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Synchronization in a Generally Coupled Dynamical Systems

Xiong Xiaohua, Hong Shaonan Hong* & Zhou Tianshou**

Abstract: In this paper we propose a *mode decomposition* method in studying synchronization in a generally coupled dynamical system. By using this method the stability of the synchronous state equals to the stability of zero solutions of the linearized variational equations. The investigation leads to a sufficient condition which guarantees, if satisfied, the stability of the synchronization of the coupled systems.

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1. INTRODUCTION

Coupled dynamical systems are synthesized from simpler, low-dimensional systems, such as chaos systems, to form new and complex systems. The study of the synchronous state of the coupled systems has attracted great attention. Up to now, there are many good methods in studying synchronization of coupled systems. Among them are Lyapunov function method [1,2,3], master stability function method [4] and constriction method [5]. In [6,7,8,9,10], mode decomposition method was used in studying two and above coupled systems. In this paper we use this method to investigate a generally coupled systems, sufficient conditions for synchronization of the coupled systems as well as numerical simulations are obtained.

2. SYSTEMS UNDER STUDY

We consider systems coupled by$N$ identical cells with mass exchange, the reaction of these cells are illustrated in Figure 1.

Suppose every cell has m different substances. The substances in a cell satisfies following equation:

$$\frac{dx}{dt} = F(x), \tag{1}$$

where $x \in \mathbf{R}^m$. Suppose cells are linearly coupled, the coupled systems are:

$$\frac{dX}{dt} = \mathcal{F}(X) + \mathcal{D}X, \tag{2}$$

where

$$\mathcal{D} = \begin{pmatrix} D_1 & D_2 & D_3 & \cdots & D_N \\ D_N & D_1 & D_2 & \cdots & D_{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ D_2 & D_3 & D_4 & \cdots & D_1 \end{pmatrix}, \mathcal{F}(X) = \begin{pmatrix} F(X_1) \\ F(X_2) \\ \vdots \\ F(X_N) \end{pmatrix},$$

* Department of Computer and Information Engineering, Jiangxi Normal University, Nanchang, 330027 *E-mail: xhxiong8899@yahoo.com.cn*

** Department of Mathematics and Computational Science, Zhongshan University, Guangzhou 510275

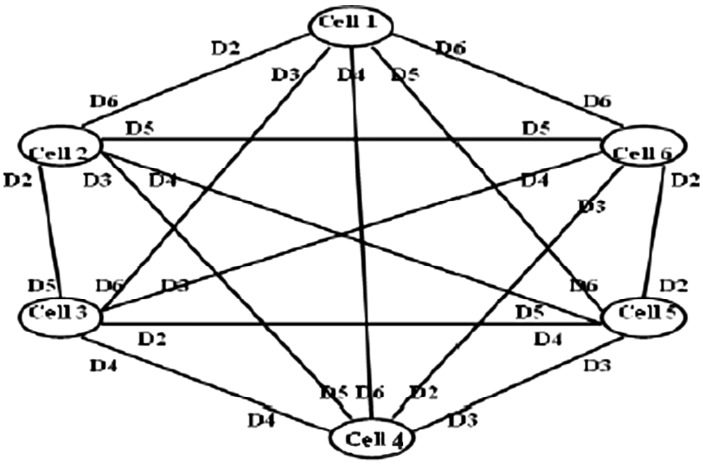

Figure 1: A Network of Six Coupled Cells, here, D_j (j = 1, 2, … 6) is the Matrix Governing the Coupled Cells

$D_j \in \mathbf{R}^{m\times m}$ $(1 \le j \le N)$) and $\mathcal{D}$ is circular matrix [11]. Notice that equation (2) can be written simply as:

$$\frac{dX}{dt} = \mathcal{F}(X) + \sum_{k=1}^{N} D_k \mathcal{J}^{k-1} X, \tag{3}$$

where $\mathcal{J} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ I & 0 & 0 & \cdots & 0 \end{pmatrix}$ and I is identical matrix of order m. If mass exchange between cells are conservative, we may suppose:

$$\sum_{j=1}^{N} \mathcal{D}_j = \mathcal{O} \text{ (zero matrix).} \tag{4}$$

This condition is called conservative condition. Next we consider synchronization of the coupled systems (2). The starting point in this analysis is the linearized systems of (2) about the synchronized state $s(t)$. This leads to a set of linear variational equations given by:

