V, s)-dissipative (resp., locally feedback stricly (V, s)-dissipative) with the functions V and s if there exists a locally regular static state feedback control law of the form $u = \alpha(x, v)$, with v as the new input, such that the feedback transformed system is locally (V, s)-dissipative (resp., locally strictly (V, s)-dissipative). A system of the form (1) is said to be locally feedback V-passive if it is locally feedback (V, s)-dissipative with $s(y,v) = yv$.

3. LOCAL FEEDBACK DISSIPATIVITY THROUGH THE ENERGY INVARIANCE SITUATION

3.1 Description of the Methodology

Consider system (1). Suppose that there exists a control $u^* : \mathcal{X} \to \mathcal{U}$ such that

$$V(f(x, u^*)) - V(x) = 0, \ \forall (x, u^*) \in \mathcal{X} \times \mathcal{U} \tag{5}$$

with V a storage function. Let a function $\delta u^*(x, u^*v)$ such that $\delta u^* : \mathcal{X} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$. Define the following state-dependent input coordinate transformation,

$$u = u^*(x) + \delta u^*(x, u^*, v) \tag{6}$$

where,

$$\delta u^*(x, u^*, v) = \frac{s(h(x, u^*), v) - \phi(x, u^*)}{\frac{\partial}{\partial u}\left[V(f(x, u)) + \phi(x, u) - s(h(x, u), v)\right]\Big|_{u = u^*}}, \tag{7}$$

with v the new input to the system, s a supply function, and ϕ acting as a dissipation rate function.

Proposition 1: Consider $V(x)$ as a storage function. Let $(x_0, u_0^*) \in \mathcal{X} \times \mathcal{U}$. Suppose that the following conditions are satisfied:

$$\exists (x_0, u_0^*) : V(f(x_0, u_0^*)) - V(x_0) = 0 \tag{8}$$

$$\frac{\partial}{\partial u^*} V(f(x, u^*))\Big|_{(x_0, u_0^*)} \neq 0 \tag{9}$$

Then, there exists a unique static state feedback control law of the form $u^* = \alpha(x)$ defined in a neighbourhood of x_0, $\widetilde{\mathcal{X}} \subset \mathcal{X}$, and valued in a neighbourhood of u_0^*, $\widetilde{\mathcal{U}} \subset \mathcal{U}$, such that equation (5) is satisfied with $u^* = \alpha(x)$, for all $x \in \widetilde{\mathcal{X}}$.

Proof: The proof follows from the implicit function theorem.

Proposition 2: Let $V(x)$ and $s(y,v)$ be a storage function and a supply function, respectively. Suppose that conditions (8)-(9) are satisfied. Let $\bar{x}$ be an isolated fixed point of $f(x, \bar{u})$, with $\bar{u}$ a constant. Let $\widetilde{\mathcal{X}} \subset \mathcal{X}$ and $\widetilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{\mathcal{P}} \subset \mathcal{U}$ be compact sets containing $\bar{x}$ and $\bar{u}$, respectively. Then, system (1) is locally feedback (V, s)-dissipative with the functions V and s, by means of a feedback of the form (6), with $|\delta u^*| \cdot \rho, \rho > 0$ small enough, $u^* : \tilde{\mathcal{X}} \to \tilde{\mathcal{U}}$ obtained from (5), $\delta u^* : \widetilde{\mathcal{X}} \times \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \to \tilde{\mathcal{P}}$ given by (7) and $u^* + \delta u^*$ defined in a neighbourhood of u^* if there exists a dissipation rate function $\phi(x, u)$ for which

$$\left| \frac{s(h(x,u^*),v) - \phi(x,u^*) - \min(R_V + R_\phi) + \max R_s}{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\Big|_{u=u^*}} \right| \le \rho \tag{10}$$

with R_V, R_ϕ and Rs the remainder of the Taylor expansion of $V(f(x,u^* + \delta u^*))$, $\phi(x, u^* + \delta u^*)$ and $s(h(x,u^* + \delta u^*),v)$ at u^*, respectively.

