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New Exact Solutions of the Kawahara Equation using Generalized F-expansion Method

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Abstract: In this paper, a generalized F-expansion method is used to construct exact solutions of the Kawahara equation. As a result, many new and more general exact travelling wave solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, solitary wave solutions and trigonometric function solutions. Compared with the most existing F-expansion methods, the proposed method gives new and more general exact solutions. More importantly, the method with the help of symbolic computation provides a powerful mathematical tool for solving a great many nonlinear partial differential equations in mathematical physics.

Keywords: Generalized F-expansion method, Jacobi elliptic function solutions, Solitary wave solutions, Trigonometric function solutions.

1. INTRODUCTION

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented, such as inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine-cosine method [5], Adomian Pade approximation [6], homogenous balance method [7], homotopy perturbation method [8–10], variational method [11–17], asymptotic methods [18], non-perturbative methods [19], algebraic method [20–22], tanh function method [23–26] and so on.

Recently, F-expansion method [27–29] was proposed to construct periodic wave solutions of NLPDEs, which can be thought of as an over-all generalization of Jacobi elliptic function expansion method [30–32]. F-expansion method was later extended in different manners [33–40]. Very recently, by using a new and more general ansätz we proposed a generalized F-expansion method [41] which generalized the work made in [27–29,35,38–40] to obtain new and more general exact solutions of NLPDEs. With the help of *Mathematica* the generalized F-expansion method can be applied to a great many NLPDEs.

The present paper is motivated by the desire to extend the generalized F-expansion method [41] to the Kawahara equation:

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{1}$$

which occurs in the theory of magneto-acoustic waves in a plasmas [42] and in the theory of shallow water waves with surface tension [43]. Sirendaoreji [44] obtained some travelling wave solutions of Eq. (1) by using a direct algebraic method. Wazwaz [45] obtained solitary wave solutions by means of tanh method.

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The rest of this paper is organized as follows: in Section 2, we give the description of the generalized *F*-expansion method; in Section 3, we extend this method to the Kawahare equation; in Section 4, some conclusions are given.

2. DESCRIPTION OF THE GENERALIZED F-EXPANSION METHOD

For a given NLPDE with independent variables $x=(t,x_1,x_2,\ldots,x_m)$ and dependent variable *u*:

$$F(u,u_t,u_{x_1},u_{x_2},\ldots,u_{x_m},u_{x_1t},u_{x_2t}\ldots,u_{x_mt},u_{tt},u_{x_1x_1},u_{x_2x_2},\ldots,u_{x_mx_m},\ldots)=0, \tag{2}$$

we seek its solutions in the more general form:

$$u=a_0+\sum_{i=1}^{n}\{a_iF^i(\xi)+b_iF^{-i}(\xi)+c_iF^{i-1}(\xi)F'(\xi)+d_iF^{-i}(\xi)F'(\xi)\}, \tag{3}$$

where $a_0=a_0(x)$, $a_i=a_i(x)$, $b_i=b_i(x)$, $c_i=c_i(x)$, $d_i=d_i(x)$ $(i=1,2,\ldots,n)$ and $\xi=\xi(x)$ are all functions to be determined later, $F(\xi)$ and $F'(\xi)$ satisfy

$$F'^2(\xi)=PF^4(\xi)+QF^2(\xi)+R, \tag{4}$$

and hence holds for $F(\xi)$ and $F'(\xi)$

$$\begin{cases} F''(\xi)=2PF^3(\xi)+QF(\xi), \\ F^{(3)}(\xi)=(6PF^2(\xi)+Q)F'(\xi), \\ F^{(4)}(\xi)=24P^2F^5(\xi)+20PQF^3(\xi)+(Q^2+12PR)F(\xi), \\ \cdots \end{cases} \tag{5}$$

where *P*, *Q* and *R* are all parameters, the prime denotes $\mathrm{d}/\mathrm{d}\xi$. Given different values of *P*, *Q* and *R*, the different Jacobi elliptic function solutions $F(\xi)$ can be obtained from Eq. (4) (see Appendix A).