$$\frac{dW}{dt} = \left(\Gamma + D\right) W, \tag{5}$$

where $\Gamma = \text{diag}\,(A, A, \ldots, A)$ and A $A = DF(s)$. Let $\omega_j = e^{\frac{2\pi i(j-1)}{N}}$ (where $1 \le j \le N$, $i = \sqrt{-1}$), it is easy to see ω_j is a solution of algebra equation: $\lambda^N - 1 = 0$. Denote Vandermonde matrix $\begin{pmatrix} I & I & \cdots & I \\ \omega_1 I & \omega_2 I & \cdots & \omega_N I \\ \vdots & \vdots & & \vdots \\ \omega_1^{N-1} I & \omega_2^{N-1} I & \cdots & \omega_N^{N-1} I \end{pmatrix} = \mathcal{P}$, then we can proof

$$\mathcal{P}^{-1}(\Gamma + \mathcal{D})\mathcal{P} = diag(d(\omega_1 I), \cdots, d(\omega_N I)), \tag{6}$$

where $d(x) = \mathcal{A} + \sum_{j=1}^{N} \mathcal{D}_j x^{j-1}$. From the conservative condition, we have $d(\omega_1) = d(1) = \mathcal{A}$. Because the matrix $\Gamma + \mathcal{D}$ is similar with a block matrix, that is, there exists a real invertiable matrix $\mathcal{Q}$, such that

$$\mathcal{Q}^{-1}\mathcal{D}\mathcal{Q} = \begin{pmatrix} \mathcal{A} & \mathcal{O} \\ \mathcal{O} & \mathcal{R} \end{pmatrix}, \tag{7}$$

where $\mathcal{R}$ is a real matrix. To simplify the analysis we divide the problem into two cases.

Case 1: $N = 2K + 1$. Define two functions of $m \times m$ matrix:

$$\begin{aligned} \mathcal{D}_R(\vartheta) &= \mathcal{A} + \mathcal{D}_1 + \mathcal{D}_2 \cos\vartheta + \mathcal{D}_3 \cos 2\vartheta + \cdots + \mathcal{D}_N \cos(N-1)\vartheta, \\ \mathcal{D}_I(\vartheta) &= \mathcal{D}_2 \sin\vartheta + \mathcal{D}_3 \sin 2\vartheta + \cdots + \mathcal{D}_N \sin(N-1)\vartheta \end{aligned}$$

and two functions of $Nm \times m$ matrix:

$$\begin{aligned} \mathcal{P}_R(\vartheta) &= (I, I\cos\vartheta, I\cos 2\vartheta, \cdots, I\cos(N-1)\vartheta)^t, \\ \mathcal{P}_I(\vartheta) &= (\mathcal{O}, I\sin\vartheta, I\sin 2\vartheta, \cdots, I\sin(N-1)\vartheta)^t, \end{aligned}$$

where t means transition of matrix. Now we can express $\mathcal{Q}$ and $\mathcal{R}$ as,

$$\mathcal{R} = diag\left(\begin{pmatrix} \mathcal{D}_R(\vartheta_1) & -\mathcal{D}_I(\vartheta_1) \\ \mathcal{D}_I(\vartheta_1) & \mathcal{D}_R(\vartheta_1) \end{pmatrix}, \cdots, \begin{pmatrix} \mathcal{D}_R(\vartheta_K) & -\mathcal{D}_I(\vartheta_K) \\ \mathcal{D}_I(\vartheta_K) & \mathcal{D}_R(\vartheta_K) \end{pmatrix} \right)$$

$$\mathcal{Q} = \left(\mathcal{P}_R(0); \mathcal{P}_R(\vartheta_1), \mathcal{P}_I(\vartheta_1); \cdots; \mathcal{P}_R(\vartheta_K), \mathcal{P}_I(\vartheta_K) \right)$$

where $\vartheta_j = 2\pi j/(2K+1)$ and $1 \le j \le K$.

Case 2: $N = 2K$. Similarly, we have:

$$\mathcal{R} = diag\left(\mathcal{D}_R(0), \mathcal{D}_R(\pi), \begin{pmatrix} \mathcal{D}_R(\vartheta_1) & -\mathcal{D}_I(\vartheta_1) \\ \mathcal{D}_I(\vartheta_1) & \mathcal{D}_R(\vartheta_1) \end{pmatrix}, \cdots, \begin{pmatrix} \mathcal{D}_R(\vartheta_K) & -\mathcal{D}_I(\vartheta_K) \\ \mathcal{D}_I(\vartheta_K) & \mathcal{D}_R(\vartheta_K) \end{pmatrix} \right)$$

$$\mathcal{Q} = \left(\mathcal{P}_R(0), \mathcal{P}_R(\pi); \mathcal{P}_R(\vartheta_1), \mathcal{P}_I(\vartheta_1); \cdots; \mathcal{P}_R(\vartheta_K), \mathcal{P}_I(\vartheta_K) \right)$$

where $\vartheta_j = 2\pi j/(2K)(1 \le j \le K-1)$, $\mathcal{D}_R(0) = \mathcal{A}, \mathcal{D}_R(\pi) = \mathcal{A} + \sum_{i=1}^{N} (-1)^{i+1} \mathcal{D}_i$, $\mathcal{P}_R(0) = (I, I, \ldots, I)^t$, $\mathcal{P}_R(\pi) = (I, -1, I, \ldots, I, -I)^t$.