Proof: Let consider the control $u = u^*(x) + \delta u^*(x,u^*,v)$ which is applied to system (1), then

$$x(k+1) = f(x(k),u^*(k) + \delta u^*(k))$$
$$y(k) = h(x(k),u^*(k) + \delta u^*(k)) \tag{11}$$

with $u^*(x)$ such a control that makes the system orbits lie on the level surfaces of V and $|\delta u^*| \le \rho$, $\rho > 0$. Control δu^* is proposed in such a way that makes the system (11) be (V, s)-dissipative, i.e.,

$$V(f(x,u^*+\delta u^*)) - V(x) = s(h(x,u^*+\delta u^*), v) - \phi(x,u^*+\delta u^*) \tag{12}$$

with v the new input. Considering the first-order Taylor approximation at u^* of (12), it is obtained

$$V(f(x,u^*)) + \frac{\partial}{\partial u}V(f(x,u))\Big|_{u=u^*} \delta u^* + R_V - V(x) =$$

$$= s(h(x,u^*),v) + \frac{\partial}{\partial u}s(h(x,u),v)\Big|_{u=u^*} \delta u^* + R_s - \tag{13}$$

$$- \left\{\phi(x,u^*) + \frac{\partial}{\partial u}\phi(x,u)\Big|_{u=u^*} \delta u^* + R_\phi\right\}$$

Taking into account that for $u^* = \alpha(x)$, $V(f(x, u^*)) - V(x) = 0$, $\forall x \in \widetilde{\mathcal{W}}$, then the control δu^* which renders the system locally (V, s)-dissipative is obtained from (13) and takes the form,

$$\delta u^* = \frac{s(h(x,u^*),v) - \phi(x,u^*) - (R_V + R_\phi - R_s)}{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\Big|_{u=u^*}} \tag{14}$$

First of all, for the existence of control δu^*, it is necessary that the denominator of (14) is not equal to zero $\forall (x, u^*, v) \in \widetilde{\mathcal{W}} \subset \mathcal{X} \times \mathcal{U} \times \mathcal{U}$. If condition (9) holds, this fact will be satisfied for an open neighbourhood $\widetilde{\mathcal{W}}$.

Besides, control δu^* is needed to be bounded and small enough in order to have $u^* + \delta u^*$ defined in a neighbourhood of u^*. Therefore, from (14),

$$\left|\delta u^*\right| \le \left| \frac{s(h(x,u^*),v) - \phi(x,u^*) - \min(R_V + R_\phi) + \max R_s}{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\Big|_{u=u^*}} \right| \le \rho$$

Indeed, function $|\delta u^*|$ is bounded if s, ϕ, the residues R_V, R_s, R_ϕ and the denominator of (14) are bounded. Due to the fact that functions V, s, ϕ are smooth and defined in compact sets, they are bounded.

To conclude with, if condition (10) is satisfied then $u^* + \delta u^*$ with u^* obtained from equation (5) and δu^* given by (7) is valid and system (1) is rendered locally (V, s)-dissipative. Note that in addition to condition (10), which assures that δu^* is bounded, control δu^* must be small enough.

Remark 1: The orbits of the feedback transformed system are assured not to leave the compact $\widetilde{\mathcal{X}}$, where (V, s)-dissipativity (strictly (V, s)-dissipativity) is achieved, if they start in $\widetilde{\mathcal{X}}$. The idea is the following one. If for a $(x^*, v^*) \in \widetilde{\mathcal{X}} \times \widetilde{\mathcal{V}}$ of the system $x(k+1) = \bar{f}(x, v)$, $y = \bar{h}(x, v)$, we have that $s(\bar{h}(x^*,v^*),v^*) = 0$, and (x^*,v^*) is closed enough to the fixed point $(\bar{x},\bar{u})$, then there exists a neighbourhood $\mathcal{W} \subset \widetilde{\mathcal{X}} \times \widetilde{\mathcal{V}}$ of (x^*,v^*) containing $(\bar{x},\bar{u})$ where $s(\bar{h}(x, v),v) = 0$, $\forall (x,v) \in \mathcal{W}$, and ≤ 0 (resp., $V(x(k+1)) - V(x(k)) < 0)$, then there is a region where the fixed point is stable (resp., asymptotically stable). If $v^* \neq 0$, the function v should be bounded, and will be established from the relation $s(\bar{h}(x,v), v) = 0$. The bounds of v will depend on the compacts where (V, s)-dissipativity is assured.

Remark 2: For the validity of this method, it is necessary to check how good the first-order Taylor approximations at u^* used for $V(f(x, u^* + \delta u^*))$, $\phi(x,u^* + \delta u^*)$ and $s(h(x,u^* + \delta u^*),v)$ are. Then, the validity of the method can be also tested by means of the boundedness of R_V, R_ϕ and R_s.

Remark 3: If $\delta u^*(k) \to 0$ *as* $k \to \infty$ then R_V, R_s *and* R_ϕ tend to zero [17], and consequently, the approximation of the feedback dissipativity equality is valid.

