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An SEIR Epidemic Model with Profitless Latent Period Time Delay and Pulse Vaccination

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Abstract: In this paper, we formulate a robust SEIR epidemic disease model with a profitless delay and nonlinear incidence, and analyze the dynamics behaviors of the model under pulse vaccination. By use of the discrete dynamical system determined by the stroboscopic map, we obtain an 'infection-free' periodic solution, further, show that the 'infection-free' periodic solution is globally attractive when some parameters of the model are under appropriate conditions. Using the theory on delay functional and impulsive differential equation, we obtain sufficient condition with time delay for the permanence of the system, and prove that time delays, pulse vaccination and nonlinear incidence can bring obvious effects on the dynamics behaviors of the model. Our results indicate that the delay is "profitless". The theoretical results are confirmed by numerical simulations.

Keywords: Permanence; Pulse vaccination; Nonlinear incidence; Latent period time delay; Global attractivity

Mathematics Subject Classification: 92D30, 34K45, 34K25.

1. INTRODUCTION

The SEIR infections disease model is a very important biologic model and has been studied by [1-9]. An epidemic model based on these assumptions is customarily called SIR(susceptible, infectious, recovered) model. However, many diseases incubate inside the hosts for a period of time before the hosts become infectious. We assume that a susceptible individual first goes through a latent period after infection before becoming infectious. For example, the incubation time for canine madness is anything from a few months to years after the patient has got canine madness virus. To include the latent period it is necessary to consider a new group in the population (i.e. exposed). The resulting model is SEIR (susceptible, exposed, infectious, recovered) epidemic type. For a schematic representation of the flow of individuals between the four epidemiological subclasses [5] where the authors model the course of an Ebola outbreak via an SEIR model.

In recent years, pulse vaccination, the repeated application of vaccine over a defined age range, is gaining prominence as a strategy for the elimination of childhood viral infectious such as measles hepatitis, parotitis, smallpox and phthisis. Pulse vaccination strategy (PVS)[1,10-12], consists of periodical repetitions of impulsive vaccinations in a population, on all the age cohorts, differently from the traditional constant continuous vaccination.

Epidemic models with nonlinear incidence were proposed by a number of authors [13-15]. For example, the nonlinear incidence $\frac{\beta SI}{1+\alpha S}$, βSI measures the infection force of the disease and $1/(1+\alpha S)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from

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the crowding effect of the infective individuals.This incidence rate seems more reasonable than βSI because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters.

Generally, death for the infectious during a latent period occurs continuously, which is called the phenomena of 'time delay', so, time delay has important biologic meaning in epidemic models. Recently, many epidemic models with time delay were extensively studied by [3,8,16-18]. The investigation of impulsive delay differential equations is inchoate, and most of impulsive delay differential equations are analyzed in theory [19-21], however the good results on global qualitative analysis for delay biologic models with impulse effect are not extensive. Time delay and impulse are introduced into epidemic disease models, which greatly enriches biologic background. Therefore, in present paper, we propose a delay SEIR epidemic model with nonlinear incidence and pulse vaccination, study their dynamic behaviors (the 'infection-free' periodic solution, the permanence, global attractive behavior) under pulse vaccination.

The organization of this paper is as follows. In the next section, we introduce the SEIR epidemic model with time delay and pulse vaccination. In Section 3 and 4, we investigate the dynamic behaviors of the model and obtain the sufficient condition for global attractivity of 'infection-free' periodic solution and the permanence of the population. Lastly, we give pulse vaccination strategy and show the effect of pulse vaccination rate, period of pulsing, latent period of the disease, and nonlinear incidence on the dynamics behaviors of model by numerical analysis.

2. SEIR EPIDEMIC MODEL

In the following model, we study a population that is partitioned into four classes, the susceptible, exposed, infectious and recovered, with sizes denote by S; E; I and R, respectively, and consider nonlinear incidence, the latent period of disease, and pulse vaccination strategy. Motivated by [1-4,6,7], we give the following assumptions:

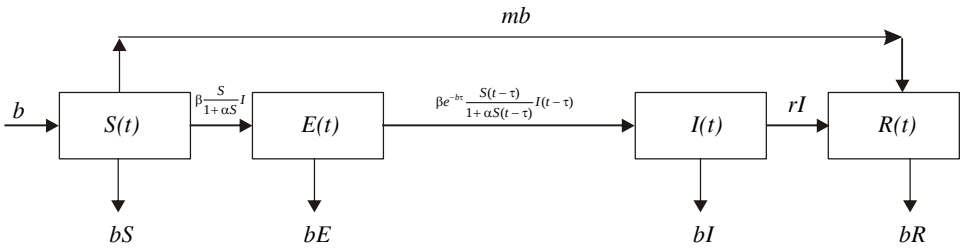
(i) Death rate (d) is equal to birth rate (b), that is, the population has a constant size, which is normalized to unity $S(t)+E(t)+I(t)+R(t)=1$;

(ii) All new-born transfer to the 'susceptible' class (S) as soon as who are born, and the treatment with pulse vaccination is taken successfully to a proportion (with m) of those to all newborns;

(iii) The nonlinear incidence in this model is $\beta\frac{S}{1+\alpha S}I$, where α is called saturation.

(iv) Suppose that the latent period of disease is τ (latent period time delay) and consider the death toll of the exposed individuals during latent period of disease, that is the $\beta e^{-\mu\tau}\frac{S(t-\tau)}{1+\alpha S(t-\tau)}I(t-\tau)$ term.

Therefore, under assumptions above, we consider a new SEIR model with time delay and pulse vaccination as follows:



$$\left\{\begin{array}{c} \left.\begin{array}{c} \dot{S}(t)=b-\beta\dfrac{S(t)}{1+\alpha S(t)}I(t)-bS(t), \\ \dot{E}(t)=\beta\dfrac{S(t)}{1+\alpha S(t)}I(t)-\beta e^{-b\tau}\dfrac{S(t-\tau)}{1+\alpha S(t-\tau)}I(t-\tau)-bE(t), \\ \dot{I}(t)=\beta e^{-b\tau}\dfrac{S(t-\tau)}{1+\alpha S(t-\tau)}I(t-\tau)-(r+b)I(t), \\ \dot{R}(t)=rI(t)-bR(t), \end{array}\right\} t\neq nT, \\ \left.\begin{array}{c} S(t^+)=S(t)-mb, \\ E(t^+)=E(t), \\ I(t^+)=I(t), \\ R(t^+)=R(t)+mb, \end{array}\right\} \end{array}\right. \tag{1}$$

where mb (with $0 < m < 1$) is those new-born vaccinated successfully. The details for system (1) can be seen in [3, 9, 13-15].

