Journal of Mathematical Control Science and Applications (JMCSA)
Vol. 1 No. 1 (January-June, 2015), ISSN : 0974-0570

# A Cutting Point Technique for Singular Perturbation Problems

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**Abstract:** In this paper, a cutting point technique is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The method consists of the following steps: (i) The original singularly perturbed two-point boundary value problem is divided into two problems, namely inner and outer region problems. (ii) A boundary condition in implicit form at the cutting point is derived from the inner region problem. (iii) Then, the outer region problem is solved as a two-pint boundary value problem, and an explicit boundary condition is derived at the cutting point. (iv) Using stretching transformation, a modified inner region problem is constructed and is solved as a two-point boundary value problem using explicit boundary condition at the cutting point. (v) Finally, we combine the solutions of both the problems to obtain an approximate solution to the original problem. The proposed method is iterative on the cutting point. The process is to be repeated for various choices of the cutting point, until the solution differ materially from iteration to iteration. Several linear and nonlinear problems have been solved to demonstrate the applicability of the method.

**Key Words.** Two-point boundary value problems, singular perturbation problems, boundary layer, cutting point.

## 1. INTRODUCTION

The numerical treatment of singular perturbation problems is far from the trivial, because of the boundary layer behavior of solutions. These problems arise very frequently in fluid mechanics, elasticity, and other branches of Applied Mathematics, Science and Engineering. A few notable examples are boundary layer problems, WKB problems, convective heat transport problems with large peclet number etc. Pearson [ 9 ] was the first person who solved the singular perturbation problems using variable mesh finite difference scheme. Several authors have investigated solving singular perturbation problems by numerically constructing asymptotic solutions. The general motivation is to provide simpler efficient computational techniques to solve singular perturbation problems. A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these, we mention Bellman [1], Bender [2], Kevorkian & Cole [6 ], Nayfeh [7 ], Van Dyke [11 ], O' Malley [8 ], Eckhaus [3 ], Hemker [4 ], Robert [10 ], Kadalbajoo & Reddy [5 ].

In this paper, a cutting point technique is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The method consists of the following steps: (i) The original singularly perturbed two-point boundary value problem is divided into two problems, namely inner and outer region problems. (ii) A boundary condition in implicit form at the cutting point is derived from the inner region problem. (iii) Then, the outer region problem is solved as a two-pint boundary value problem, and an explicit boundary condition is derived at the cutting point. (iv) Using stretching transformation, a modified inner region problem is constructed and is solved as a two-point boundary value problem using explicit boundary condition at the cutting point. (v) Finally, we combine the solutions of both the problems to obtain an approximate solution to the original problem.

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The proposed method is iterative on the cutting point. The process is to be repeated for various choices of the cutting point, until the solution differ materially from iteration to iteration. Several linear and nonlinear problems have been solved to demonstrate the applicability of the method.

## 2. CUTTING POINT TECHNIQUE

To describe the method, we first consider a linear singularly perturbed two-point boundary value problem of the form:

$$ey''(x) + a(x)\, y'(x) + b(x)\, y(x) = f(x) \quad x \in [0, 1] \tag{1}$$

$$\text{with } y(0) = \alpha \tag{2a}$$

$$\text{and } y(1) = \beta \tag{2b}$$

where $\varepsilon$ is a small positive parameter ($0<\varepsilon<<1$) and $\alpha$, $\beta$ are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \geq M>0$ throughout the interval [0, 1], where $M$ is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$.

Consider $\delta = O(\varepsilon)$ be the cutting point or thickness of the boundary layer (inner region). Now we divide the original problem into two problems, an inner region problem and an outer region problem. The inner region problem is defined in the interval $0 \leq x \leq \delta$ and the outer region problem is defined in the interval $\delta \leq x \leq 1$.

### Mixed Condition at the Cutting Point

We now derive the mixed condition at the cutting point $\delta$ as follows:

From the theory of singular perturbations it is well known that the inner region problem is:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad 0 \leq x \leq \delta$$

By using the stretching transformation $t = \dfrac{x}{\varepsilon}$, we have

$$y(x) = y(t\varepsilon) = Y(t),$$

$$y'(x) = \frac{1}{\varepsilon} y'(t\varepsilon) = \frac{1}{\varepsilon} Y'(t),$$

$$y''(x) = \left(\frac{1}{\varepsilon^2}\right) y''(t\varepsilon) = \left(\frac{1}{\varepsilon^2}\right) Y''(t),$$

$$a(x) = a(t\varepsilon) = A(t)$$

$$b(x) = b(t\varepsilon) = B(t)$$

$$f(x) = f(t\varepsilon) = F(t)$$

$$\text{and } y(0) = \alpha = Y(0).$$

We get the inner region problem as:

$$Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \tag{3}$$

As $\varepsilon \to 0$; equation (3) becomes

$$Y''(t) + a(0)\ Y'(t) = 0$$

This equation in terms of x we have :

$$\varepsilon y''(x) + a(0)y'(x) = 0 \tag{4}$$

By Taylor's series expansion for $0 < \delta << 1$, we have:

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{5}$$

By substituting (5) in (4) we get

$$2\varepsilon y(x-\delta) + (\delta^2 a(0) + 2\varepsilon\delta)y'(x) - 2\varepsilon y(x) = 0 \tag{6}$$

By putting $x = \delta$ in (6), we get

$$2\varepsilon y(0) + (\delta^2 a(0) + 2\varepsilon\delta)y'(\delta) - 2\varepsilon y(\delta) = 0$$

Using $y(0)= \alpha$ and denoting $c = \dfrac{-\delta(\delta a(0) + 2\varepsilon)}{2\varepsilon}$,

we get the mixed boundary condition at the cutting point as:

$$y(\delta) + cy'(\delta) = \alpha \tag{7}$$

**Outer Region Problem**

Using the implicit condition (7), we get the outer region problem as a two-point boundary value problem,

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x);\ \delta \le x \le 1 \tag{8}$$

$$\text{with } \alpha\ y(\delta) + cy'(\delta) = \alpha \tag{9}$$

$$\text{and } y(1) = \beta \tag{10}$$

We solve the outer region problem to obtain solution over the interval $\delta \le x \le 1$. Hence, the solution of the outer region problem will provide the explicit boundary condition at the cutting point. Let us denote it by

$$y(\delta) = \gamma \text{ (say)} \tag{11}$$

**Inner Region Problem**

Since the cutting point is common to both the inner and outer regions, we can formulate the inner region problem as:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad 0 \le x \le \delta \tag{12}$$

$$\text{with } y(0) = \alpha \text{ and } y(\delta) = \gamma. \tag{13}$$

In terms of the inner variable $t = \dfrac{x}{\varepsilon}$ the equation (12) becomes

$$Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \tag{14}$$

and the boundary conditions becomes

$$Y(0) = \alpha \tag{15a}$$

and

$$Y(t_p) = \gamma \tag{15b}$$

where

$$t_p = \frac{\delta}{\varepsilon}.$$

We solve this new inner region problem (14)-(15) to obtain the solution over the interval $0 \le t \le t_p$.

**Solution of the Original Problem**

In order to solve the two-point boundary value problems given by the equations (8) –(9&10) ( outer region problem ) and (14)–(15) (inner region problem), we used second order central finite difference scheme. In fact, any standard analytical or numerical method can be used. After solving both the outer and inner region problems, we combine the solutions of these problems to obtain an approximate solution to the original problem (1)–(2).

***Remark:*** In this method Inner region problem provides the condition for the outer region problem and outer region problem provides the condition for the inner region problem. This serves as the link between inner region problem and outer region problem.

We repeat the process for various choices of $\delta$, until the solution profiles do not differ materially from iteration to iteration. For computational purposes we use an absolute error criterion, namely

$$\left| Y(t)^{m+1} - Y(t)^m \right| \le \sigma \,; \; 0 \le t \le t_p$$

where $Y(t)^m$ is the $m^{th}$ iterate of the inner region solution and $\sigma$ is the prescribed tolerance bound.

**2.1 Linear Examples**

To demonstrate the applicability of the method we have applied it on three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

**Example 2.1.1.** Consider the following homogeneous singular perturbation problem from Bender and Orszag [[ 2 ], page 480; problem 9.17 with a=0]

$$\varepsilon y''(x) + y'(x) - y(x) = 0\,; \quad x \in [0,1]$$

with $y(0) = 1$ and $y(1) = 1$.

The outer region problem is given by

$$\varepsilon y''(x) + y'(x) - y(x) = 0\,; \quad \delta \le x \le 1$$

with $y(\delta) - \dfrac{\delta(\delta + 2\varepsilon)}{2\varepsilon} y'(\delta) = 1$, $y(1)=1$.

The inner region problem is given by

$$Y''(t) + Y'(t) - \varepsilon Y(t) = 0\,; \quad 0 \le t \le t_p$$

with $Y(0) = 1$, $Y(t_p) = y(\delta)$

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$$

Where $m_1 = (-1 + \sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1 - \sqrt{1+4\varepsilon})/(2\varepsilon)$

The numerical results are given in tables 1(a), 1(b) for e = $10^{-3}$ and $10^{-4}$ respectively.

**Example 2.1.2.** Now consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x\,; \quad x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 1$.