3.2 An Illustrative Example

The feedback dissipativity methodology shown above will be illustrated by means of the feedback passivity of the following system extracted from [18],

$$\begin{aligned}
x_1(k+1) &= \left[x_1^2(k) + x_2^2(k) + u(k) \right] \cos\left[x_2(k) \right] \\
x_2(k+1) &= \left[x_1^2(k) + x_2^2(k) + u(k) \right] \sin\left[x_2(k) \right] \\
y(k) &= x_1^2(k) + x_2^2(k) + u(k)
\end{aligned} \tag{15}$$

System (15) will be rendered locally V-passive with the storage function $V = x_1^2 + x_2^2$ and the supply function $s(y,v) = yv$. The dissipation rate function $\phi(x, u)$ is chosen in order to collect the positive terms appearing in $V(x(k + 1))$, that is

$$\phi(x,u) = \mu\left[(x_1^2 + x_2^2)^2 + u^2 + x_1^2 + x_2^2 \right] \tag{16}$$

with μ a positive constant. Let $x_1 \in [-\varepsilon_{x_1}, \varepsilon_{x1}]$, $x_2 \in [-\varepsilon_{x_2}, \varepsilon_{x_2}]$, $u, u^*, v \in [-\varepsilon_u, \varepsilon_u]$, $\delta u^* \in [-\rho, \rho]$, with $\varepsilon_{x_1}, \varepsilon_{x_2}, \varepsilon_u$, ρ positive constants. In order to obtain the control u^*, equation (5) for the example, takes the following form,

$$a_{u^*}(u^*)^2 + b_{u^*}(x_1, x_2)u^* + c_{u^*}(x_1, x_2) = 0 \tag{17}$$

with $a_{u^*} = 1$, $b_{u^*}(x_1, x_2) = 2(x_1^2 + x_2^2)$, $c_{u^*}(x_1, x_2) = (x_2^1 + x_2^2)^2 - (x_1^2 + x_2^2)$.

If sufficient conditions (8)-(9) for the existence of control u^* are met for (15) for some (x_1, x_2, u^*) then there exists a control u^* satisfying (5). This u^* can be obtained from the explicit solution of (17), that is,

$$u_{11}^*(x_1,x_2) = \frac{-b_{u^*} + \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}}, u_{12}^*(x_1,x_2) = \frac{-b_{u^*} - \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}} \tag{18}$$

It is necessary that $b_{u^*}^2 - 4a_{u^*}c_{u^*} \geq 0$, which will be always achieved.

Concerning the computation of control δu^*, condition (10) is needed to be assured. In other words, it is necessary to verify that all the terms appearing in (10) are bounded. As V, s and ϕ are continuous with continuous derivatives, and their variables vary in a compact set, they are bounded [17]. The remainders R_V, R_s and R_ϕ must be also bounded. The expressions for these errors are the following ones: $R_V = (\delta u^*)^2$, $R_s = 0$, $R_\phi = \mu(\delta u^*)^2$. It can be noticed that the errors of the approximation depends on δu^*. If δu^* is bounded then R_V and R_ϕ are bounded in the sets where x, u^* and v are defined. The errors of the approximation tend to zero as long as δu^* does.

In addition, an upper bound for μ is needed. The maximum value for μ for control δu^* to be valid is such that relation (10) is met. This value will depend on the maximum and minimum values of the states and the controls, i.e., on the constants ε_{x1}, ε_{x2}, ε_u, ρ. Then, it can be concluded that for any $\rho > 0$, $\varepsilon_{x1} > 0$, $\varepsilon_{x2} > 0$, $\varepsilon_u > 0$ there exists $\bar{\mu} > 0$ such that,

$$0 < \mu < \bar{\mu}\,(\rho,\ \varepsilon_{x1},\ \varepsilon_{x2},\ \varepsilon_u) \tag{19}$$

Therefore, control δu^* is given by expression (7), i.e.,

$$\delta u^*(x_1, x_2, u^*, v) = \frac{(x_1^2 + x_2^2 + u^*)v - \mu\left[(x_1^2 + x_2^2)^2 + (u^*)^2 + x_1^2 + x_2^2\right]}{2u^*(1+\mu) + 2\left(x_1^2 + x_2^2\right) - v} \tag{20}$$

with $\mu > 0$ and satisfying (10), and control u^* as defined in (18). The control which renders the system locally V-passive is given by $u = \alpha\,(x,\ v) = u^* + \delta u^*$. Two solutions for the feedback passivity control can be considered: one for $u^* = u_{11}^*$, another for $u^* = u_{12}^*$.

Remark 4: The response of the feedback transformed system depends on μ appearing in function ϕ. Parameter μ acts as the damping coefficient of the system.