Note that the variables E and R do not appear in the first and third equations of system (1), hence we only need to consider the subsystem of (1) as follows:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dot{S}(t)=b-\beta\dfrac{S(t)}{1+\alpha S(t)}I(t)-bS(t), \\ \dot{I}(t)=\beta e^{-b\tau}\dfrac{S(t-\tau)}{1+\alpha S(t-\tau)}I(t-\tau)-(r+b)I(t), \end{array}\right\} \quad t\neq nT, n\in N, \\ \left.\begin{array}{l} S(t^+)=S(t)-mb, \\ I(t^+)=I(t), \end{array}\right\} \quad t=nT, n\in N \end{array}\right. \tag{2}$$

We note that if the initial epidemic state (S(0), I(0)) is such that: $S(0)$-m b ≤ 0 (when the pulsing rate m is very large), then the fraction of susceptible subjects after the pulsing would be nonpositive: $S(0^+) \leq 0$, which implies all the susceptibles transfer to the recovered class right now, so, our aim is obtained. However, from the fact and the standpoint of economy, we hope the minimum amount of the susceptibles are vaccinated such that the epidemic disease is eradicated eventually. Therefore, we will find the conditions which the epidemic disease is eradicated eventually provided that $S(0) - mb > 0$ in the rest of this paper.

The initial conditions for (2) are

$$(\phi_1(s),\phi_2(s))\in C_+ = C([-\tau,0],R_+^2),\phi_i(0)>0(i=1,2). \tag{3}$$

From the biological point of view, we only consider system (2) in the biological meaning region: $D=\{(S,I)\,|\,S,I\geq 0\}$.

Lemma 2.1. (see [22]) *Consider the following impulse differential inequalities:*

$$\begin{aligned} w'(t) &\le (\ge)\, p(t)w(t) + q(t), && t \ne t_k, \\ w(t_k^+) &\le (\ge) d_k w(t_k) + b_k, && t = t_k, k \in N, \end{aligned}$$

where $p(t), q(t) \in C(R_+, R), d_k \ge 0,$ *and* b_k *are constants.*

Assume

(A_0) *the sequence* $\{t_k\}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots,$ *with* $\lim_{t\to\infty} t_k = \infty;$

(A_1)$w \in PC'(R_+, R)$ *and w(t) is left-continuous at* t_k, $k \in N$.

Then

$$\begin{aligned} w(t) \le (\ge) w(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) b_k \\ + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(\theta)d\theta\right) q(s)ds, t \ge t_0. \end{aligned}$$

When pulse vaccination and delays are incorporated into SEIR models, the systems become nonautonomous, which make it difficult for us in studying the model. Therefore, the literature on this one is not extensive. Now we study the global attractivity of 'infection-free' solution (infection-eradication) and the permanence (endemic) of the systems (2) with initial conditions (3) under the influence of impulse, time delay and nonlinear incidence.

3. 'INFECTION-FREE' PERIODIC SOLUTION

3.1 Existence of 'Infection-free' Solution

Before starting our theorem, we give the following lemma.

Lemma 3.1. (see [23]) *Consider the following delay differential equation*

$$\frac{dx(t)}{dt} = r_1 x(t-\tau) - r_2 x(t),$$

where a, b, τ *are all positive constants and x(t) > 0 for* $t \in [-\tau, 0]$.

(i) If $r_1 < r_2,$ *then* $\lim_{t\to\infty} x(t) = 0.$

(ii) If $r_1 > r_2,$ *then* $\lim_{t\to\infty} x(t) = +\infty.$

We begin the analysis of (2) by first demonstrating the existence of an 'infection-free' solution, in which infectious individuals are entirely absent from the population permanently, i.e.,

$$I(t) = 0, t \ge 0. \tag{4}$$

This is motivated by the fact that $I^* = 0$ is an equilibrium solution for the variable $I(t)$, as it leaves $I'(t) = 0$. Under these conditions, we show below that the susceptible population oscillates with period T in synchronization with the periodic pulse vaccination. In the section that follows, we determine the global attractivity of this 'infection-free' periodic solution.

Assuming (4), we know that the growth of the susceptible in the time-interval $nT < t \le (n+1)T$ and give some basic properties of the following subsystem of (2)

$$\begin{cases} \dot{S}(t) = -bS(t) + b, & t \ne nT, n \in N \\ S(t^+) = S(t) - mb, & t = nT, n \in N. \end{cases} \tag{5}$$

For integrating and solving equation (5) between pulses, yields

$$S(t) = 1 - (1 - S(nT))e^{-b(t-nT)}, nT < t \le (n+1)T,$$

where $S(nT)$ be the initial value at time nT. Using the second equation of system (5), we deduce the stroboscopic map on the discrete dynamical system (see [22]) such that

$$S((n+1)T) = 1 - (1 - S(nT))e^{-bT} - mb = f(S(nT)), \tag{6}$$

where $f(S) = 1 - (1 - S)e^{-bT} - mb$. If $1 - mb - e^{-bT} > 0,$ then it is easy to know that difference equation (6) has unique a positive equilibrium $S^* = 1 - \frac{mb}{1 - e^{-bT}}$. Since $f(S)$ is a straight line with respect to S with slope less than 1, we obtain that S^* is globally asymptotically stable. This implies that the corresponding periodic solution of system (5), is globally asymptotically stable.

Then we have

$$\begin{cases} S^*(t) = 1 - \dfrac{mb}{1 - e^{-bT}} e^{-b(t-nT)}, & t \in (nT, (n+1)T], n \in N, \\ S^*(0^+) = S^*(nT^+) = 1 - \dfrac{mb}{1 - e^{-bT}}, \end{cases}$$

which is unique a globally asymptotically stable positive periodic solution of system (5).