By using above results we can apply mode decomposition to variational equations (5), that is, it can be decoupled by independent variables. On the other hand, recall next useful inequality [8]:

$$\| W(t_0) \|_2 \, e^{\int_{t_0}^{t} \alpha(\tau) d\tau} \le \| W(t) \|_2 \le \| W(t_0) \|_2 \, e^{\int_{t_0}^{t} \beta(\tau) d\tau}, \tag{8}$$

where α and β are minimum and maximum eigenvalues of the symmetric part of $\Gamma + \mathcal{A}$. Then the decoupled variational equations are given by:

$$\frac{dU}{dt} = J_{ks}U, \tag{9}$$

where $0 \le k \le [N/2]$, $J_{ks} = \frac{\mathcal{D}_R(\vartheta_k)+\mathcal{D}_R^T(\vartheta_k)}{2}$.

Obviously, the stability of zero solutions of these systems governs the stability of the synchronous state. The $k = 0$ mode decides motion on the synchronization manifold, it does not contribute to the stability of the synchronous state since the Floquet multiplier is 1. Let $\lambda_{\max}^{(k)}$ denotes the maximum eigenvalue of J_{ks} and let $\lambda_{\max} = \max_{1\le k\le[N/2]} \lambda_{\max}^{(k)}$. Then the sign of $\lambda_{\max}$ decides the stability of the synchronous state, that is, if $\lambda_{\max} < 0$, the synchronous state is stable; if $\lambda_{\max} > 0$, it is unstable.

3. APPLICATION

3.1 Coupled Brusselator Systems

Consider three coupled Brusselator systems corresponding to systems (2) we have:

$$\mathcal{D} = \begin{pmatrix} D_1 & D_2 & D_3 \\ D_3 & D_1 & D_2 \\ D_2 & D_3 & D_1 \end{pmatrix}, \mathcal{F}(X) = \begin{pmatrix} F(X_1) \\ F(X_2) \\ F(X_3) \end{pmatrix} \tag{10}$$

where the single Brusselator system is:

$$\frac{dx}{dt} = F(x) = \begin{pmatrix} a-(\mu+1)x_1 + x_1^2 x_2 \\ \mu x_1 - x_1^2 x_2 \end{pmatrix}, \tag{11}$$

where parameters $a > 0$ and $\mu > 0$.

The steady state of the single Brusselator model is $E = (a, \mu/a)$. As well known, for $\mu > a^2 + 1$, it has only one limit cycle [9] and for $(a - 1)^2 < \mu \le a^2 + 1$, the steady state is stable.

From above section we know the stability of the synchronous state can be determined by the stability of zero solutions of the following equations:

$$\frac{dU}{dt} = J_{ks}U \tag{12}$$

where

$$J_{0s} = \begin{pmatrix} -(\mu+1)+2x_1x_2 & x_1^2 \\ \mu-2x_1x_2 & -x_1^2 \end{pmatrix}, J_{1s} = J_{2s} = J_{0s} - \frac{3}{4}\left(D_2 + D_2^T + D_3 + D_3^T\right).$$

To make the numerical simulation simple, we choose $D_2 = D_3 = d\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Notice that for $\mu > a^2 + 1$, the single limit cycle of Brusselator is contained in a bounded square, suppose the domain is $\Omega = \{(x, y) || x |< m, | y |< m\}$. So to make the maximum eigenvalue of J_{1s} or J_{2s} is less than zero, for parameter d, we have a sufficient condition $d > \frac{m-(\mu+1)}{3}$. The numerical simulation in Figure 2. shows this conclusion.