4. LOCAL FEEDBACK DISSIPATIVITY THROUGH THE FEEDBACK LOSSLESSNESS PROPERTY

The methodology proposed in this section in order to render a system locally $(V,\ s)$-dissipative by means of a static feedback follows a similar approach than the one presented in Section 3. However, it is based on the basic dissipativity inequality (4), indeed, on an approximation of the dissipativity inequality by using the first-order Taylor series expansion at u of $V(f\,(x,\ u))$ and $s(h(x,\ a),\ v)$. It consists of two steps. First, the system is rendered lossless. Second, the control which renders the system locally $(V,\ s)$-dissipative will be a "perturbation" of the one which renders it locally $(V,\ s)$-lossless.

The feedback losslessness methodology proposed is derived from the feedback dissipativity one presented in [7].

4.1 Description of the Methodology

Before proposing the control which renders system (1) locally $(V,\ s)$-dissipative, the feedback losslessness problem is defined in terms of the dissipativity inequality.

Definition 6: Consider a system of the form (1) and two scalar functions $V(x)$ and $s(y,\ v)$ considered as a storage function and a supply function, respectively. The system is said to be locally feedback $(V,\ s)$-lossless with the functions V and s, if there exists a regular static state feedback control law of the form, $u = \alpha(x,\ v)$, with v the new input, such that the feedback transformed system is locally $(V,\ s)$-lossless.

The existence of a feedback control law of the form $u^* = \alpha(x, v)$ for which the system is rendered locally (V, s)-lossless must be assessed from the existence of solutions, for the control input u^*, of the following equation,

$$V(f(x,u^*)) - V(x) = s(h(x, u^*), v) \tag{21}$$

The following proposition states sufficient conditions under which local feedback losslessness is possible.

Proposition 3: Consider a system of the form (1) and two scalar functions $V(x)$ and $s(y, v)$ considered as a storage function and a supply function, respectively. Let $(x_0, u_0^*, v_0) \in \mathcal{X} \times \mathcal{U} \times \mathcal{U}$. Suppose that the following two conditions are satisfied:

$$\exists (x_0, u_0^*, v_0): V(f(x_0, u_0^*)) - V(x_0) - s(h(x_0, u_0^*), v_0) = 0 \tag{22}$$

$$\frac{\partial}{\partial u^*}\left[V(f(x,u^*)) - s(h(x,u^*), v)\right]\Bigg|_{(x_0, u_0^*, v_0)} \neq 0 \tag{23}$$

Then, there exists a unique static state feedback control law of the form $u^* = \alpha^*(x, v)$ defined in a neighbourhood of (x_0, v_0) and valued in a neighbourhood of u_0^* such that the feedback transformed system $x(k + 1) = \bar{f}(x(k), v(k))$, $y(k) = \bar{h}(x(k), v(k))$ is locally (V, s)-lossless.

Proof: The proof follows from the implicit function theorem.

Consider system (1). Suppose that there exists a regular static state feedback $u^*: \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{U}$ which renders the system locally (V, s)-lossless. Let a function $\delta u^*(x, u^*, v)$ such that $\delta u^*: \mathcal{X} \times \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{U}$. Define the following state-dependent input coordinate transformation,

$$u = u^*(x, v) + \delta u^*(x, u^*, v) \tag{24}$$

with,

$$\delta u^*(x, u^*, v) = -\mu \frac{\partial}{\partial u}\left[V(f(x,u)) - s(h(x,u), v)\right]\Bigg|_{u=u^*} \tag{25}$$

where μ is a positive constant.

Proposition 4: Let $V(x)$ and $s(y, v)$ be a storage function and a supply function, respectively. Suppose that conditions (22)-(23) are satisfied. Let $\widetilde{\mathcal{X}} \subset \mathcal{X}$ and $\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}, \widetilde{\mathcal{P}} \subset \mathcal{U}$ be compact sets containing $\bar{x}$ and $\bar{u}$, respectively. Then, system (1) is locally feedback (V, s)-dissipative with the functions V and s by means of a feedback of the form (24), with $u^*: \widetilde{\mathcal{X}} \times \widetilde{\mathcal{V}} \rightarrow \widetilde{\mathcal{U}}$ obtained from (21), $\delta u^*: \widetilde{\mathcal{X}} \times \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \rightarrow \widetilde{\mathcal{P}}$ given by (25), and $u^* + \delta u^*$ defined in a neighbourhood of u^* if there exists a positive constant μ, for which the following conditions are satisfied

$$\left|\max R_V - \min R_s\right| \leq \mu \left\{\frac{\partial}{\partial u}\left[V(f(x,u)) - s(h(x,u), v)\right]\Bigg|_{u=u^*}\right\}^2 \tag{26}$$

$$\mu \left|\frac{\partial}{\partial u}\left[V(f(x,u)) - s(h(x,u), v)\right]\Bigg|_{u=u^*}\right| \leq \rho \tag{27}$$

with ρ a positive constant small enough, and R_V, R_s the remainder of the Taylor series expansion of $V(f(x, u^* + \delta u^*))$ and $s(h(x, u^* + \delta u^*), v)$ at u^*, respectively.