When the pulsing rate m is very large, then the fraction of susceptible subjects after the pulsing would be nonpositive: $S^*(0^+) \le 0$, which implies all the susceptibles transfer to the recovered class right now, so, our aim is obtained. However, from the fact and the standpoint of economy, we hope the minimum amount of the susceptibles are vaccinated such that the epidemic disease is eradicated eventually. Therefore, we will find the conditions which the epidemic disease is eradicated eventually provided that $1 - \frac{mb}{1 - e^{-bT}} > 0 (1 - mb - e^{-bT} > 0)$ in the rest of this paper.

Since the solution of (5) is

$$\begin{cases} S(t)=\left(S(0^+)-\left(1-\dfrac{mb}{1-e^{-bT}}\right)\right)e^{-bt}+S^*(t), t\in(nT,(n+1)T], \\ S^*(0^+)=1-\dfrac{mb}{1-e^{-bT}}, \end{cases} \tag{7}$$

we have the following Lemma (3.2).

Lemma 3.2. *System (5) has unique a positive periodic solution* $S^*(t)$*, that is to say the system (2) has an 'infection-free' periodic solution* $(S^*(t),0)$ *for* $t\in(nT,(n+1)T], n\in N$*, for any solution* $(S(t),I(t))$ *of (2) we have* $S(t)\to S^*(t)$ *as* $t\to\infty$.

Now we give the conditions which assure the global attractivity of the infection-free periodic solution $(S^*(t),0)$.

3.2 Global Attractive of 'Infection-free' Periodic Solution

Denote

$$\mathscr{R}=\frac{\hbar\left[e^{bT}-(1+mb)\right]}{(1+\alpha)(e^{bT}-1)-\alpha mb}, \tag{8}$$

where $\hbar=\frac{\beta e^{-b\tau}}{r+b}$.

Theorem 3.2. *If* $\mathscr{R}_1<1$ *and* $1-mb-e^{-bT}$*, then the 'infection-free' periodic solution* $(S^*(t),0)$ *of system (2) is globally attractive.*

Proof. Let $(S(t), I(t))$ be any solution of system (2) with initial condition (3). Since $\mathscr{R}_1<1$, we have

$$\beta e^{-b\tau}\frac{1-\frac{mbe^{-bT}}{1-e^{-bT}}}{1+\alpha\left(1-\frac{mbe^{-bT}}{1-e^{-bT}}\right)}-(r+b)<0.$$

Since $\frac{x}{1+\alpha x}$ is a monotonically increasing function with respect to x, we can choose a sufficiently small positive constant ε such that

$$\beta e^{-b\tau}\frac{1-\frac{mbe^{-bT}}{1-e^{-bT}}+\varepsilon}{1+\alpha\left(1-\frac{mbe^{-bT}}{1-e^{-bT}}+\varepsilon\right)}-(r+b)<0. \tag{9}$$

Note that $dS(t)/dt\le b-bS(t), S(t^+)=S(t)-mb$ for $nT<t\le(n+1)T$, then we consider the following impulsive impulse differential inequalities:

$$\begin{cases} \dfrac{dS(t)}{dt} \le b - bS(t), & t \ne nT, n \in N \\ S(t^+) = S(t) - mb, & t = nT, n \in N. \end{cases}$$

By using Lemma 2.1, we have

$$\begin{aligned} S(t) &\le S(0^+)e^{\int_0^t -bds} - mb \sum_{0<nT<t} e^{\int_{nT}^t -bds} + \int_0^t \prod_{s<nT<t} e^{\int_s^t -bd\theta} bds \\ &\le \left(S(0^+) - 1 + \frac{mbe^{-bT}}{1-e^{-bT}} \right) e^{-bt} + 1 - \frac{mbe^{-bT}}{1-e^{-bT}}, \end{aligned}$$

which implies

$$\limsup_{t\to\infty} S(t) \le 1 - \frac{mbe^{-bT}}{1-e^{-bT}}.$$

Hence, there exist a positive integer n_1 and a arbitrarily small positive constant ε such that for all $t \ge n_1 T$,

$$S(t) \le 1 - \frac{mbe^{-bT}}{1-e^{-bT}} + \varepsilon =: \eta. \tag{10}$$

From (10) and the second equation of (2), we get that, for $t > n_1 T + \tau$,

$$\dot{I}(t) \le \beta e^{-b\tau} \frac{\eta}{1+\alpha\eta} I(t-\tau) - (r+b)I(t). \tag{11}$$

Consider the following comparison equation

$$\frac{dz(t)}{dt} = \beta e^{-b\tau} \frac{\eta}{1+\alpha\eta} z(t-\tau) - (r+b)z(t). \tag{12}$$

We see that (9) i.e.

$$\beta e^{-b\tau} \frac{\eta}{1+\alpha\eta} - (r+b) < 0.$$

According to Lemma 3.1 we obtain that

$$\lim_{t\to\infty} z(t) = 0.$$

Since $I(s) = z(s) = \phi_2(s) > 0$ for all $s \in [-\tau, 0]$, by the comparison theorem in differential equation and the nonnegativity of solution (with $I(t) \ge 0$), we have that $I(t) \to 0$ as $t \to \infty$.

Without loss of generality, we may assume that $0 < I(t) < \varepsilon$ for all $t \geq 0$, by the first equation of system (2), we have

$$\frac{dS}{dt} \geq b - (\beta\varepsilon + b)S(t).$$

Then we have $\tilde{z}_1(t) \leq S(t)$ and $\tilde{z}_1(t) \to S^*(t)$, as $\varepsilon \to 0$, where $\tilde{z}_1(t)$ is a unique positive periodic solution of

$$\begin{cases} \dfrac{dz_1(t)}{dt} = b - (\beta\varepsilon + b)z_1(t), t \neq nT, n \in N, \\ z_1(t^+) = z_1(t) - mb, \qquad t = nT, n \in N, \\ z_1(0^+) = S(0^+). \end{cases} \tag{13}$$

From (13), we have that, for $nT < t \leq (n+1)T$,

$$\tilde{z}_1(t) = \frac{b}{\beta\varepsilon + b} - \frac{mb}{1 - \exp\{-(\beta\varepsilon + b)T\}} \exp\{-(\beta\varepsilon + b)(t - nT)\}.$$