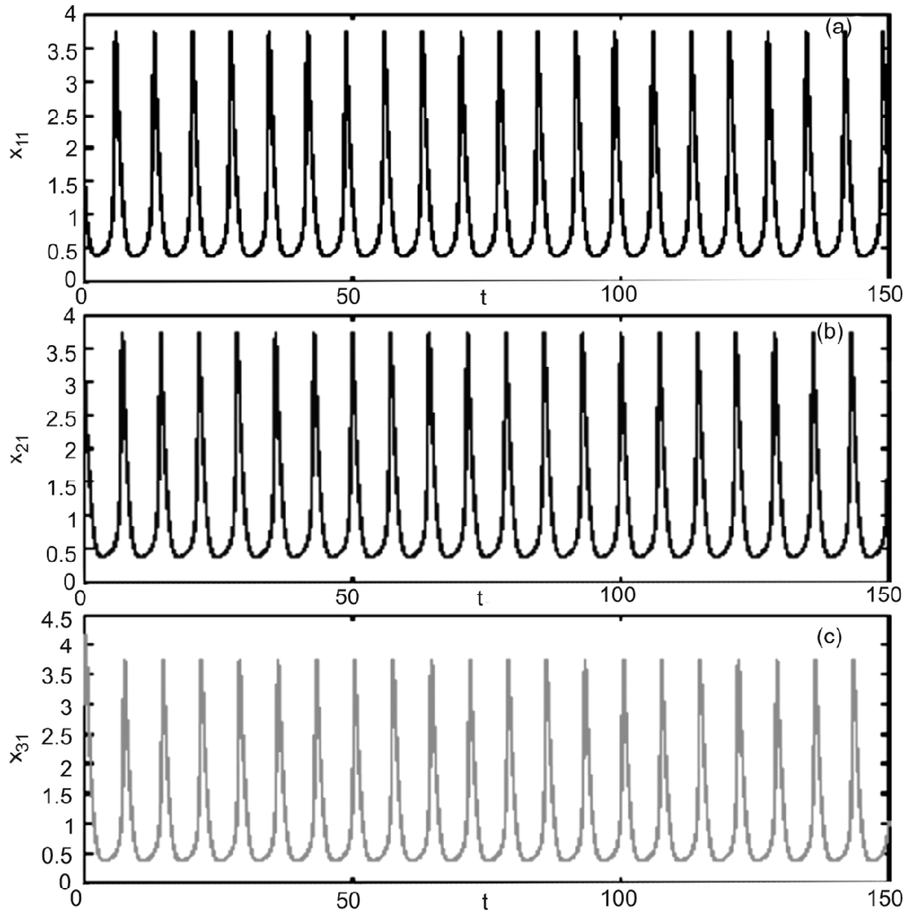

Figure 2: Synchronization of Three Coupled Brusselator Systems. The First Variable via Time *t*, where $\mu = 3$, $a = 1$ and $d = 5.0$

3.2 Coupled Lorenz Systems

Now we consider four coupled Lorenz systems. Relating to systems (2) we choose:

$$D = \begin{pmatrix} D_1 & D_2 & D_3 & D_4 \\ D_4 & D_1 & D_2 & D_3 \\ D_3 & D_4 & D_1 & D_2 \\ D_2 & D_3 & D_4 & D_1 \end{pmatrix}, \mathcal{F}(X) = \begin{pmatrix} F(X_1) \\ F(X_2) \\ F(X_3) \\ F(X_4) \end{pmatrix} \tag{13}$$

The single Lorenz system is

$$\begin{cases} \frac{dx}{dt} = a(y-x) \\ \frac{dy}{dt} = cx - xz - y \\ \frac{dz}{dt} = xy - bz, \end{cases} \tag{14}$$

where $a > 0$, b and c are parameters.

To guarantee the stability of the synchronous state, according to above section we consider the following three independent equations:

$$\frac{dU}{dt} = J_{ks}U \tag{15}$$

where

$$J_{0s} = \begin{pmatrix} -a & \frac{a+c-z}{2} & \frac{y}{2} \\ \frac{a+c-z}{2} & -1 & 0 \\ \frac{y}{2} & 0 & -b \end{pmatrix}, \; J_{1s} = J_{0s} - \left(\frac{D_2 + D_2^T}{2} - \frac{D_4 + D_4^T}{2} \right),$$

$$J_{2s} = J_{3s} = J_{0s} + \frac{D_1 + D_1^T}{2} - \frac{D_3 + D_3^T}{2}.$$

The stability of zero solutions of these systems governs the stability of the synchronous state. For the numerical simulation we choose $D_3 = D_4 = d\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_2 = 2D_4$. From the conservative condition we have: $D_1 = -4D_3$. It is well known for parameters $a = 10$, $b = 8/3$, $c = 28$, the attractor of Lorenz systems $\frac{dx}{dt} = f(x)$, is contained in a bounded domain :$\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + (z - a - c)^2 \le b^2 (a + c)^2/(4(b-1))\}$ [14]. So to make the maximum eigenvalues of J_{1s}, J_{2s} and J_{3s} less than zero, for parameter d, we have a sufficient condition: $d > \frac{b(a+c)}{2\sqrt{b-1}} \simeq 39.25$. The numerical simulation illustrates this conclusion in Figure 3.