It can be easily found that the ansätz (3) is more general than those in [27–29,35,38–40], to be more precise, if $b_i=c_i=d_i=0$, a_0 and a_i are constants, and ξ is merely a linear function of $t,x_1,x_2,\ldots,x_m$, then the ansätz (3) becomes that introduced in [27–29]:

$$u=\sum_{i=0}^{n}a_iF^i(\xi).$$

If $c_i=d_i=0$, a_0 and a_i are constants, and ξ is merely a linear function of $t,x_1,x_2,\ldots,x_m$, then the ansätz (3) reduces to that constructed in [38,39]:

$$u=\sum_{i=-n}^{n}a_iF^i(\xi).$$

If $c_i = d_i = 0$, then the ansätz (3) changes into that used in [35]:

$$u = a_0 + \sum_{i=1}^{n} \{a_i F^i(\xi) + b_i F^{-i}(\xi)\}.$$

If $d_i = 0$, then the ansätz (3) converts into that proposed in [40]:

$$u = a_0 + \sum_{i=1}^{n} \{a_i F^i(\xi) + b_i F^{-i}(\xi) + c_i F^{i-1}(\xi) F'(\xi)\}.$$

So we may obtain new and more general exact solutions of NLPDEs by using the ansätz (3). In order to determine u explicitly, we take the following four steps:

Step 1. Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Eq. (2).

Step 2. Substitute (3) along with (4) and (5) into Eq. (2) and collect coefficients of $F'^l(\xi) F^j(\xi)$ $(l = 0,1;\ j = 0, \pm 1, \pm 2, \ldots)$, then set each coefficient to zero to derive a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, n)$ and ξ.

Step 3. Solve the system of over-determined partial differential equations obtained in Step 2 for a_0, a_i, b_i, c_i, d_i and ξ by use of *Mathematica*.

Step 4. Select P, Q, R and $F(\xi)$ from Appendix A and substitute them along with a_0, a_i, b_i, c_i, d_i and ξ into (3) to obtain Jacobi elliptic function solutions of Eq. (2) (see Appendix B for $F'(\xi)$), from which hyperbolic function solutions and trigonometric function solutions can be obtained in the limit cases when $m \to 1$ and $m \to 0$ (see Appendix C).

Remark 1. In order to determine the explicit solutions of the partial differential equations derived in Step 2, we may choose special forms of a_0, a_i, b_i, c_i, d_i and ξ (As we do in Section 3).

3. EXACT SOLUTIONS OF THE KAWAHARA EQUATION

By balancing uu_x and u_{xxxxx} in Eq. (1), we get $n = 4$. We suppose that Eq. (1) has the following formal solution:

$$\begin{aligned} u = a_0 &+ a_1 F(\xi) + a_2 F^2(\xi) + a_3 F^3(\xi) + a_4 F^4(\xi) + b_1 F^{-1}(\xi) + b_2 F^{-2}(\xi) + b_3 F^{-3}(\xi) + b_4 F^{-4}(\xi) \\ &+ c_1 F'(\xi) + c_2 F'(\xi) F(\xi) + c_3 F'(\xi) F^2(\xi) + c_4 F'(\xi) F^3(\xi) + d_1 F'(\xi) F^{-1}(\xi) \\ &+ d_2 F'(\xi) F^{-2}(\xi) + d_3 F'(\xi) F^{-3}(\xi) + d_4 F'(\xi) F^{-4}(\xi), \end{aligned} \quad (6)$$

where a_0, a_i, b_i, c_i, d_i ($i = 1, 2, 3, 4$) and ξ are all function of x and t to be further determined.