By using comparison theorem of impulsive equation (see Theorem 3.1.1 in [22]), for any $\varepsilon_1 > 0$ there exists such a $T_1 > 0$ that, for $t > T_1$,

$$S(t) > \tilde{z}_1(t) - \varepsilon_1. \tag{14}$$

On the other hand, from the first equation of (2), it follows that

$$\frac{dS}{dt} \leq b - bS(t).$$

Then we have $S(t) \leq \tilde{z}_2(t)$ and $\tilde{z}_2(t) \to S^*(t)$, as $t \to \infty$, where $\tilde{z}_2(t)$ is a unique positive periodic solution of

$$\begin{cases} \dfrac{dz_2(t)}{dt} = b - bz_2(t), t \neq nT, n \in N, \\ z_2(t^+) = z_2(t) - mb, \quad t = nT, n \in N, \\ z_2(0^+) = S(0^+). \end{cases} \tag{15}$$

From (15), we have that, for $nT < t \leq (n+1)T$,

$$\tilde{z}_2(t) = 1 - \frac{mb}{1-e^{-bT}} e^{-b(t-nT)}.$$

By using comparison theorem of impulsive equation, for any $\varepsilon_1 > 0$ there exists a $T_2 > 0$ such that

$$S(t) < \tilde{z}_2(t) + \varepsilon_1 \text{for} t > T_2. \tag{16}$$

Let $\varepsilon \to 0$, then it follows from (14) and (16) that

$$S^*(t) - \varepsilon_1 < S(t) < S^*(t) + \varepsilon_1, \tag{17}$$

for *t* large enough, which implies $S(t) \to S^*(t)$ as $t \to \infty$. This completes the proof.

Corollary 3.2. (i) *If* $\beta e^{-b\tau} \le (1+\alpha)(r+b)$, *then 'infection-free' periodic solution* $(S^*(t), 0)$ *is globally attractive*. (ii) *If* $\beta e^{-b\tau} > (1+\alpha)(r+b)$ *and* $m > m^*$ *or* $T < T_*$ *or* $\tau > \tau^*$ *or* $\alpha > \alpha^*$, *then 'infection-free' periodic solution* $(S^*(t), 0)$ *is globally attractive, where the critical values* m^*, T_*, τ^*, *and* α^* *are listed in Table 1 for system (2).*

Table 1
Critical Values of Some Parameters of System (2) ($\mathcal{R}_1 < 1$ must be satisfied)

The conditions for global attractivity of $(S^*(t), 0)$	
$m > m^*$	$m^* = \frac{(e^{bT}-1)[\beta e^{-b\tau} - (1+\alpha)(r+b)]}{b\left[\beta e^{-b\tau} - \alpha(r+b)\right]}$
$T < T_*$	$T_* = \frac{1}{b} In \frac{(1+mb)\left[\beta e^{-b\tau} - \alpha(r+b)\right] - (r+b)}{\beta e^{-b\tau} - (1+\alpha)(r+b)}$
$t > t^*$	$\tau^* = \frac{1}{b} \ln \frac{\beta\left[e^{bT} - (1+mb)\right]}{(r+b)\left[(1+\alpha)e^{bT} - 1 - \alpha(1+mb)\right]}$
$a > a^*$	$\alpha^* = \frac{\left(e^{bT}-1\right)\left[\beta e^{-b\tau} - (r+b)\right] - mb\beta e^{-b\tau}}{(r+b)\left[e^{bT} - (1+mb)\right]}$

Remark 3.2: From Theorem 3.2, we can see that a short period of pulsing (with T) or a large pulse vaccination rate (with *m*) or a long latent period of the disease (with τ) or a large saturation (with α) is sufficient condition for the global attractivity of the 'infection-eradication' periodic solution.

4. PERMANENCE

4.1. Uniform Persistence of the Population

Before starting our theorem, we give the following definition.

Definition 4.1. System(2) is said to be uniformly persistent if there are positive constants m_i, $i = 1,2$ and a finite time T_0 such that for all solutions $(S(t), I(t))$ with initial values $S(0^+) > 0$, $I(0^+) > 0$, $m_1 \le S(t)$, $m_2 \le I(t)$ holds for all $t \ge T_0$.

Definition 4.2. System(2) is said to be permanent if there exists a compact region $D \in \text{int}\,\Omega$ such that every solution of system (2) with initial conditions (3) will eventually enter and remain in region D.

Denote

$$\mathscr{R}_2 = \frac{(\hbar-\alpha)(1-e^{-bT})}{1+mb(\hbar-\alpha)-e^{-bT}}, \text{ where } \hbar = \frac{\beta e^{-b\tau}}{r+b}. \tag{18}$$

Theorem 4.1. *If $\mathscr{R}_2 > 1$, then the system (2) is uniformly persistent, that is, there exist two positive constants m_1 and m_2 such that $S(t) \geq m_1$, $I(t) \geq m_2$ for t large enough.*

Proof. Suppose that $X(t) = (S(t), I(t))$ is any positive solution of system (2) with initial conditions (3). The second equation of system (2) may be written as follow:

$$\dot{I}(t) = \left[\beta e^{-b\tau}\frac{S(t)}{1+\alpha S(t)} - (r+b)\right] I(t) - \beta e^{-b\tau}\frac{d}{dt}\int_{t-\tau}^{t}\frac{S(\theta)}{1+\alpha S(\theta)} I(\theta)d\theta. \tag{19}$$

Define

$$V(t) = I(t) + \beta e^{-b\tau}\int_{t-\tau}^{t}\frac{S(\theta)}{1+\alpha S(\theta)} I(\theta)d\theta.$$

Calculating the derivative of $V(t)$ along the solution of (2), it follows from (19) that

$$\begin{aligned}\frac{dV(t)}{dt} &= (r+b)\left[\frac{\beta e^{-b\tau}}{r+b}\frac{S(t)}{1+\alpha S(t)} - 1\right] I(t) \\ &= (r+b)\left[\hbar\frac{S(t)}{1+\alpha S(t)} - 1\right] I(t).\end{aligned} \tag{20}$$

Set

$$m_2^* = \frac{b}{\beta}(\mathscr{R}_2 - 1).$$

Since $\mathscr{R}_2 > 1$, then $m_2^* > 0$ and there exists a positive constant ε_1 small enough such that

$$\hbar\frac{\varrho}{1+\alpha\varrho} > 1, \tag{21}$$

where

$$\varrho = \frac{b}{m_2^*\beta + b} - \frac{mb}{1-\exp\{-(m_2^*\beta+b)T\}} - \varepsilon_1 > 0.$$