The outer region problem is given by

$$\varepsilon y''(x) + y'(x) = 1 + 2x\,; \quad \delta \le x \le 1$$

with $y(\delta) - \dfrac{\delta(\delta + 2\varepsilon)}{2\varepsilon} y'(\delta) = 0$, $y(1) = 1$.

The inner region problem is given by

$$Y''(t) + Y'(t) = \varepsilon(1 + 2\varepsilon t)\,; \quad 0 \le t \le t_p$$

with Y(0) = 0, $Y(t_p) = y(\delta)$ $\mathrm{Y}(0) = 0$

The exact solution is given by

$$\mathrm{y(x) = x(x + 1\text{-}2\varepsilon)} + \frac{(2\varepsilon - 1)\left(1 - e^{-x/\varepsilon}\right)}{\left(1 - e^{-1/\varepsilon}\right)}$$

The numerical results are given in tables 2(a), 2(b) for ε = $10^{-3}$ and $10^{-4}$ respectively.

**Example 2.1.3.** Now we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole[[6], page 33; equations 2.3.26 and 2.3.27 with $\alpha = -1/2$]

$$\varepsilon y''(x) + (1 - \frac{x}{2}) y'(x) - \frac{1}{2} y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

The outer region problem is given by

$$\varepsilon y''(x) + (1 - \frac{x}{2}) y'(x) - \frac{1}{2} y(x) = 0; \quad d \le x \le 1$$

with
$$y(\delta) - \frac{\delta(\delta + 2\varepsilon)}{2\varepsilon} y'(\delta) = 0, \quad y(1) = 1.$$

The inner region problem is given by

$$Y''(t) + \left(1 - \frac{t\varepsilon}{2}\right) Y'(t) - \frac{\varepsilon}{2} Y(t) = 0; \quad 0 \le t \le t_p$$

with
$$Y(0) = 0, \quad Y(t_p) = y(\delta)$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [[5], page 148; equation 4.2.32] as our 'exact' solution:

$$y(x) = \frac{1}{2 - x} - \frac{1}{2} e^{-(x - x^2/4)/\varepsilon}$$

The numerical results are given in tables 3(a), 3(b) for $\varepsilon = 10^{-3}$ and $10^{-4}$ respectively.

### 2.2 Nonlinear Examples

We have applied the present method on three nonlinear singular perturbation problems with left-end boundary layer. Nonlinear singular perturbation problems are first converted as a sequence of linear singular perturbation problems by using Quasilinearization method. The solution of the reduced problem is taken as initial approximation.

**Example 2.2.1.** Consider the following singular perturbation problem from Bender and Orszag [[2], page 463; equations: 9.7.1]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 0$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

The outer region problem is given by

$$\varepsilon y''(x)+2y'(x)+\frac{2}{x+1}y(x)=\left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right)-1\right];\quad \delta\le x\le 1$$

with $$y(\delta)-\frac{\delta(\delta+\varepsilon)}{\varepsilon}y'(\delta)=0,\ y(1)=0.$$

The inner region problem is given by

$$Y''(t)+2Y'(t)+\frac{2\varepsilon}{1+t\varepsilon}Y(t)=\left(\frac{2\varepsilon}{1+t\varepsilon}\right)\left[\log_e\left(\frac{2}{1+t\varepsilon}\right)-1\right];\quad 0\le t\le t_p$$

with $Y(0)=0$, $Y(t_p)=y(\delta)$

We have chosen to use Bender and Orszag's uniformly valid approximation [[2], page 463; equation: 9.7.6] for comparison,

$$y(x)=\log_e\left(\frac{2}{x+1}\right)-(\log_e 2)e^{-2x/\varepsilon}$$

The numerical results are given in tables 4(a), 4(b) for $\varepsilon = 10^{-3}$ and $10^{-4}$ respectively.

**Example 2.2.2.** Now consider the following singular perturbation problem from Kevorkian and Cole [[6], page 56; equation 2.5.1]

$$\varepsilon y''(x)+y(x)y'(x)-y(x)=0;\quad x\in[0,1]$$

with $y(0)=-1$ and $y(1)=3.9995$

The linear problem concerned to this example is

$$\varepsilon y''(x)+(x+2.9995)y'(x)=x+2.9995$$

The outer region problem is given by

$$\varepsilon y''(x)+(x+2.9995)y'(x)=x+2.9995;\quad \delta\le x\le 1$$

with $y(\delta)-\dfrac{\delta(2.9995\delta+2\varepsilon)}{2\varepsilon}y'(\delta)=-1$, $y(1)=3.9995$

The inner region problem is given by

$$Y''(t)+(t\varepsilon+2.9995)Y'(t)=\varepsilon(t\varepsilon+2.9995);\quad 0\le t\le t_p$$

with $$Y(0)=-1,\ Y(t_p)=y(\delta)$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation [[4], pages 57 and 58; equations (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right)$$

Where $c_1 = 2.9995$ and $c_2 = (1/c_1) \log_e[(c_1\text{-}1)/(c_1\text{+}1)]$

For this example also we have a boundary layer of width O(ε) at x = 0 [cf. Kevorkian and Cole [4], pages 56-66].

