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On Gaver's Parallel System: The Idle Time

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Abstract: We consider Gaver's parallel system attended by a single repairman. We analyse the idle time of the repairman during the lifetime of the system. Our analysis involves a stochastic identity and an integral equation of the renewal-type. Finally, as an example, we consider the particular case of deterministic repair (replacement).

Keywords: Parallel system, idle time, lifetime, stochastic identity, deterministic repair.

1. INTRODUCTION

Duplex systems are often employed to increase the reliability, availability, quality and safety of operational plants, e.g. Birolini [1]. A basic duplex system is Gaver's parallel system attended by a single repairman, henceforth called a **G**-system, e.g. Vanderperre *et al.* [2].

The **G**-system consists of a single operative unit (the o-unit) endowed with an identical unit in *hot* standby (the s-unit). The notion of "hot" standby signifies that the s-unit has the *same* failure rate as the o-unit. Note that the hot standby mode is often indispensable to perform an instantaneous switch from standby into the operative state allowing continuous operation of a technical system upon failure of the on-line unit.

The **G**-system acts as a *closed* queueing system evolving in time, i.e. any failed unit goes immediately into repair provided that the repairman is idle. Otherwise, the failed unit has to queue for repair. On the other hand, a repaired unit lines-up in hot standby or becomes immediately operative if the remaining unit is down.

An example of a **G**-system is a twin-generator attending the light-plant of a tunnel. Our **G**-system satisfies the usual condition, i.e. any repair is perfect. The o-unit (s-unit) has a constant failure rate, e.g. Gnedenko *et al.* [3], $\lambda > 0$ and an *arbitrary* repair time distribution $R(\cdot), R(0) = 0.$ The failure-free time of the o-unit (s-unit) is denoted by f and the repair time by r. The random variables f and r are supposed to be *statistically* independent.

Apart from a general reliability analysis, see Vanderperre *et al.* [2], the stochastic analysis of the **G**-system is far from complete. At present, we analyse the idle time of the repairman *during* the survival time of the **G**-system. Our analysis involves a *stochastic* identity and an integral equation of the convolution-type. The solution is constructed by means of the Laplace-transform. Finally, as an example, we consider the particular case of deterministic repair (replacement).

2. FORMULATION

Consider the **G**-system starting to operate at some time origin $t = 0$. In order to describe the random behaviour of the **G**-system, we employ a stochastic process $\{N_t, t \geq 0\}$ with discrete state space $\{A, B, D\} \subset [0, \infty)$, characterized by the following mutually exclusive events:

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$\{N_t = A\}$: "The repairman is idle at time *t*."
$\{N_t = B\}$: "One unit is operative and the other unit is under progressive repair at time *t*."
$\{N_t = D\}$: "The system is down at time *t*."
State *D* is called the system-down state.

The non-Markovian stochastic process $\{N_t, t \geq 0\}$ is defined on some filtered probability space $\{\Omega, \mathcal{A}, \mathbf{P}, \Im\}$, where the history $\Im := \{\Im_t, t \geq 0\}$ satisfies the Dellacherie conditions:

- $\Im_0$ contains the **P**-null sets of $\mathcal{A}$,
- $\forall t \geq 0, \Im_t = \bigcap_{u>t} \Im_u$, *i.e. the family* $\Im$ is right-continuous.

Consider the $\Im$-*stopping* time

$$\theta_G := \inf\{t > 0 : N_t = D \mid N_0 = A\}.$$

In Statistical Reliability Engineering, e.g. Gnedenko *et al.* [3], θ_G (measured from the time origin onwards) is usually called the *survival* time (lifetime) of the **G**-system starting to operate at time $t = 0$ in state *A*.

We introduce the following notations:

$\mathbf{1}\{\cdot\}$: the indicator of an event $\{\cdot\} \in \mathcal{A}$,

$[t]$: the greatest integer function,

i_G : the *total* idle time of the repairman during θ_G,

$\mathcal{B}[0, \infty)$: the Borel algebra on $[0, \infty)$.