With the aid of *Mathematica*, substituting (6) along with (4) and (5) into Eq. (1), the left-hand side of Eq. (1) is converted into a polynomial of $F'^l(\xi) F^j(\xi)$ $(l = 0,1;\ j = 0, \pm 1, \pm 2, \ldots)$, then setting each coefficient to zero, we get a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i and ξ. However, it is very difficult for us to solve the set of over-determined partial differential equations. As the calculation goes on, in order to simplify the work or make the work feasible, we choose special forms by setting $a_0 = a_0(t)$, $a_i = a_i(t)$, $b_i = b_i(t)$, $c_i = c_i(t)$, $d_i = d_i(t)$, $\xi = kx + \eta$, $\eta = \eta(t)$ and k = constant, then we get the following results:

Case 1

$$a_0 = -\frac{31k + 910k^3Q \pm 5640k^3\sqrt{PR}(13k^2Q-1) + 1183k^5(Q^2-108PR) + 507\omega}{507k}, \quad a_1 = 0, \tag{7}$$

$$a_2 = \frac{140}{13}k^2P(52k^2Q-1), \quad a_3 = 0, \quad a_4 = 840k^4P^2, \quad b_1 = 0, \quad b_2 = \frac{140}{13}k^2R(52k^2Q-1), \tag{8}$$

$$b_3 = 0, \quad b_4 = 840k^4R^2, \quad c_1 = \pm\frac{140}{13}k^2\sqrt{P}(13k^2Q-1), \quad c_2 = 0, \quad c_3 = \pm 840k^4P\sqrt{P}, \tag{9}$$

$$c_4 = 0, \quad d_1 = 0, \quad d_2 = \pm\frac{140}{13}k^2\sqrt{R}(13k^2Q-1), \quad d_3 = 0, \quad d_4 = \pm 840k^4R\sqrt{R}, \quad \eta = \omega t + c, \tag{10}$$

$$31 - 3549k^4(Q^2+132PR) - 21970k^6Q(Q^2-1044PR) \pm 212940k^4\sqrt{PR}(13k^2(Q^2+12PR) - Q) = 0, \tag{11}$$

where ω and c are arbitrary constants. The sign "±" means that all possible combinations of "+" and "–" can be taken in c_3 and d_4. If the same sign is used in c_3 and d_4, then "+" must be used in a_0, d_2 and (11). If different signs are used in c_3 and d_4, then "–" must be used in a_0, d_2 and (11). Furthermore, the same sign must be used in c_1 and c_3. Hereafter, the sign "±" always stands for this meaning in the similar circumstances.

Case 2

$$a_0 = -\frac{31k + 910k^3Q + 1183k^5(Q^2-108PR) + 507\omega}{507k}, \quad a_1 = 0, \quad a_2 = \frac{140}{13}k^2P(52k^2Q-1), \tag{12}$$

$$a_3 = 0, \quad a_4 = 840k^4P^2, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \quad b_4 = 0, \quad c_1 = \pm\frac{140}{13}k^2\sqrt{P}(13k^2Q-1), \tag{13}$$

$$c_2 = 0, \quad c_3 = \pm 840k^4P\sqrt{P}, \quad c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad \eta = \omega t + c, \tag{14}$$

$$31 - 3549k^4(Q^2+132PR) - 21970k^6Q(Q^2-36PR) = 0, \tag{15}$$

where ω and c are arbitrary constants.

Case 3

$$a_0 = -\frac{31k + 3640k^3Q + 18928k^5(Q^2-18PR) + 507\omega}{507k}, \quad a_1 = 0, \quad a_2 = \frac{280}{13}k^2P(52k^2Q-1), \tag{16}$$

$$a_3 = 0, \quad a_4 = 1680k^4P^2, \quad b_1 = 0, \quad b_2 = \frac{280}{13}k^2R(52k^2Q-1), \quad b_3 = 0, \quad b_4 = 1680k^4R^2, \tag{17}$$

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad \eta = \omega t + c, \tag{18}$$

$$31 - 56784k^4(Q^2 + 12PR) - 1406080k^6Q(Q^2 - 36PR) = 0, \tag{19}$$

where ω and c are arbitrary constants.