For any positive constant t_0, we claim that the inequality $I(t) < m_2^*$ cannot hold for all $t \geq t_0$. Otherwise, there is a positive constant t_0, such that $I(t) < m_2^*$ for all $t \geq t_0$. From the first and third equation of system (2), we have

$$\begin{cases} \dfrac{dS(t)}{dt} \geq b - (m_2^*\beta + b)S(t), t \neq nT, \\ S(t^+) = S(t) - mb, \qquad\qquad t = nT. \end{cases}$$

Similarly, by Lemma 2.1, there exists such $T_1 \geq t_0 + \tau$, for $t \geq T_1$ that

$$S(t) > \frac{b}{m_2^*\beta + b} - \frac{mb}{1 - \exp\{-(m_2^*\beta + b)T\}} - \varepsilon_1 =: \varrho. \tag{22}$$

From (20) and (22), we see that

$$\frac{dV(t)}{dt} > (r+b)\left[\hbar\frac{\varrho}{1+\alpha\varrho} - 1\right] I(t), t \geq T_1. \tag{23}$$

Let

$$I^l = \min_{t\in[T_1, T_1+\tau]} I(t).$$

We show that $I(t) \geq I^l$ for all $t \geq T_1$. Otherwise, there exists a nonnegative constant T_2 such that $I(t) \geq I^l$ for $t \in [T_1, T_1 + \tau + T_2]$, $I(T_1 + \tau + T_2) = I^l$ and $\dot{I}(T_1 + \tau + T_2) \leq 0$. Thus from the second equation of (2), (21) and (22), we easily see that

$$\begin{aligned} \dot{I}(T_1 + \tau + T_2) &\geq \left(\beta e^{-b\tau}\frac{\varrho}{1+\alpha\varrho} - (r+b)\right) I^l \\ &= (r+b)\left(\hbar\frac{\varrho}{1+\alpha\varrho} - 1\right) I^l \\ &> 0, \end{aligned}$$

which is a contradiction. Hence we get that $I(t) \geq I^l > 0$ for all $t \geq T_1$. From (23), we have

$$\begin{aligned} \frac{dV(t)}{dt} &> (r+b)\left[\hbar\frac{\varrho}{1+\varrho} - 1\right] I^l \\ &> 0, \end{aligned}$$

which implies $V(t) \to +\infty$ as $t \to +\infty$. This is a contradiction to $V(t) \le 1 + \beta\tau e^{-b\tau}$. Therefore, for any positive constant t_0, the inequality $I(t) < m_2^*$ cannot hold for all $t \ge t_0$.

On the one hand, if $I(t) \ge m_2^*$ holds true for all *t* large enough, then our aim is obtained. On the other hand, *I*(*t*) is oscillatory about m_2^*.

Let

$$m_2 = \min\left\{\frac{m_2^*}{2}, m_2^* e^{-(r+b)\tau}\right\} > 0.$$

In the following, we shall show that $I(t) \ge m_2$. There exist two positive constants $\bar{t}, \omega$ such that

$$I(\bar{t}) = I(\bar{t} + \omega) = m_2^*$$

and

$$I(t) < m_2^*, \text{for} \bar{t} < t < \bar{t} + \omega.$$

In the following, we only need to prove that $I(t) \ge m_2$ ($0 < m_2 < m_2^*$) for $\bar{t} < t < \bar{t} + \omega$. When $\bar{t}$ ($\bar{t} \ge T_1$) is large enough, the inequality $S(t) > \varrho$ holds true for $\bar{t} < t < \bar{t} + \omega$, since $I(t) < m_2^*$. Since *I*(*t*) is continuous and bounded and is not effected by impulses, we conclude that *I*(*t*) is uniformly continuous. Hence there exists a constant T_3 (with $0 < T_3 < \tau$ and T_3 is independent of the choice of $\bar{t}$) such that $I(t) > \frac{m_2^*}{2}$ for all $\bar{t} \le t \le \bar{t} + T_3$. If $\omega \le T_3$, our aim is obtained. If $T_3 < \omega \le \tau$, from the second equation of (2) we have that $\dot{I}(t) \ge -(r+b)I(t)$ for $\bar{t} < t \le \bar{t} + \omega$. Then we have $I(t) \ge m_2^* e^{-(r+b)\tau}$ for $\bar{t} < t \le \bar{t} + \omega \le \bar{t} + \tau$ since $I(\bar{t}) = m_2^*$. It is obvious that $I(t) \ge m_2$ for $\bar{t} < t \le \bar{t} + \omega$. Similarly, if $\omega \ge \tau$, by the second equation of (2), then we see that $I(t) \ge m_2$ for $\bar{t} \le t \le \bar{t} + \tau$. Thus, proceeding exactly as the proof for above claim, we can obtain $I(t) \ge m_2$ for $\bar{t} + \tau \le t \le \bar{t} + \omega$. Since the interval $[\bar{t}, \bar{t} + \omega]$ is arbitrarily chosen (we only need $\bar{t}$ to be large), we get that $I(t) \ge m_2$ for *t* large enough. In view of our arguments above, the choice of m_2 is independent of the positive solution of (2) which satisfies that $I(t) \ge m_2$ for sufficiently large *t*.

From the first equation of (2), we have that

$$\frac{dS}{dt} \ge b - (\beta + b)S(t).$$

Then we have $S(t) \ge \tilde{z}_3(t)$, where $\tilde{z}_3(t)$ is unique a globally asymptotically stable positive periodic solution of

$$\begin{cases} \dfrac{dz_3(t)}{dt} = b - (\beta + b) z_3(t), t \neq nT, n \in N, \\ z_3(t^+) = z_3(t) - mb, t = nT, n \in N, \\ z_3(0^+) = S(0^+) > 0. \end{cases} \tag{24}$$

There exists a $\varepsilon > 0$ small enough such that

$$S(t) \geq \tilde{z}_3(t) \geq \tilde{z}_3(0^+) = \frac{b}{\beta + b} - \frac{mb}{1 - e^{-(\beta+b)T}} =: m_1. \tag{25}$$

This completes the proof.