Finally, we introduce the (defective) probability measure

$$\varphi(u) = \int_0^u (1 - e^{-2\lambda(u-z)}) d\mathbf{P}\{r \leq z, f > r\}.$$

Remark 2.1: Note that $\varphi(\cdot)$ is Lebesgue-absolutely continuous on $[0, \infty)$ with bounded Radon-Nikodym derivative

$$\frac{d\varphi}{du}(u) = 2\lambda \int_0^u e^{-2\lambda(u-z)} d\mathbf{P}\{r \leq z, f > r\}.$$

3. IDLE TIME

Let

$$\wp_G(t) := \mathbf{P}\{N_t = A, \forall u \in (0,t), N_u \neq D \mid N_0 = A\}, t \geq 0.$$

Observe that $\wp_G(t)$ is the probability to find the repairman idle at time *t during* the lifetime of the **G**-system. Note that $\wp_G(\infty)=0$.

Property 3.1

$$\mathbf{E}i_G=\int_0^\infty \wp_G(t)dt. \tag{1}$$

Proof

Clearly,

$$i_G=\int_0^{\theta_G} 1\{N_t=A\,|\,N_0=A\}\,dt=\int_0^\infty 1\{N_t=A\,|\,N_0=A\}1\{\theta_G>t\}\,dt.$$

But

$$1\{N_t=A\,|\,N_0=A\}1\{\theta_G>t\}=1\{N_t=A,\forall u\in(0,t),N_u\neq D\,|\,N_0=A\}.$$

Hence, i_G satisfies the *stochastic* identity

$$i_G=\int_0^\infty 1\{N_t=A,\forall u\in(0,t),N_u\neq D\,|\,N_0=A\}\,dt. \tag{2}$$

However, i_G is stochastically smaller than θ_G and, see Vanderperre *et al.* [2], $\mathbf{E}\theta_G<\infty$. Moreover, the function $1\{N_t=A,\forall u\in(0,t),N_u\neq D\,|\,N_0=A\}$ is $\mathcal{A}\otimes\mathcal{B}[0,\infty)$ measurable. Hence, by (2) and Fubini-Tonelli, e.g. Doob [4],

$$\begin{aligned}\mathbf{E}i_G&=\mathbf{E}\int_0^\infty 1\{N_t=A,\forall u\in(0,t),N_u\neq D\,|\,N_0=A\}\,dt\\&=\int_0^\infty \mathbf{E}1\{N_t=A,\forall u\in(0,t),N_u\neq D\,|\,N_0=A\}\,dt\\&=\int_0^\infty \wp_G(t)dt<\infty.\end{aligned}$$

Notation

The Laplace-transform of $\wp_G(t)$ is denoted by

$$\wp_G^{\wedge}(s):=\int_0^\infty e^{-st}\wp_G(t)dt\text{ , }\operatorname{Re}s\geq 0.$$

Theorem 3.1

For Re $s\geq 0$,

$$\wp_G^{\wedge}(s)=\frac{1}{s+2\lambda}\left(1-\frac{2\lambda}{s+2\lambda}\mathbf{E}e^{-(s+\lambda)r}\right)^{-1}. \tag{3}$$

Proof: Observe that state A is *regenerative* for the process $\{N_t\}$. Therefore, by Renewal Theory,

$$\wp_G(t) = e^{-2\lambda t} + \int_0^t \wp_G(t-u)d\varphi(u).$$

Applying the convolution theorem for Laplace-transforms, entails that

$$\wp^{\wedge}(s) = \frac{1}{s+2\lambda} + \wp_G^{\wedge}(s)\int_0^\infty e^{-su}d\varphi(u).$$

But

$$\int_0^\infty e^{-su}d\varphi(u) = \frac{2\lambda}{s+2\lambda}\int_0^\infty e^{-(s+\lambda)u}dR(u).$$

Finally, note that

$$\int_0^\infty e^{-(s+\lambda)u}dR(u) = \mathbf{E}e^{-(s+\lambda)r}.$$

Hence, the Laplace-transform of $\wp_G(\cdot)$ is completely determined as an explicit functional of *f* and *r*.