The numerical results are given in tables 5(a), 5(b) for $e=10^{-3}$ and $10^{-4}$ respectively.

**Example 2.2.3.** Finally we consider the following singular perturbation problem from O' Malley [[8], page 9; equation (1.10) case 2]:

$$\varepsilon y''(x) - y(x)\, y'(x); \qquad x \in [-1,1]$$

with $y(-1) = 0$ and $y(1) = -1$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0$$

The outer region problem is given by

$$\varepsilon y''(x) + y'(x) = 0; \quad \delta \le x \le 1$$

with $y(\delta) - \dfrac{\delta(\delta+2\varepsilon)}{2\varepsilon} y'(\delta) = 0$ , $y(1) = -1$

For this example the stretching transformation is : $t = \dfrac{x+1}{\varepsilon}$ and the inner region problem is given by

$$Y''(t) + Y'(t) = 0; \text{ with } Y(0) = 0,\ Y(t_p) = y(\delta)$$

We have chosen to use O' Malley's approximate solution [[8], pages 9 and 10; equations 1.13 and 1.14] for comparison,

$$y(x) = -\frac{\left(1 - e^{-(x+1)/\varepsilon}\right)}{\left(1 + e^{-(x+1)/\varepsilon}\right)}$$

For this example, we have a boundary layer of width O(ε) at x = –1 [[cf. O' Malley [8], pages 9 and 10, eqs. (1.10), (1.13), (1.14), case 2 ].

The numerical results are given in tables 6(a), 6(b) for $\varepsilon = 10^{-3}$ and $10^{-4}$ respectively.

## 3. RIGHT-END BOUNDARY LAYER PROBLEMS

Now we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1] \tag{16}$$

with
$$y(0) = \alpha \tag{17a}$$

and
$$y(1) = \beta \tag{17b}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$) and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

Consider $\delta = O(\varepsilon)$ be the cutting point or thickness of the boundary layer (inner region). Now we divide the original problem into two problems, an inner region problem and an outer region problem. The outer region problem is defined in the interval $0 \le x \le 1-\delta$ and the inner region problem is defined in the interval $1-\delta \le x \le 1$.

**Mixed Condition at the Cutting Point**

We now derive the mixed boundary condition at the cutting point 1-δ as follows:

From the theory of singular perturbations it is well known that the inner region problem is:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad 1-\delta \le x \le 1$$

By using the stretching transformation $t = \dfrac{1-x}{\varepsilon}$

We have $y(x) = y(1-t\varepsilon) = Y(t)$,

$$y'(x) = \frac{-1}{\varepsilon} y'(1-t\varepsilon) = \frac{-1}{\varepsilon} Y'(t),$$

$$y''(x) = \left(\frac{1}{\varepsilon^2}\right) y''(1-t\varepsilon) = \left(\frac{1}{\varepsilon^2}\right) Y''(t),$$

$$a(x) = a(1-t\varepsilon) = A(t)$$
$$b(x) = b(1-t\varepsilon) = B(t)$$
$$f(x) = f(1-t\varepsilon) = F(t)$$

and $y(1) = \beta = Y(0)$.

We get the inner region problem as:

$$Y''(t) - A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \tag{18}$$

As $\varepsilon \to 0$; equation (18) becomes

$$Y''(t) - a(1)\, Y'(t) = 0$$

This equation in terms of x, we have

$$\varepsilon y''(x) + a(1)y'(x) = 0 \tag{19}$$

By Taylor's series expansion for $0 < \delta << 1$, we have

$$y(x+\delta) \approx y(x) + \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{20}$$

By substituting (20) in (19) we get

$$2\varepsilon y(x+\delta) + (\delta^2 a(1) - 2\varepsilon\delta) y'(x) - 2\varepsilon y(x) = 0 \tag{21}$$

By putting $x = 1 - \delta$ in (21), we get

$$2\varepsilon y(1) + (\delta^2 a(1) - 2\varepsilon\delta) y'(1-\delta) - 2\varepsilon y(1-\delta) = 0$$

Using y(1) = β and denoting $c = \dfrac{\delta(2\varepsilon - \delta a(1))}{2\varepsilon}$,

we get the mixed boundary condition at the cutting point as:

$$y(1-\delta) + cy'(1-\delta) = \beta \tag{22}$$

Equation (22) is considered as an implicit condition at the cutting point $x = 1 - \delta$.