Case 4

$$a_0 = -\frac{31k + 910k^3Q + 1183k^5(Q^2 - 108PR) + 507\omega}{507k}, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \tag{20}$$

$$b_1 = 0, \quad b_2 = \frac{140}{13}k^2R(52k^2Q-1), \quad b_3 = 0, \quad b_4 = 840k^4R^2, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \tag{21}$$

$$c_4 = 0, \quad d_1 = 0, \quad d_2 = \pm\frac{140}{13}k^2\sqrt{R}(13k^2Q-1), \quad d_3 = 0, \quad d_4 = \pm 840k^4R\sqrt{R}, \quad \eta = \omega t + c, \tag{22}$$

$$31 - 3549k^4(Q^2 + 12PR) - 21970k^6Q(Q^2 - 36PR) = 0, \tag{23}$$

where ω and c are arbitrary constants.

Case 5

$$a_0 = -\frac{31k + 3640k^3Q + 18928k^5(Q^2 - 18PR) + 507\omega}{507k}, \quad a_1 = 0, \quad a_2 = \frac{280}{13}k^2P(52k^2Q-1), \tag{24}$$

$$a_3 = 0, \quad a_4 = 1680k^4P^2, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \quad b_4 = 0, \quad c_1 = 0, \quad c_2 = 0, \tag{25}$$

$$c_3 = 0, \quad c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad \eta = \omega t + c, \tag{26}$$

$$31 - 56784k^4(Q^2 - 3PR) - 703040k^6Q(2Q^2 - 9PR) = 0, \tag{27}$$

where ω and c are arbitrary constants.

Case 6

$$a_0 = -\frac{31k + 3640k^3Q + 18928k^5(Q^2 - 18PR) + 507\omega}{507k}, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \tag{28}$$

$$b_1 = 0, \quad b_2 = \frac{280}{13}k^2R(52k^2Q-1), \quad b_3 = 0, \quad b_4 = 1680k^4R^2, \quad c_1 = 0, \quad c_2 = 0, \tag{29}$$

$$c_3 = 0, \quad c_4 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = 0, \quad \eta = \omega t + c, \tag{30}$$

$$31 - 56784k^4(Q^2 - 3PR) - 703040k^6 Q(2Q^2 - 9PR) = 0, \tag{31}$$

where *w* and *c* are arbitrary constants.

Substituting Cases 1–6 into (6) respectively, we have six kinds of formal solutions of Eq. (1):

$$u = -\frac{31k + 910k^3Q \pm 5640k^3\sqrt{PR}(13k^2Q - 1) + 1183k^5(Q^2 - 108PR) + 507\omega}{507k}$$

$$+\frac{140}{13}k^2P(52k^2Q-1)F^2(\xi) + 840k^4P^2F^4(\xi) + \frac{140}{13}k^2R(52k^2Q-1)F^{-2}(\xi) + 840k^4R^2F^{-4}(\xi)$$

$$\pm\frac{140}{13}k^2\sqrt{P}(13k^2Q-1)F'(\xi) \pm 840k^4P\sqrt{P}F'(\xi)F^2(\xi) \pm \frac{140}{13}k^2\sqrt{R}(13k^2Q-1)F'(\xi)F^{-2}(\xi)$$

$$\pm 840k^4R\sqrt{R}F'(\xi)F^{-4}(\xi), \tag{32}$$

where $\xi = kx + \omega t + c$, k is determined by (11).

$$u = -\frac{31k + 910k^3Q + 1183k^5(Q^2 - 108PR) + 507\omega}{507k} + \frac{140}{13}k^2P(52k^2Q-1)F^2(\xi)$$

$$+840k^4P^2F^4(\xi) \pm \frac{140}{13}k^2\sqrt{P}(13k^2Q-1)F'(\xi) \pm 840k^4P\sqrt{P}F'(\xi)F^2(\xi) \tag{33}$$

where $\xi = kx + \omega t + c$, k is determined by (15).