Proof: Let consider the control $u = u^* + \delta u^*$ which is applied to system (1), with u^* such a control that makes the system locally (V, s)-lossless with V and s as storage energy and supply functions, respectively. Control δu^* is proposed in such a way that makes the feedback transformed system be locally (V, s)-dissipative (i.e., it satisfies (4)), with v the new input. By considering the first-order Taylor approximation at u^* of $V(f(x, u^* + \delta u^*))$ and $s(h(\text{x}, u^* + \delta u^*), v)$, one yields to

$$V(f(x,u^*)) + \frac{\partial}{\partial u}V(f(x,u))\bigg|_{u=u^*} \delta u^* + R_V - V(x) \leq$$

$$\leq s(h(x,u^*),v) + \frac{\partial}{\partial u}s(h(x,u),v)\bigg|_{u=u^*} \delta u^* + R_s \qquad (28)$$

From (21) and (25), relation (28) takes the form,

$$\frac{\partial}{\partial u}\big[V(f(x,u)) - s(h(x,u),v)\big]\bigg|_{u=u^*} \delta u^* - R_s + R_V$$

$$\leq -\mu\left\{\frac{\partial}{\partial u}\big[V(f(x,u)) - s(h(x,u),v)\big]\bigg|_{u=u^*}\right\}^2 - \min R_s + \max R_V \leq 0$$

which is assured by means of condition (26).

Control δu^* must be also bounded and small enough in order to have $u^* + \delta u^*$ defined in a neighbourhood of u^*. This holds if (27) holds with r small enough. Indeed, due to the fact that the smooth functions V, s are defined in compact sets, it is the same for their derivatives and consequently, they are bounded [17].

Summing up, if conditions (26)-(27) are satisfied then u given by (24), with u^* obtained from (21) and δu^* given by (25), renders system (1) locally $(V; s)$-dissipative. In addition, the orbits of the feedback transformed system are assured not to leave the compact $\widetilde{\mathcal{X}}$, where (V, s)-dissipativity is achieved, if they start in $\widetilde{\mathcal{X}}$, as it was explained in Section 3.

Remark 5: For the validity of this method, it is necessary to check how good the first-order Taylor approximations used are, that is, the boundedness of R_V and R_s.

Remark 6: For systems which are non-affine in the control input, an explicit expression for u^* would not be possible to obtain in all cases. In this case, the control u^* should be obtained by means of iterative-like methods.

4.2 An Illustrative Example

The feedback dissipativity methodology presented will be applied to render system (15) locally V-passive. Equation (21) is used in order to obtain the control u^*,

$$a_{u*}(u^*)^2 + b_{u*}(x_1, x_2, v)u^* + c_{u*}(x_1, x_2, v) = 0 \qquad (29)$$

with, $a_{u*} = 1$, $b_{u*}(x_1, x_2, v) = 2(x_1^2 + x_2^2) - v$, $c_{u*}(x_1, x_2, v) = (x_1^2 + x_2^2)^2 - (x_1^2 + x_2^2) - (x_1^2 + x_2^2)v$. If sufficient feedback losslessness conditions (22)-(23) are met for (15) for some (x_1, x_2, u^*, v) then there exists a control u^* satisfying (29). This u^* can be obtained from the explicit solution of (29), that is,

$$u_{21}^*(x_1,x_2,v) = \frac{-b_{u^*} + \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}}, u_{22}^*(x_1,x_2,v) = \frac{-b_{u^*} - \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}} \tag{30}$$

It is necessary to assure that $b_{u^*}^2 - 4a_{u^*}c_{u^*} \geq 0$, which will be always achieved.