**Outer Region Problem**

Using the implicit condition (22), we get the outer region problem as a two-point boundary value problem,

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad 0 \le x \le 1-\delta, \tag{23}$$

with $$y(0) = \alpha \tag{24}$$

and $$y(1-\delta) + cy'(1-\delta) = \beta \tag{25}$$

We solve the outer region problem to obtain solutions over the interval $0 \le x \le 1-\delta$. Hence, the solution of the outer region problem will provide the explicit boundary condition at the cutting point. Let us denote it by

$$y(1-\delta) = \gamma \text{ (say)} \tag{26}$$

**Inner Region Problem**

Since the cutting point is common to both the inner and outer regions, we can formulate the inner region problem as:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \quad 1-\delta \le x \le 1, \tag{27}$$

with $$y(1-\delta) = \gamma \tag{28a}$$

$$\text{and } y(1) = \beta \tag{28b}$$

In terms of the inner variable $t = \dfrac{1-x}{\varepsilon}$ the equation (27) becomes

$$Y''(t) - A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t) \tag{29}$$

and the boundary conditions becomes

$$Y(0) = \beta \tag{30a}$$

and $Y(t_p) = \gamma$ (30b)

where $t_p = \dfrac{1-\delta}{\varepsilon}$.

We solve this new inner region problem (29)-(30) to obtain the solution over the interval $0 \le t \le t_p$.

**Solution of the Original Problem**

In order to solve the two-point boundary value problems given by the equations (23)–(24 & 25) (outer region problem) and (29) – (30) (inner region problem), we used second order central finite difference scheme. In fact, any standard analytical or numerical method can be used. After solving both the outer and inner region problems, we combine the solutions of these problems to obtain an approximate solution to the original problem (16)–(17).

We repeat the process for various choices of δ, until the solution profiles do not differ materially from iteration to iteration. For computational purposes we use an absolute error criterion, namely

$$\left| Y(t)^{m+1} - Y(t)^{m} \right| \le \sigma\,; \;\; 0 \le t \le t_p$$

where $Y(t)^m$ is the m$^{th}$ iterate of the inner region solution and σ is the prescribed tolerance bound.

**3.1 Examples with Right -End Boundary Layer**

To illustrate the method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval we have implemented on three examples.

**Example 3.1.1.** Consider the following singular perturbation problem

$$\varepsilon y''(\mathrm{x}) - y'(x) = 0; \quad x \in [0,1]$$

with $y(0) = 1$ and $y(1) = 0$.

Clearly, this problem has a boundary layer at $x = 1$. i.e., at the right end of the underlying interval.

The outer region problem is given by

$$\varepsilon y''(x) - y'(x) = 0; \;\; 0 \le x \le 1 - \delta$$

with $y(0)$=1 and $y(1-\delta) + \dfrac{\delta(\delta+2\varepsilon)}{2\varepsilon} y'(1-\delta) = 0$.

For this example the stretching transformation is $t = \dfrac{1-x}{\varepsilon}$ and the inner region problem is given by

$$Y''(t) + Y'(t) = 0; \text{ with } Y(0) = 0,\ \ Y(t_p) = y\ (1-\delta)$$

The exact solution is given by

$$y(x) = \frac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$$

The numerical results are given in tables 7(a), 7(b) for $\varepsilon = 10^{-3}$ and $10^{-4}$ respectively.

**Example 3.1.2.** Now we consider the following singular perturbation problem $\varepsilon y''(x) - y'(x) - (1 + \varepsilon)\, y(x) = 0$ ; $x \in [0, 1]$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$; and $y(1) = 1 + 1/e$.

Clearly this problem has a boundary layer at $x = 1$.

The outer region problem is given by

$$\varepsilon y''(x) - y'(x) - (1 + \varepsilon)\, y(x) = 0; \quad 0 \le x \le 1 - \delta$$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$ and $y(1-\delta) + \dfrac{\delta(\delta + 2\varepsilon)}{2\varepsilon} y'(1-\delta) = 1 + e^{-1}$.

For this example the stretching transformation is

$$t = \frac{1-x}{\varepsilon}$$

and the inner region problem is given by

$$Y''(t) + Y'(t) - \varepsilon(1 + \varepsilon)\, Y(t) = 0;$$

with $Y(0) = 1 + e^{-1}$, $Y(t_p) = y(1 - \delta)$

The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$

The numerical results are given in tables 8(a), 8(b) for $\varepsilon = 10^{-3}$ and $10^{-4}$ respectively.