4.2 Permanence of the Population

Theorem 4.2. *If* $\mathcal{R}_2 > 1$. *then system (2) is permanent.*

Proof. Suppose that $X(t) = (S(t), I(t))$ is any positive solution of system (2) with initial conditions (3). By Theorem 4.1, there exist positive constants m_1, m_2 and T^* such that $S(t) \geq m_1, I(t) \geq m_2$ for $t \geq T^*$. Set

$$D = \{(S, I) \in R_+^2 \mid m_1 \leq S(t), m_2 \leq I(t), S(t) + I(t) \leq 1\}.$$

Then D is a bounded compact region which has positive distance from coordinate axes. By Theorem 4.1, one obtains that every solution of system (2) with the initial condition (3) eventually enters and remains in the region D. The proof is completed.

Corollary 4.2. *If* $m < m_*$ *or* $T > T^*$ *or* $\tau < \tau_*$ *or* $\alpha < \alpha_*$, *then system (2) is permanent, that is, the disease can generate an endemic, where the critical values* m_*, T^*, τ_*, *and* α_* *are listed in Table 2 for system (2).*

Table 2
Critical Values of Some Parameters of System (2) ($\mathcal{R}_2 > 1$ must be satisfied)

The conditions for permanence of the population	
m < m*	$m_* = 1 - \frac{(\hbar-\alpha)\left(b+e^{-bT}-1\right)+1-e^{-bT}}{b(\hbar-\alpha)}$
T > T*	$T^* = -\frac{1}{b}\ln\left(1 - \frac{mb(\hbar-\alpha)}{\hbar-\alpha-1}\right)$
t < t*	$\tau_* = \frac{1}{b}\ln\left(\frac{\beta(1-e^{-bT}-mb)}{(r+b)[(1+\alpha)(1-e^{-bT})-mb\alpha]}\right)$
a < a*	$\alpha_* = \hbar - \frac{1-e^{-bT}}{1-e^{-bT}-mb}$

Remark 4.2.1. Let $m = 0$ and $\alpha = 0$ then system (1) become ([3,9])

$$\begin{cases} \dot{S}(t) = b - \beta S(t)I(t) - bS(t), \\ \dot{E}(t) = \beta S(t)I(t) - \beta e^{-b\tau} S(t-\tau)I(t-\tau) - bE(t), \\ \dot{I}(t) = \beta e^{-b\tau} S(t-\tau)I(t-\tau) - (r+b)I(t), \\ \dot{R}(t) = rI(t) - bR(t). \end{cases} \tag{26}$$

If $\mathscr{R} = \frac{\beta e^{-b\tau}}{r+b} < 1$, then the 'infection-free' equilibrium of (26) is globally attractive. If $\mathscr{R} = \frac{\beta e^{-b\tau}}{r+b} > 1$, then the disease is permanent. The homologous results were also given in [3] and [9].

Remark 4.2.2. From Theorem 4.1 and 4.2, we can see that a long period of pulsing (with *T*) or a small pulse vaccination rate (with *m*) or a short latent period of the disease (with τ) or a small saturation (with α) is sufficient condition for the permanence of the epidemic disease.

5. NUMERICAL ANALYSIS AND ECOLOGICAL DISCUSSION

In this paper, we introduce a time delay, pulse vaccination and nonlinear incidence into the SEIR model, and theoretically analyze the influence of pulse vaccination, the latent period of disease, nonlinear incidence on infection-eradication and the permanence of epidemic disease. Theorem 3.2, 4.1 and 4.2 show that $\mathscr{R}_1$ and $\mathscr{R}_2$ depend on time delay τ, so, we call it "profitless" delay. By Theorem 3.2, 4.1 and 4.2, we can choose a smaller vaccination period (with *T*) or increase the proportion (with *m*) of those vaccinated successfully to all of the newborn population such that $\mathscr{R}_1 < 1$ in order to prevent the epidemic disease from generating endemic.

In the following, we will analyze the influence of pulse vaccination rate, period of pulsing, latent period of the disease, nonlinear incidence rate on the system (2) by numerical analysis. We consider the hypothetical set of parameter values as $b = 0.5, r = 0.3, \beta = 2, \alpha = 0.3, \tau = 1, m = 0.2, T = 0.5$.

Table 3. The influence of pulse vaccination rate (with *m*), period of pulsing (with *T*), latent period of the disease (with τ), and saturation (α) for nonlinear incidence on the global attractivity of the 'infection-eradication' periodic solution and the permanence of epidemic disease.

m	*T*	*τ*	*α*	$\mathscr{R}_1$	$\mathscr{R}_2$	*Attractivity*	*Permanence*
0.2	0.5	1	0.3	0.8226	0.7848	yes	no
0.05	0.5	1	0.3	1.0858	1.0694	no	yes
0.2	6	1	0.3	1.1618	1.0784	no	yes
0.2	0.5	0.1	0.3	1.2900	1.0714	no	yes
0.2	2	1	0.5	0.9709	0.8756	yes	no
0.2	2	1	0.1	1.3052	1.1571	no	yes
0.4	0.5	0	0.3	0.7359	0.6791	yes	no
0.2	0.5	0	0.3	1.3561	1.1029	no	yes
0.4	0.5	1	0	0.6395	0.4485	yes	no
0		2	0.3	0.7075	0.6198	yes	no

By Theorem 3.2, we know that when $\mathcal{R}_1 = 0.8226 < 1$, the disease dies out ultimately (see Fig.1 (1)), the susceptible population (S) periodically oscillate (see Fig.1 (2)), and the 'infection-eradication' periodic solution is globally attractive (see Fig. 1 (3)). By Theorem 4.1 and 4.2, when $\mathcal{R}_2 = 1.0694 > 1$, the system (2) is permanent (see Fig.2 (4) and (5)). Fig. 2 also shows periodic fluctuating of the population for system (2) which implies it can generate an endemic (see Fig. 2 (6)).