$$u = -\frac{31k + 3640k^3Q + 18928k^5(Q^2 - 18PR) + 507\omega}{507k} + \frac{280}{13}k^2P(52k^2Q-1)F^2(\xi)$$

$$+1680k^4P^2F^4(\xi) + \frac{280}{13}k^2R(52k^2Q-1)F^{-2}(\xi) + 1680k^4R^2F^{-4}(\xi), \tag{34}$$

where $\xi = kx + \omega t + c$, k is determined by (19).

$$u = -\frac{31k + 910k^3Q + 1183k^5(Q^2 - 108PR) + 507\omega}{507k} + \frac{140}{13}k^2R(52k^2Q-1)F^{-2}(\xi)$$

$$840k^4R^2F^{-4}(\xi) \pm \frac{140}{13}k^2\sqrt{R}(13k^2Q-1)F'(\xi)F^{-2}(\xi) \pm 840k^4R\sqrt{R}F'(\xi)F^{-4}(\xi), \tag{35}$$

where $\xi = kx + \omega t + c$, k is determined by (23).

$$u = -\frac{31k + 3640k^3 Q + 18928k^5(Q^2 - 18PR) + 507\omega}{507k} + \frac{280}{13}k^2 P(52k^2 Q - 1)F^2(\xi)$$
$$+1680k^4 P^2 F^4(\xi), \tag{36}$$

where $\xi = kx + \omega t + c$, k is determined by (27).

$$u = -\frac{31k + 3640k^3 Q + 18928k^5(Q^2 - 18PR) + 507\omega}{507k} + \frac{280}{13}k^2 R(52k^2 Q - 1)F^{-2}(\xi)$$
$$+1680k^4 R^2 F^{-4}(\xi), \tag{37}$$

where $\xi = kx + \omega t + c$, k is determined by (31).

From Appendix A, choosing $F(\xi) = \mathrm{ns}\xi$, $P = 1$, $Q = -(1+m^2)$, $R = m^2$, inserting them into (32) and using Appendix B, we obtain combined non-degenerate Jacobi elliptic function solution of Eq.(1):

$$u = -\frac{31k - 910k^3(1+m^2) \mp 5640k^3 m(13k^2(1+m^2)+1) + 1183k^5((1+m^2)^2 - 108m^2) + 507\omega}{507k}$$
$$-\frac{140}{13}k^2(52k^2(1+m^2)+1)\mathrm{ns}^2\xi + 840k^4\mathrm{ns}^4\xi - \frac{140}{13}k^2 m^2(52k^2(1+m^2)+1)\mathrm{sn}^2\xi$$
$$+840k^4 m^4\mathrm{sn}^4\xi \pm \frac{140}{13}k^2(13k^2(1+m^2)+1)\mathrm{cs}\xi\mathrm{ds}\xi \mp 840k^4\mathrm{cs}\xi\mathrm{ds}\xi\mathrm{ns}^2\xi$$
$$\pm\frac{140}{13}k^2 m(13k^2(1+m^2)+1)\mathrm{cn}\xi\mathrm{dn}\xi \mp 840k^4 m^3\mathrm{cn}\xi\mathrm{dn}\xi\mathrm{sn}^2\xi, \tag{38}$$

where $\xi = kx + \omega t + c$, k is determined by (11) with $P = 1$, $Q = -(1+m^2)$ and $R = m^2$.