Concerning the computation of control δu^*, conditions (26)-(27) must be verified. They will be achieved by means of choosing an appropriate value of μ. Considering $R_v = (\delta u^*)^2$, $R_s = 0$ and taking into account (25), conditions (26) and (27) take the following form, respectively,

$$\mu \leq \frac{\min\left\{\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\}}{\max\left\{\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\}} \tag{31}$$

$$\mu \leq \frac{\rho}{\max\left|2(u^*+x_1^2+x_2^2)-v\right|} \tag{32}$$

for some positive constant ρ. An upper bound of constant μ can be established from (31) and (32), and will depend on the bounds of the states and the controls. It is also necessary to give a bound for v in order to ensure that the orbits of the feedback transformed system will remain in the compacts where the feedback dissipativity is considered. An option is proposing v in such a way to have $s(\bar{h}(x,v), v) = (x_1^2+x_2^2+u^*+\delta u^*)v = 0$. The control u^* is approximated by its linearization at $x_1 = x_2 = v = 0$, obtaining $u^* = \frac{1}{2}v$, and using this in $s(\bar{y}, v) = 0$, it is obtained that $v = 0$ or,

$$v = -2(1-2\mu)(x_1^2+x_2^2) \tag{33}$$

The control v will be bounded by means of μ, x_1 and x_2. Considering (32) and the fact that ρ must be small enough, the denominator appearing in (32) is concluded to be greater than ρ and, consequently, μ will be less than one. Then, from (33), one yields to,

$$|v| < 2(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2) \tag{34}$$

A value for ε_v is proposed as $\varepsilon_v = 2(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2)$, and relations (31) and (32) yield to,

$$\mu \leq \left[\frac{-\varepsilon_u}{2(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2)+\varepsilon_u}\right]^2 = \bar{\mu}_1 \tag{35}$$

$$\mu \leq \frac{\rho}{4(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2)+2\varepsilon_u} = \bar{\mu}_2 \tag{36}$$

Finally, an upper bound for μ is proposed as follows,

$$0 < \mu \leq \min(\bar{\mu}_1, \bar{\mu}_2) \tag{37}$$

This bound of μ ensures conditions (26)-(27) to hold and control v to be bounded.

Remark 7: Inequality (37) is only a sufficient condition for conditions (26)-(27) to be fulfilled. There is possible to be higher values of μ for the method to be applicable.

Under the conditions studied, control δu^* is given by expression (25), i.e.,

$$\delta u^*(x_1, x_2, u^*, v) = -\mu \left[2(u^* + x_1^2 + x_2^2) - v \right] \tag{38}$$

with μ satisfying (37), and control u^* as defined by equation (29), for which two possibilities are the controls (30). The control which renders the system locally V-passive is given by $u = \alpha(x, v) = u^* + \delta u^*$.

4.3 A Note on the Local Stability Properties of the Passified System

If a system of the form (1) is rendered locally (V, s)-dissipative (locally strict (V, s)- dissipative), with s satisfying the ZIO property, by means of a nonlinear regular static state feedback control law of the form $u = \alpha(x,v)$, the control law $u = \alpha(x, 0)$ makes the system fixed point locally stable (locally asymptotically stable), see [7]. In other words, the control $u = \alpha(x,v)$ induces an implicit damping injection which renders the system fixed point locally stable (locally asymptotically stable) when the new input v is zero.

In order to illustrate this stability property, the feedback dissipativity methodology presented in Section 4.1 will be used in the example. The feedback passivity control $\alpha(x, v)$ with $v = 0$ is applied to system (15) using $\alpha_1(x, 0) = u_{21}^*(x, 0) + \delta u_{21}^*(x, u_{21}^*, 0)$, and the system converges to the fixed point $(0, 0)$, which is unstable in open loop. It is considered $\varepsilon_{x_1} = 1$, $\varepsilon_{x_2} = 0.01$, $\varepsilon_u = 1.9$, $\rho = 1.8$ and the values for the upper bounds of μ are given by $\bar{\mu}_1 = 0.2373195$ and $\bar{\mu}_2 = 0.2307574$, consequently, admissible values of μ for which the feedback passivity method is valid are $\mu \in (0, \bar{\mu}_2]$. The response of the system for $\alpha_1(x, 0)$ and different values for the constant μ is depicted in Figure 1 considering $x_0 = (-1, 0.01)^T$. Changes in the constant μ influence the response of the feedback transformed system.
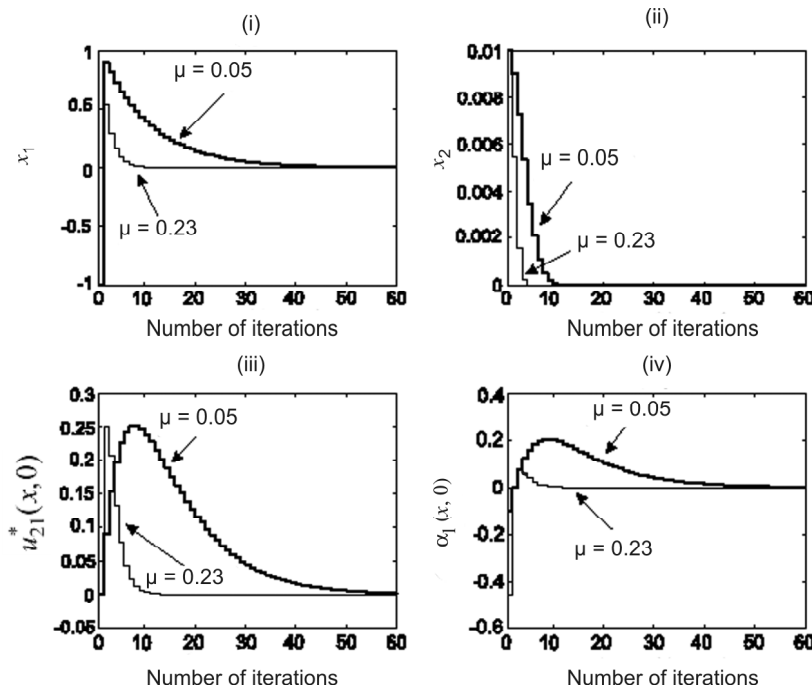