From the second to eighth line of Table 3, we can see that a smaller pulse vaccination rate or a longer period of pulsing or a shorter latent period of the disease or a smaller saturation (α) for nonlinear incidence could cause global attractive 'infection-eradication' periodic solution to lose and generate an endemic. From the seventh and eighth line of Table 3, we can see that when pulse vaccination rate ($m = 0.4$) is larger or the period of pulsing is shorter, although $\tau = 0$ or $\alpha = 0$, the epidemic disease can not be permanent. The last line of Table 3 show that when the latent period of disease are larger, although $m = 0$, the epidemic disease couldn't be permanent yet. ***Ultimately, when the latent period time delay is too long, the permanence of system disappears and the the epidemic disease dies out. This shows the sensitivity of the model dynamics on latent period time delay. The ultimate scenario makes intuitive biological sense: if the susceptible takes too long to transfer to the infectious, then the infectious population will suffer low survival and, as a result, the highest possible***

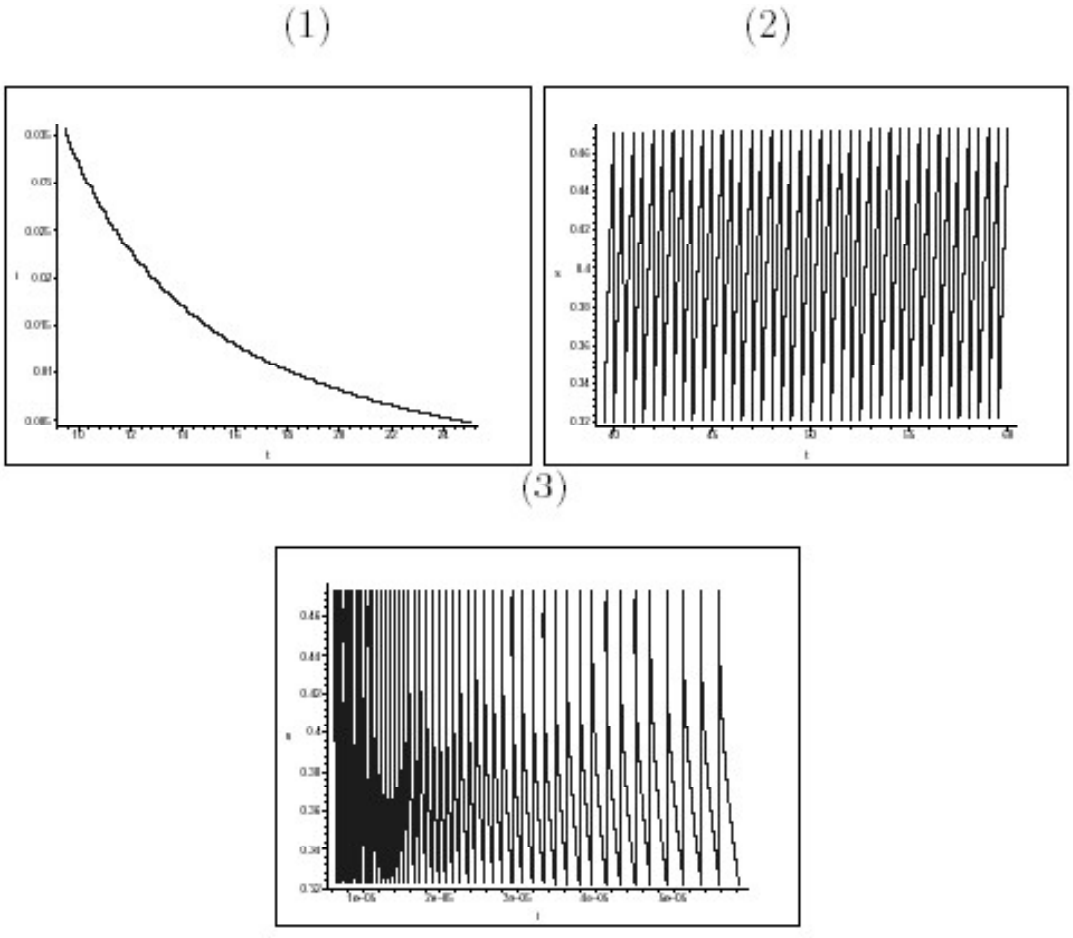


Fig. 1: Dynamical Behavior of the System (2) with $b = 0.5, r = 0.3, \beta = 2, \alpha = 0.3, \tau = 0, m = 0.4, T = 0.5$ and $R_1 = 0.7359 < 1$.
(1)Time-series of the Infective Population (I) of System (2). (2)Time-series of the Susceptible Population (S) of System (2). (3) Phase Portrait (Global Attractivity of the 'infection-eradication' Periodic Solution) of the Susceptible and Infectious Population of the System (2).

recruitment rate to the infectious ($\beta e^{-b\tau}$) ***will drop below the the infectious death rate b, leading to the extinction of I.*** This implies that pulse vaccination and time delays brings determinant effect on the dynamics behaviors of the model, the otherwise effects are accidental. Therefore, it is very important to consider time delays for latent period of the disease in SEIR epidemic disease models. This give us much help in presenting a pulse vaccination strategy.