In the limit case when $m \to 1$, from (38) we obtain solitary wave solution of Eq. (1):

$$u = -\frac{31k - 1820k^3 \mp 5640k^3(26k^2+1) - 123032k^5 + 507\omega}{507k} - \frac{140}{13}k^2(104k^2+1)\coth^2\xi$$
$$+840k^4\coth^4\xi - \frac{140}{13}k^2(104k^2+1)\tanh^2\xi + 840k^4\tanh^4\xi \pm \frac{140}{13}k^2(26k^2+1)\mathrm{csch}^2\xi$$
$$\mp 840k^4\cosh^2\xi\coth^2\xi \pm \frac{140}{13}k^2(26k^2+1)\mathrm{sech}^2\xi \mp 840k^4\mathrm{sech}^2\xi\tanh^2\xi, \tag{39}$$

where $\xi = kx + \omega t + c$, k is determined by (11) with $P = 1$, $Q = -2$ and $R = 1$.

When $m \to 0$, from (38) we obtain trigonometric function solution of Eq. (1):

$$u = -\frac{31k - 910k^3 + 1183k^5 + 507\omega}{507k} - \frac{140}{13}k^2(52k^2+1)\csc^2\xi + 840k^4\csc^4\xi$$

$$\pm\frac{140}{13}k^2(13k^2+1)\cot\xi\csc\xi \mp 840k^4\cot\xi\csc^3\xi, \tag{40}$$

where $\xi = kx + \omega t + c$, k is determined in (11) with $P = 1$, $Q = -1$ and $R = 0$.

Choosing $F(\xi) = \mathrm{ns}\xi \pm \mathrm{cs}\xi$, $P = 1/4$, $Q = (1-2m^2)/2$, $R = 1/4$, inserting them into (32) and using Appendix B, we obtain combined non-degenerate Jacobi elliptic function solution of Eq. (1):

$$u = -\frac{134k + 1820k^3(1-2m^2) \pm 2820k^3(13k^2(1-2m^2)-2) + 1183k^5((1-2m^2)^2 - 27) + 2028\omega}{2028k}$$

$$+\frac{35}{13}k^2(26k^2(1-2m^2)-1)(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2 + \frac{105}{2}k^4(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^4$$

$$+\frac{35}{13}k^2(26k^2(1-2m^2)-1)\frac{1}{(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2} + \frac{105}{2}k^4\frac{1}{(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^4}$$

$$\mp\frac{35}{13}k^2(13k^2(1-2m^2)-2)(\mathrm{cs}\xi\mathrm{ds}\xi \pm \mathrm{ns}\xi\mathrm{ds}\xi) \mp 105k^4(\mathrm{cs}\xi\mathrm{ds}\xi \pm \mathrm{ns}\xi\mathrm{ds}\xi)(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2$$

$$\pm\frac{35}{13}k^2(13k^2(1-2m^2)-2)\frac{\mp\mathrm{ds}\xi}{\mathrm{ns}\xi \pm \mathrm{cs}\xi} \pm 105k^4\frac{\mp\mathrm{ds}\xi}{(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^3}, \tag{41}$$

where $\xi = kx + \omega t + c$, k is determined by (11) with $P = 1/4$, $Q = (1-2m^2)/2$ and $R = 1/4$.

In the limit case when $m \to 1$, from (41) we obtain solitary wave solution of Eq. (1):

$$u = -\frac{134k - 1820k^3 \mp 2820k^3(13k^2+2) - 30758k^5 + 2028\omega}{2028k} - \frac{35}{13}k^2(26k^2+1)(\coth\xi \pm \mathrm{csch}\xi)^2$$

$$+\frac{105}{2}k^4(\coth\xi \pm \mathrm{csch}\xi)^4 - \frac{35}{13}k^2(26k^2+1)\frac{1}{(\coth\xi \pm \mathrm{csch}\xi)^2} + \frac{105}{2}k^4\frac{1}{(\coth\xi \pm \mathrm{csch}\xi)^4}$$

$$\pm\frac{35}{13}k^2(13k^2+2)(\mathrm{csch}^2\xi \pm \coth\xi\mathrm{csch}\xi) \mp 105k^4(\mathrm{csch}^2\xi \pm \coth\xi\mathrm{csch}\xi)(\coth\xi \pm \mathrm{csch}\xi)^2$$

$$\mp\frac{35}{13}k^2(13k^2+2)\frac{\mp\mathrm{csch}\xi}{\coth\xi \pm \mathrm{csch}\xi} \pm 105k^4\frac{\mp\mathrm{csch}\xi}{(\coth\xi \pm \mathrm{csch}\xi)^3}, \tag{42}$$

where $\xi = kx + wt + c$, k is determined by (11) with $P = 1/4$, $Q = -1/2$ and $R = 1/4$.