Figure 1: Response of System (15) Passified by Means of Feedback (24) considering $v = 0$ for μ = 0.05 and μ= 0.23: (i) x_1 (ii) x_2 (iii) Control which Renders the System V-lossless $u_{21}^*(x, 0)$ (iv) Feedback Passivity Control for $v = 0$, $\alpha_1(x, 0) = u_{21}^*(x, 0) + \delta u_{21}^*(x, u_{21}^*, 0)$

Now, the approximation made for function $V(f(x, u))$, with $u = \alpha(x, 0)$ is analysed. As $v = 0$, the function $s(h(x,\alpha(x,0)),0) = 0$ and there is no need to study its first-order Taylor approximation. Let us define functions,

$$V_1 = V(f(x,u^* + \delta u^*))$$

$$V_2 = V(f(x,u^*)) + \frac{\partial}{\partial u}V(f(x,u))\bigg|_{u=u^*} \delta u^* \tag{39}$$

As δu^* tends to zero in the steady state, the approximation errors are zero and the approximations made are valid. In Figure 2, functions V_1, V_2 are compared graphically, for different values of μ: $\mu = 0.05$, $\mu = 0.23$, with a1$(x, 0)$. It can be noticed that unless the steady state is reached, V_2 is not equal to V_1. When the system has reached its steady state, $\delta u^* = 0$, $V_2 = V_1$ is obtained. The smaller in modulus the value of δu^* is, the better the approximation is, that is why the feedback transformed system response with $\mu = 0.23$ gives the worst approximation V_2, however, with $\mu = 0.23$, $V_1 - V_2$ gets zero sooner due to the fact that the response of the system is faster than the response with $\mu = 0.05$.
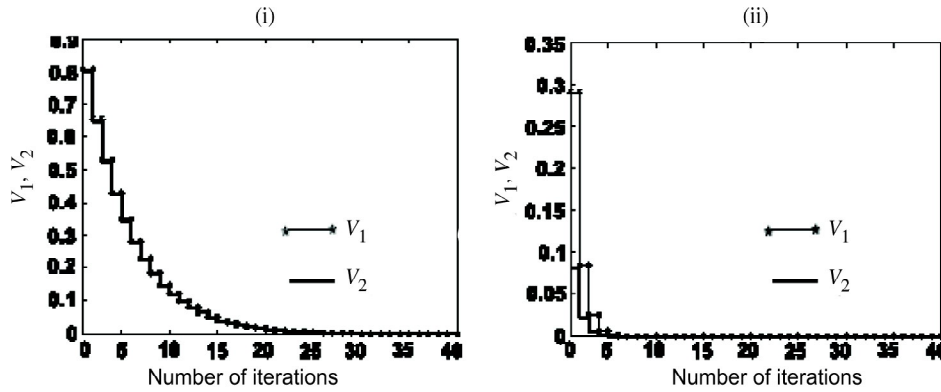

Figure 2: Analysis of the Approximation for $V(f(x, \alpha_1(x, 0))$: **(i)** V_1, V_2 **for** $\mu = 0.05$ **(ii)** V_1, V_2 **for** $\mu = 0.23$,

with $V_1 = V(f(x, u^* + \delta u^*))$, $V_2 = V(f(x,u^*)) + \frac{\partial}{\partial u}V(f(x,u))\big|_{u=u^*} \delta u^*$

5 CONCLUSIONS

Two approaches for dealing with the local feedback dissipativity problem in a class of nonlinear discrete-time systems which are non-affine in the control input and the states have been proposed. The feedback dissipativity schemes are of approximate type since they are based on a first-order Taylor approximation of the basic dissipativity inequality. The first one achieves the feedback dissipativity goal by means of the storage energy invariance, whereas the second one proposes dissipativity as a "perturbation" of the system losslessness situation. Sufficient conditions under which the approximation considered is valid have been presented.