## 4. DISCUSSION AND CONCLUSIONS

A Cutting point technique is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. The solution of the given singular perturbed two-point boundary value problem is computed numerically by dividing it into inner region problem and outer region problem. A boundary condition in implicit form at the cutting point is derived from the inner region problem. Then, the outer region problem is solved as a two-pint boundary value problem, and an explicit boundary condition is obtained at the cutting point. Using stretching transformation, a modified inner region problem is constructed and is solved as a two-point boundary value problem using explicit boundary condition. In this method Inner region problem provides the condition for the outer region problem and outer region problem provides the condition for the inner region problem. This serves as the link between inner region problem and outer region problem. To solve the inner and outer region problems, we have used the second order central finite difference scheme. In fact any standard analytical or numerical method can be used. The proposed method is iterative on the cutting point. The process is to be repeated for various values of cutting point, until the solution stabilizes in both the regions. We have implemented the present method on three linear examples, three non-linear example with left-end boundary layer and two examples with right-end boundary layer by taking different values of e. Numerical results are presented in tables. It can be observed from the tables that the present method approximates the exact solution very well.

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**Table 1(a)**
**Numerical Results of Example 2.1.1., ε = $10^{-3}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.0000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.0005 | 0.6059501 | 0 .7461787 | 0.7476856 | 0.7476733 | 0.7514189 |
| 0.0010 | 0.3696413 | 0.5940351 | 0.5964465 | 0.5964269 | 0.6007948 |
| 0.0025 | | 0.4155980 | 0.4190756 | 0.4190473 | 0.4209001 |
| 0.0050 | | 0.3711139 | 0.3748699 | 0.3748394 | 0.3743310 |
| 0.0075 | | | 0.3722975 | 0.3722667 | 0.3713678 |
| 0.0100 | | | 0.3729566 | 0.3729257 | 0.3719772 |
| 0.0200 | | | | 0.3766484 | 0.3756831 |
| 0.1000 | 0.4069371 | 0.4069371 | 0.4069371 | 0.4069371 | 0.4069397 |
| 0.2000 | 0.4496898 | 0.4496898 | 0.4496898 | 0.4496898 | 0.4496925 |
| 0.3000 | 0.4969341 | 0.4969341 | 0.4969341 | 0.4969341 | 0.4969368 |
| 0.4000 | 0.5491418 | 0.5491418 | 0.5491418 | 0.5491418 | 0.5491446 |
| 0.5000 | 0.6068350 | 0.6068350 | 0.6068350 | 0.6068350 | 0.6068373 |
| 0.6000 | 0.6705891 | 0.6705891 | 0.6705891 | 0.6705891 | 0.6705912 |
| 0.7000 | 0.7410411 | 0.7410411 | 0.7410411 | 0.7410411 | 0.7410430 |
| 0.8000 | 0.8188952 | 0.8188952 | 0.8188952 | 0.8188952 | 0.8188963 |
| 0.9000 | 0.9049281 | 0.9049281 | 0.9049281 | 0.9049281 | 0.9049289 |
| 1.0000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Table 1(b)**
**Numerical Results of Example 2.1.1., ε = $10^{-4}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.00000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.00005 | 0.6050918 | 0.7457117 | 0.7472387 | 0.7472475 | 0.7513128 |
| 0.00010 | 0.3681588 | 0.5931537 | 0.5955968 | 0.5956109 | 0.6004911 |
| 0.00025 | | 0.4137938 | 0.4173146 | 0.4173348 | 0.4199244 |
| 0.00050 | | 0.3683060 | 0.3721015 | 0.3721232 | 0.3724053 |
| 0.00075 | | | 0.3686716 | 0.3686934 | 0.3685901 |
| 0.00100 | | | 0.3684900 | 0.3685118 | 0.3683615 |
| 0.00200 | | | | 0.3688580 | 0.3687013 |
| 0.10000 | 0.4067090 | 0.4067090 | 0.4067090 | 0.4067090 | 0.4066546 |
| 0.20000 | 0.4494657 | 0.4494657 | 0.4494657 | 0.4494657 | 0.4494124 |
| 0.30000 | 0.4967175 | 0.4967175 | 0.4967175 | 0.4967175 | 0.4966660 |
| 0.40000 | 0.5489371 | 0.5489371 | 0.5489371 | 0.5489371 | 0.5488880 |
| 0.50000 | 0.6066467 | 0.6066467 | 0.6066467 | 0.6066467 | 0.6066011 |
| 0.60000 | 0.6704227 | 0.6704227 | 0.6704227 | 0.6704227 | 0.6703823 |
| 0.70000 | 0.7409034 | 0.7409034 | 0.7409034 | 0.7409034 | 0.7408698 |
| 0.80000 | 0.8187935 | 0.8187935 | 0.8187935 | 0.8187935 | 0.8187687 |
| 0.90000 | 0.9048721 | 0.9048721 | 0.9048721 | 0.9048721 | 0.9048584 |
| 1.00000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Table 2(a)**
**Numerical Results of Example 2.1.2., ε = $10^{-3}$**