Remarks 3.1

By (1) and (3) we obtain

$$\mathbf{E}i_G = \frac{1}{2\lambda}(1-\mathbf{E}e^{-\lambda r})^{-1}.$$

4. DETERMINISTIC REPAIR

As an application, we consider the particular but important case of deterministic repair (replacement), i.e. let

$$R(t) = \begin{cases} 1, \ if \ \ t \ge t_0 > 0, \\ 0, \ if \ t < t_0. \end{cases}$$

Clearly,

$$\mathbf{E}e^{-(s+\lambda)r} = \mathbf{E}(e^{-(s+\lambda)r}1\{r = t_0\}) = e^{-(s+\lambda)t_0}.$$

Hence,

$$\wp_G^{\wedge}(s) = \frac{1}{s+2\lambda}\frac{1}{1-\frac{2\lambda}{s+2\lambda}e^{-(s+\lambda)t_0}}, \ \mathrm{Re}\ s \ge 0.$$

Note that Remark 2.1 implies that $\wp_G(t)$ is Lebesgue-absolutely continuous on $(0,\infty)$ and of bounded variation on $[0,\infty)$. Hence, by the inversion theorem,

$$\wp_G(t) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{-iT}^{iT} e^{st}\wp_G^{\wedge}(s)ds \ , \ t>0.$$

Or by (3),

$$\wp_G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{1}{1-\alpha(s)}\frac{ds}{s+2\lambda},$$

where $\alpha(s) := 2\lambda(s+2\lambda)^{-1}e^{-(s+\lambda)t_0}$. An application of the maximum-modulus theorem entails that

$$|\alpha(s)| \le \max_{\operatorname{Re} s\ge 0} |\alpha(s)| = e^{-\lambda t_0} < 1.$$

Consequently, the identity

$$\frac{1}{1-\alpha(s)} = \sum_{k=0}^{[tt_0^{-1}]}\alpha^k(s) + \frac{\alpha^{[tt_0^{-1}]+1}(s)}{1-\alpha(s)}$$

holds for Re $s \ge 0$ and $t \ge 0$. Furthermore, $\alpha(s)$ is analytic in the half-plane $\{s\text{: Re } s > 0\}$ and boundedly continuous on the closed half-plane $\{s\text{: Re } s \ge 0\}$.

In addition,

$$\lim_{\substack{|s|\to\infty \\ -\frac{\pi}{2}\le \arg s \le \frac{\pi}{2}}} \left(e^{st}\frac{\alpha^{[tt_0^{-1}]+1}(s)}{1-\alpha(s)}\right) = 0.$$

Hence, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{\alpha^{[tt_0^{-1}]+1}(s)}{1-\alpha(s)}\frac{ds}{s+2\lambda} = 0.$$

On the other hand,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\sum_{k=0}^{[tt_0^{-1}]}\alpha^k(s)\frac{ds}{s+2\lambda} = \sum_{k=0}^{[tt_0^{-1}]}(2\lambda)^k e^{-k\lambda t_0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(t-kt_0)}\frac{ds}{(s+2\lambda)^{k+1}}.$$

However, by the residue theorem,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(t-kt_0)}\frac{ds}{(s+2\lambda)^{k+1}} = \frac{1}{k!}(t-kt_0)^k e^{-2\lambda(t-kt_0)}.$$

Hence,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\sum_{k=0}^{[tt_0^{-1}]}\alpha^k(s)\frac{ds}{s+2\lambda} = e^{-2\lambda t}\sum_{k=0}^{[tt_0^{-1}]}\frac{1}{k!}e^{\lambda k t_0}(2\lambda(t-kt_0))^k.$$

We formulate the following

Property 4.1

For $t \geq 0$,

$$\wp_G(t) = e^{-2\lambda t} \sum_{k=0}^{[tt_0^{-1}]} \frac{1}{k!} e^{\lambda k t_0} (2\lambda(t - kt_0))^k.$$

$$\mathbf{E} i_G = \frac{1}{2\lambda(1 - e^{-\lambda t_0})}.$$

5. CONCLUSION

The stochastic behaviour of Gaver's duplex system can be analysed by elegant methods of Renewal Theory combined with a stochastic identity. The particular but important case of deterministic repair (replacement) provides explicit and *exact* results for the survival function and for the probability to find the repairman idle at any instant of time t during the lifetime of the system.

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