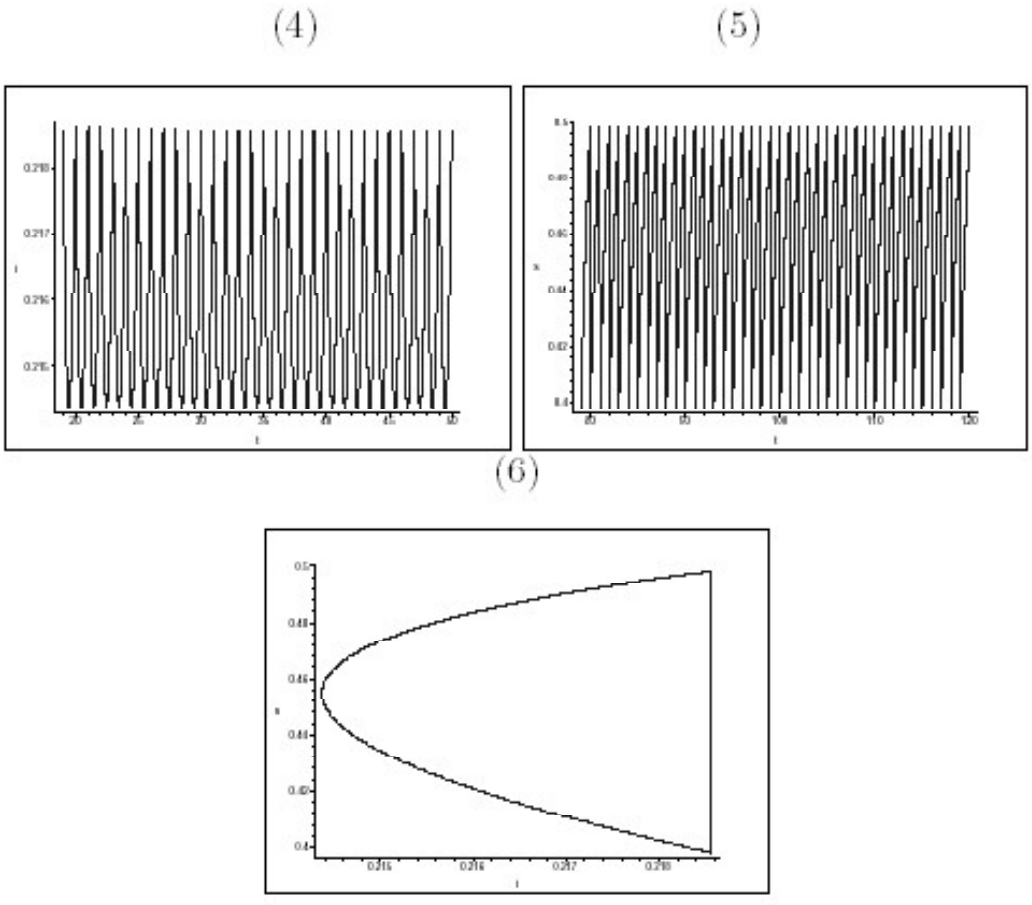


Fig. 2: Dynamical Behavior of the System (2) with $b = 0.5, r = 0.3, \beta = 2, \alpha = 0.3, \tau = 0, m = 0.2, T = 0.5$ **and** $R_2 = 1.1029 > 1$**. (4)Time-series of the Infective Population of System (2). (5)Time-series of the Susceptible Population of System (2). (6)Phase Portrait (*T*–periodic Solution) of the Susceptible and Infectious Population of the System (2).**

REFERENCES

[1] A. D'Onofrio, Stability properties of pulse vaccination strategy in SEIR epidemic model. Math.Biosci. Vol. **179**, pp. 57-72, 2002.

[2] H. Hethcote, H. Stech, and P. Van den Driessche, Periodicity and stability in epidemic models: A survey, in Differential Equations and Applications in Ecology, Epidemics,and Population Problems, K. L. Cook, ed., Academic Press, New York, pp. 65-85, 1981.

[3] K. Cooke and P. Van Den Driessche, Analysis of an SEIRS epidemic model with two delays. J. Math. Biol. Vol. **35**, pp. 240-260, 1996.

[4] Y. Li, R. John, L. Wang, and K. János, Global dynamics of a SEIR model with varying total population size. Math. Biosci. Vol. **160**, pp. 191-213, 1999.

[5] G. Chowell, N. Hengartner *et al.*, The basic reproductive number of Ebola and the effects of public health measures: the case of Congo and Uganda. J. Theoret. Biol. Vol. **229**, pp. 119-126, 2004.

[6] L. Esteva and C. Vargas, Coexistence of different serotypes of dengue virus. J. Math. Biol. Vol. **46**, pp. 31-47, 2003.

[7] M. Schuette, A qualitative analysis of a model for the transmission of varicellazoster virus. Math. Biosc. Vol. **182**, pp. 113-126, 2003.

[8] A. Murray, M. O'Callaghan, and B. Jones, Simple models of massive epidemics of herpesvirus in Australian (and New Zealand) pilchards. Env. Int. Vol. **27**, pp. 243-248, 2001.

[9] W. Wang, Global behavior of an SEIRS epidemic model with time delays. Appl. Math. Letters, Vol. **15**, pp. 423-428, 2002.

[10] D. Nokes and J. Swinton, The control of childhood viral infections by pulse vaccination. IMA J. Math. AppI. Biol. Med. 12: 29-53 (1995).

[11] B. Shulgin *et al.*, Pulse vaccination strategy in the SIR epidemic model. Bulletin of Math. Bio. Vol. **60**, pp. 1-26, 1998.

[12] Z. Lu, X. Chi, and L. Chen, The Effect of Constant and Pulse Vaccination on SIR Epidemic Model with Horizontal and Vertical Transmission. Math. Comp. Model. Vol. **36**, pp. 1039-1057, 2002.

[13] R. May and R. Anderson, Regulation and stability of host parasite population interactions, II. destabilizing process. J. Anim. Ecol. Vol. **47**, pp. 219-267, 1978).

[14] R. Anderson and R. May, Infectious Diseases of Humans, Dynamics and Control. Oxford University Press, Oxford, 1992.

[15] S. Ruan and W. Wang, Dynamical behavior of an epidemic model with a nonlinear incidence rate. J. Diff. Equat. Vol. **188**, pp. 135-163, 2003.

[16] E. Beretta and Y. Takeuchi, Global stability of an SIR epidemic model with time delays. J. Math. Biol. Vol. **33**, pp. 250-260, 1995.

[17] W. Ma, M. Song, and Y. Takeuchi, Global Stability of an SIR Epidemic Model with Time Delay. Applied Mathematics Letters, Vol. **17**, pp. 1141-1145, 2004.

[18] Z. Jin and Z. Ma, The Stability of an SIR Epidemic Model with Time Delays. Math.Biosci.Engin. Vol. **3**, No. 1, pp. 101-109, 2006.

[19] J. Yan, Stability for impulsive delay differential equations. Nonlinear Analysis, Vol. **63**, pp. 66-80, 2005.

[20] B. Leonid and B. Elena, Linearized oscillation theory for a nonlinear delay impulsive equation. J. Comp. Appl. Math. Vol. **161**, pp. 477-495, 2003.

[21] X. Liu and G. Ballinger, Boundedness for impulsive delay differential equations and applications to population growth models. Nonlinear Analysis, Vol. **53**, pp. 1041-1062, 2003.

[22] V. Lakshmikantham, D. Bainov, and P. Simeonov, Theory of Impulsive Differential Equations. World Scienti.c, Singapore, 1989.

[23] Y. Kuang, Delay differential equations with applications in population dynamics. Academic Press, INC. San Diego, CA, 1993.