When $m \to 0$, from (41) we obtain trigonometric function solution of Eq. (1):

$$u = -\frac{134k + 1820k^3 \pm 2820k^3(13k^2 - 2) - 30758k^5 + 2028\omega}{2028k} + \frac{35}{13}k^2(26k^2 - 1)(\csc\xi \pm \cot\xi)^2$$

$$+\frac{105}{2}k^4(\csc\xi \pm \cot\xi)^4 + \frac{35}{13}k^2(26k^2 - 1)\frac{1}{(\csc\xi \pm \cot\xi)^2} + \frac{105}{2}k^4\frac{1}{(\csc\xi \pm \cot\xi)^4}$$

$$\mp\frac{35}{13}k^2(13k^2 - 2)(\cot\xi\csc\xi \pm \csc^2\xi) \mp 105k^4(\cot\xi\csc\xi \pm \csc^2\xi)(\csc\xi \pm \cot\xi)^2$$

$$\pm\frac{35}{13}k^2(13k^2 - 2)\frac{\mp\csc\xi}{\csc\xi \pm \cot\xi} \pm 105k^4\frac{\mp\csc\xi}{(\csc\xi \pm \cot\xi)^3}, \qquad (43)$$

where $\xi = kx + \omega t + c$, k is determined by (11) with $P = 1/4$, $Q = 1/2$ and $R = 1/4$.

With the aid of Appendices A, B and C, from (32)–(37) we can obtain other single and combined Jacobi elliptic function solutions, solitary wave solutions and trigonometric function solutions of Eq. (1), we omit them here for simplicity.

Remark 2. All solutions obtained from Cases 1 and 4 can not been obtained by the F-expansion methods [27–29,33–40]. To the best of our knowledge, they are new and have not been reported in former literature. With the aid of *Mathematica*, we have verified all solutions obtained in this paper by putting them back into the original Eq. (1).

4. CONCLUSION

In this paper, the generalized F-expansion method has been successfully used to obtain new and more general exact solutions of the Kawahara equation. These exact solutions include single and combined non-degenerate Jacobi elliptic function solutions, solitary wave solutions and trigonometric function solutions. To our best knowledge, these solutions have not been reported in former literature. It may be important to explain some physical phenomena. It is shown that the generalized F-expansion method with the help of *Mathematica* provides a powerful mathematical tool for obtaining more general exact solutions of a great many NLPDEs in mathematical physics, such as the (3+1)-dimensional Kadomtsev–Petviashvili (KP) equation, the (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) equations, Nizhnik–Novikov–Vesselov (NNV) equations and so on. Compared with the most existing F-expansion methods [27–29,35,38–40], the proposed method gives new and more general exact solutions. Its applications are worth further studying.

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APPENDIX A

Given different values of P, Q and R, the different Jacobi elliptic function solutions $F(\xi)$ of Eq. (4) can be obtained, which are listed in Table 1.