| *x* | *y(x)* $t_p=1$ | *y(x)* $t_p=5$ | *y(x)* $t_p=10$ | *y(x)* $t_p=20$ | *Exact Sol.* |
|---|---|---|---|---|---|
| 0.0000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.0005 | -0.6227753 | -0.4008161 | -0.3983886 | -0.3983214 | -0.3921832 |
| 0.0010 | -0.9962402 | -0.6411055 | -0.6372215 | -0.6371140 | -0.6298574 |
| 0.0025 | | -0.9227713 | -0.9171744 | -0.9170195 | -0.9135780 |
| 0.0050 | | -0.9922068 | -0.9861747 | -0.9860078 | -0.9862605 |
| 0.0075 | | | -0.9892095 | -0.9890416 | -0.9899068 |
| 0.0100 | | | -0.9871032 | -0.9869353 | -0.9878747 |
| 0.0200 | | | | -0.9766920 | -0.9776400 |
| 0.1000 | -0.8881958 | -0.8881958 | -0.8881958 | -0.8881958 | -0.8882000 |
| 0.2000 | -0.7584007 | -0.7584007 | -0.7584007 | -0.7584007 | -0.7584000 |
| 0.3000 | -0.6086050 | -0.6086050 | -0.6086050 | -0.6086050 | -0.6086000 |
| 0.4000 | -0.4388080 | -0.4388080 | -0.4388080 | -0.4388080 | -0.4388000 |
| 0.5000 | -0.2490100 | -0.2490100 | -0.2490100 | -0.2490100 | -0.2490000 |
| 0.6000 | -0.0392108 | -0.0392108 | -0.0392108 | -0.0392108 | -0.0392000 |
| 0.7000 | 0.1905896 | 0.1905896 | 0.1905896 | 0.1905896 | 0.1906001 |
| 0.8000 | 0.4403915 | 0.4403915 | 0.4403915 | 0.4403915 | 0.4404000 |
| 0.9000 | 0.7101949 | 0.7101949 | 0.7101949 | 0.7101949 | 0.7102001 |
| 1.0000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Table 2(b)**
**Numerical Results of Example 2.1.2., ε = $10^{-4}$**