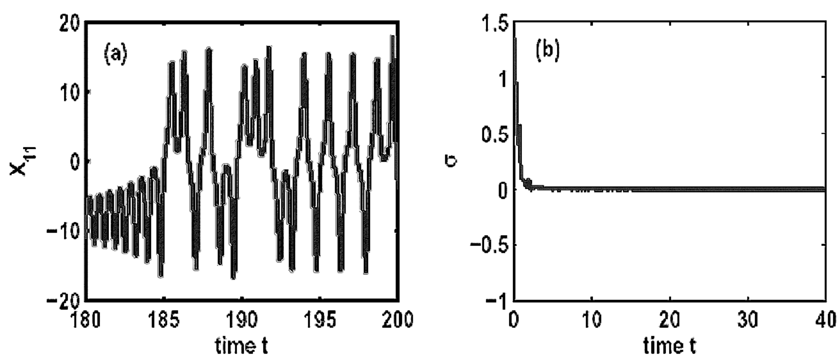

Figure 3. Synchronization of Four Coupled Lorenz Systems. LEFT: The First Variable via Time t in the First Lorenz Systems of the Four Coupled Systems ; RIGHT: Error: $\sigma = \sqrt{\sum_{i=1}^{4}\sum_{j=1}^{4} |x_{i1} - x_{j1}|^2/4}$, where $a = 10$, $b = 8/3$, $c = 28$, $d = 50$

4. CONCLUSIONS

Based on mode composition, we have derived a set of sufficient conditions on the stability of synchronous states in coupled systems including chaotic and non-chaotic systems. Such an approach can be applied to cases of various couplings, such as linear or nonlinear coupling, partial or global coupling, time-dependent or time-independent coupling and star-type coupling. Finally, it can be also straightforwardly applied to different types of neural network systems, interconnected physical or chemical systems, etc, so as to determine synchronizing behaviors and collective phenomena.

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REFERENCES

[1] R. He and P. G. Vaidya, Analysis and synthesis of synchronous periodic and chaotic systems, *Physical Review A*, Vol. **46**, pp. 7387–7392, 1992.

[2] C. W. Wu and L. O. Chua, A unified framework for synchronization and control of dynamical systems, *International Journal of Bifurcation and Chaos*, Vol. 4, pp. 979–998, 1994.

[3] N. F. Rulkov, A. R. Volkoskii, A. Rodriguez-Lozano, E. Delrio and M. G. Velarde, Mutual synchronization of chaotic self-oscillators with dissipative coupling, *International Journal of Bifurcation and Chaos*, Vol. **2**, pp. 669–676, 1992.

[4] L. M. Pecora and T. L. Carroll, Master Stability Functions for Synchronized Coupled Systems, *Physical Review Letters*, Vol. **80,** pp. 2109–2112, 1998.

[5] J. J. E. Slotine and W. Wang, Lecture notes in control and information science, Vol. 309, Spinger, Berlin heidelberg New York, 2004.

[6] T. S. Zhou and S. C. Zhang, Stability of In-Phase Solution of Coupled Equations, Acta Methematica Sinica, Vol. **44**, pp. 1121–1126, 2001.

[7] T. S. Zhou, and S. C. Zhang, Stability of In-Phase Solutions in Symmetrical Coupled Oscillations, *Journal of System Science and Mathematical Science*, Vol. **21**, pp. 65–70, 2003.

[8] T. S. Zhou and S. C. Zhang, Dynamical behavior in linearly coupled Oregonators, Physica D, Vol.**151**, pp. 199–216, 2001.

[9] T. S. Zhou, J. H. Lu, G. Chen and Y. Tang, Synchronization stability of three chaotic systems with linear coupling, *Physics Letters A*, Vol. **301**, pp. 231–240, 2002.

[10] X. Xiong, J. Wang, Y. Zhang and T. Zhou, Mode decomposition for a synchronous state and its applications, *Chaos, Solitons and Fractals*, Vol. **31**, pp. 718–725, 2007.

[11] R. A. Horn and C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, UK, 1985.

[12] J. L. Dialecii and M. G. Krein, Stability of differential equations in Banach space, Translations of Mathematical Monographs, Vol. **43**, *American Mathematical Society*, New York, 1974.

[13] C. F. Zhang, Theory of Differential Equations, Beijing Scientific Press House, 1985.

[14] T. S. Zhou, L. N. Chen and R. Q. Wang, Excitation functions of coupling, *Physical Review E*, Vol. **71**, pp. 066211 (1-9), 2005.