Table 1
Relations between values of (P, Q, R) and corresponding F(ξ) in Eq. (4)

P	Q	R	$F(\xi)$
m^2	$-(1+m^2)$	1	$\text{sn}\xi$, $\text{cdx}=\frac{\text{cn}\xi}{\text{dn}\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\text{cn}\xi$
-1	$2-m^2$	m^2-1	$\text{dn}\xi$
1	$-(1+m^2)$	m^2	$\text{ns}\xi=(\text{sn}\xi)^{-1}$, $\text{dc}\xi=\frac{\text{dn}\xi}{\text{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\text{nc}\xi=(\text{cn}\xi)^{-1}$
m^2-1	$2-m^2$	-1	$\text{nd}\xi=(\text{dn}\xi)^{-1}$
$1\text{-}m^2$	$2-m^2$	1	$\text{sc}\xi=\frac{\text{sn}\xi}{\text{cn}\xi}$
$-m^2\,(1-m^2)$	$2m^2-1$	1	$\text{sd}\xi=\frac{\text{sn}\xi}{\text{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\text{cs}\xi=\frac{\text{cn}\xi}{\text{sn}\xi}$
1	$2m^2-1$	$-m2\,(1-m^2)$	$\text{ds}\xi=\frac{\text{dn}\xi}{\text{sn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\text{ns}\xi\pm\text{cs}\xi$
$\frac{1-m^2}{4}$	$\frac{1-m^2}{2}$	$\frac{1-m^2}{4}$	$\text{nc}\xi\pm\text{sc}\xi$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{ns}\xi\pm\text{ds}\xi$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{sn}\xi\pm i\text{cn}\xi$

APPENDIX B

Derivatives of Jacobi elliptic functions

$$\text{sn}'\xi=\text{cn}\xi\text{dn}\xi,\quad \text{cd}'\xi=-(1-m^2)\text{sd}\xi\text{nd}\xi,\quad \text{cn}'\xi=-\text{sn}\xi\text{dn}\xi,\quad \text{dn}'\xi=-m^2\text{sn}\xi\text{cn}\xi,$$

$$\text{ns}'\xi=-\text{cs}\xi\text{ds}\xi,\quad \text{dc}'\xi=(1-m^2)\text{nc}\xi\text{sc}\xi,\quad \text{nc}'\xi=\text{sc}\xi\text{dc}\xi,\quad \text{nd}'\xi=m^2\text{cd}\xi\text{sd}\xi,$$

$$\text{sc}'\xi=\text{dc}\xi\text{nc}\xi,\quad \text{cs}'\xi=-\text{ns}\xi\text{ds}\xi,\quad \text{ds}'\xi=-\text{cs}\xi\text{ns}\xi,\quad \text{sd}'\xi=\text{nd}\xi\text{cd}\xi.$$

APPENDIX C

Jacobi elliptic functions degenerate into hyperbolic functions when m $\to$ 1:

$$\mathrm{sn}\xi \to \tanh\xi,\quad \mathrm{cn}\xi \to \mathrm{sech}\xi,\quad \mathrm{dn}\xi \to \mathrm{sech}\xi,\quad \mathrm{sc}\xi \to \sinh\xi,\quad \mathrm{sd}\xi \to \sinh\xi,\quad \mathrm{cd}\xi \to 1,$$
$$\mathrm{ns}\xi \to \coth\xi,\quad \mathrm{nc}\xi \to \cosh\xi,\quad \mathrm{nd}\xi \to \cosh\xi,\quad \mathrm{cs}\xi \to \mathrm{csch}\xi,\quad \mathrm{ds}\xi \to \mathrm{csch}\xi,\quad \mathrm{dc}\xi \to 1.$$

Jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$:

$$\mathrm{sn}\xi \to \sin\xi,\quad \mathrm{cn}\xi \to \cos\xi,\quad \mathrm{dn}\xi \to 1,\quad \mathrm{sc}\xi \to \tan\xi,\quad \mathrm{sd}\xi \to \sin\xi,\quad \mathrm{cd}\xi \to \cos\xi,$$
$$\mathrm{ns}\xi \to \csc\xi,\quad \mathrm{nc}\xi \to \sec\xi,\quad \mathrm{nd}\xi \to 1,\quad \mathrm{cs}\xi \to \cot\xi,\quad \mathrm{ds}\xi \to \csc\xi,\quad \mathrm{dc}\xi \to \sec\xi.$$

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