The main problem of the feedback dissipativity methodologies presented is that control $u^* + \delta u^*$ is locally valid in a neighbourhood of u^*. The approximations of $V(f(x, u))$, $s(h(x, u), v)$ could be improven, using a higher order Taylor approximation type. Alternative methods in order to compute u^* may be proposed. A geometric interpretation of the underlying idea of the two methods would be interesting.

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REFERENCES

[1] J. C. Willems, Dissipative dynamical systems. Part I: general theory. Part II: linear systems with quadratic supply rates, *Archive for Rational Mechanics and Analysis*, Vol. **45** (5), pp. 321–351, 352–393, 1972.

[2] C. I. Byrnes, A. Isidori and J.C. Willems, Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems, *IEEE Transactions on Automatic Control*, Vol. **36**, pp. 1228–1240, 1991.

[3] D. J. Hill and P. Moylan, Dissipative dynamical systems: basic input-output and state properties, *Journal of the Franklin Institute*, Vol. **309**, pp. 327–357, 1980.

[4] C. I. Byrnes and W. Lin, Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems, *IEEE Transactions on Automatic Control*, Vol. **39**(1), pp. 83–98, 1994.

[5] E. M. Navarro-López, *QSS*-dissipativity and feedback *QS*-passivity of nonlinear discrete-time systems, Dynamics of Continuous, Discrete and Impulsive Systems-Series B: Applications and Algorithms, Vol. **14**(1), pp. 47–63, 2007.

[6] E. M. Navarro-López and E. Fossas-Colet, Feedback passivity of nonlinear discrete-time systems with direct input-output link, Automatica, Vol. **40**(8), pp. 1423–1428, August, 2004.

[7] E. M. Navarro-López, H. Sira-Ramírez and E. Fossas-Colet, Dissipativity and feedback dissipativity properties of general nonlinear discrete-time systems, *European Journal of Control. Special Issue: Dissipativity of Dynamical Systems. Application in Control*, Vol. **8**(3), pp. 265–274, 2002.

[8] E. M. Navarro-López, Dissipativity and Passivity-related Properties in Nonlinear Discrete-time systems, Ph.D. Thesis, Universitat Politècnica de Catalunya, Barcelona, Spain, 2002. ISBN: 84-688-1941-7. Available online at: www.tdx.cesca.es/TDCat-0328103-104942/.

[9] H. Sira-Ramírez, A general canonical form for feedback passivity, International Journal of Control, Special Issue: Recent Advances in the Control of Nonlinear Systems, Vol. **71**(5), pp. 891–905, 1998.

[10] Z. P. Jiang, Y. Lin and Y. Wang, Nonlinear small-gain theorems for discretetime feedback systems and applications, *Automatica*, Vol. **40**(12), pp. 2129–2136, December, 2004.

[11] W. Lin and C.I. Byrnes, Passivity and absolute stabilization of a class of discrete-time nonlinear systems, *Automatica*, Vol. **31**(2), pp. 263–267, 1995.

[12] C. E. de Souza and L. Xie, On the discrete-time bounded real lemma with application in the characterization of static state feedback H_∞ controllers, *Systems and Control Letters*, Vol. **18**, pp. 61–71, 1992

[13] C. E. de Souza, M. Fu and L. Xie, H_∞ analysis and synthesis of discrete-time system with time-varying parameter uncertainty, *IEEE Transactions on Automatic Control*, Vol. **38**, pp. 459–462, 1993.

[14] Z. Tan, Y. C. Soh and L. Xie, Dissipative control for linear discrete-time systems, *Automatica,* Vol. **35**, pp. 1557–1564, 1999.

[15] Z. Tan, Y. C. Soh and L. Xie, Dissipative control of linear discrete-time systems with dissipative uncertainty, *International Journal of Control*, Vol. **73**(4), pp. 317–328, 2000.

[16] E. M. Navarro-López, Several dissipativity and passivity implications in the linear discrete-time setting, *Mathematical Problems in Engineering*, Vol. **5**, pp. 599–616, 2005.

[17] J. E. Marsden and M. J. Hoffman, Elementary Classical Analysis. W. H. Freeman & Co., 1993, 2nd edn.

[18] H. Sira-Ramírez, Non-linear discrete variable structure systems in quasisliding mode, *International Journal of Control*, Vol. **54**(5), pp. 1171–1187, 1991.