| *x* | *y(x)* $t_p=1$ | *y(x)* $t_p=5$ | *y(x)* $t_p=10$ | *y(x)* $t_p=20$ | *Exact Sol.* |
|---|---|---|---|---|---|
| 0.00000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.00005 | -0.6247783 | 0.4022727 | -0.3998544 | -0.3998392 | -0.3933407 |
| 0.00010 | -0.9996252 | 0.6436164 | -0.6397471 | -0.6397227 | -0.6318942 |
| 0.00025 | | 0.9273452 | -0.9217695 | -0.9217344 | -0.9174814 |
| 0.00050 | | 0.9992249 | -0.9932157 | -0.9931778 | -0.9925632 |
| 0.00075 | | | -0.9985405 | -0.9985024 | -0.9984967 |
| 0.00100 | | | 0.9987237 | -0.9986856 | -0.9987538 |
| 0.00200 | | | | -0.9977195 | -0.9977964 |
| 0.10000 | -0.8898202 | -0.8898202 | -0.8898202 | -0.8898202 | -0.8898200 |
| 0.20000 | -0.7598401 | -0.7598401 | -0.7598401 | -0.7598401 | -0.7598400 |
| 0.30000 | -0.6098604 | -0.6098604 | -0.6098604 | -0.6098604 | -0.6098601 |
| 0.40000 | -0.4398801 | -0.4398801 | -0.4398801 | -0.4398801 | -0.4398801 |
| 0.50000 | -0.2499001 | -0.2499001 | -0.2499001 | -0.2499001 | -0.2499000 |
| 0.60000 | -0.0399201 | -0.0399201 | -0.0399201 | -0.0399201 | -0.0399201 |
| 0.70000 | 0.1900598 | 0.1900598 | 0.1900598 | 0.1900598 | 0.1900600 |
| 0.80000 | 0.4400399 | 0.4400399 | 0.4400399 | 0.4400399 | 0.4400399 |
| 0.90000 | 0.7100198 | 0.7100198 | 0.7100198 | 0.7100198 | 0.7100199 |
| 1.00000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Table 3(a)**
**Numerical Results of Example 2.1.3., ε = $10^{-3}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.0000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.0005 | 0.3132026 | 0.2016952 | 0.2004722 | 0.2004418 | 0.1968407 |
| 0.0010 | 0.5011806 | 0.3227485 | 0.3207916 | 0.3207430 | 0.3162644 |
| 0.0025 | | 0.4653576 | 0.4625360 | 0.4624658 | 0.4595191 |
| 0.0050 | | 0.5021793 | 0.4991345 | 0.4990587 | 0.4978630 |
| 0.0075 | | | 0.5025740 | 0.5024978 | 0.5016016 |
| 0.0100 | | | 0.5034251 | 0.5033487 | 0.5024893 |
| 0.0200 | | | | 0.5059060 | 0.5050505 |
| 0.1000 | 0.5270703 | 0.5270703 | 0.5270703 | 0.5270704 | 0.5263158 |
| 0.2000 | 0.5563172 | 0.5563172 | 0.5563172 | 0.5563173 | 0.5555556 |
| 0.3000 | 0.5889978 | 0.5889978 | 0.5889979 | 0.5889980 | 0.5882353 |
| 0.4000 | 0.6257542 | 0.6257542 | 0.6257543 | 0.6257544 | 0.6250000 |
| 0.5000 | 0.6673998 | 0.6673998 | 0.6673999 | 0.6673999 | 0.6666667 |
| 0.6000 | 0.7149782 | 0.7149782 | 0.7149782 | 0.7149783 | 0.7142857 |
| 0.7000 | 0.7698525 | 0.7698525 | 0.7698525 | 0.7698525 | 0.7692308 |
| 0.8000 | 0.8338373 | 0.8338373 | 0.8338373 | 0.8338373 | 0.8333333 |
| 0.9000 | 0.9094033 | 0.9094033 | 0.9094033 | 0.9094033 | 0.9090909 |
| 1.0000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Table 3(b)**
**Numerical Results of Example 2.1.2., ε = $10^{-4}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.00000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.00005 | 0.3126384 | 0.2013088 | 0.2000988 | 0.2000920 | 0.1967453 |
| 0.00010 | 0.5002271 | 0.3220977 | 0.3201616 | 0.3201507 | 0.3160807 |
| 0.00025 | | 0.4641703 | 0.4613802 | 0.4613645 | 0.4590136 |
| 0.00050 | | 0.5003271 | 0.4973196 | 0.4973027 | 0.4967540 |
| 0.00075 | | | 0.5001729 | 0.5001559 | 0.4999107 |
| 0.00100 | | | 0.5004521 | 0.5004352 | 0.5002274 |
| 0.00200 | | | | 0.5007018 | 0.5005005 |
| 0.10000 | 0.5264980 | 0.5264980 | 0.5264980 | 0.5264980 | 0.5263158 |
| 0.20000 | 0.5557351 | 0.5557351 | 0.5557351 | 0.5557351 | 0.5555555 |
| 0.30000 | 0.5884110 | 0.5884110 | 0.5884110 | 0.5884110 | 0.5882353 |
| 0.40000 | 0.6251694 | 0.6251694 | 0.6251694 | 0.6251694 | 0.6250000 |
| 0.50000 | 0.6668264 | 0.6668264 | 0.6668264 | 0.6668264 | 0.6666667 |
| 0.60000 | 0.7144316 | 0.7144316 | 0.7144316 | 0.7144316 | 0.7142857 |
| 0.70000 | 0.7693568 | 0.7693568 | 0.7693568 | 0.7693568 | 0.7692308 |
| 0.80000 | 0.8334317 | 0.8334317 | 0.8334317 | 0.8334317 | 0.8333333 |
| 0.90000 | 0.9091488 | 0.9091488 | 0.9091488 | 0.9091488 | 0.9090909 |
| 1.00000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

**Table 4(a)**
**Numerical Results of Example 2.2.1., ε = $10^{-3}$**

| *x* | *y(x)* $t_p=1$ | *y(x)* $t_p=5$ | *y(x)* $t_p=10$ | *y(x)* $t_p=20$ | *Exact Sol.* |
|---|---|---|---|---|---|
| 0.0000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.0005 | 0.5195726 | 0.4617175 | 0.4617096 | 0.4617096 | 0.4376527 |
| 0.0010 | 0.6924879 | 0.6153672 | 0.6153566 | 0.6153565 | 0.5983404 |
| 0.0025 | | 0.6881402 | 0.6881283 | 0.6881282 | 0.6859799 |
| 0.0050 | | 0.6884965 | 0.6884847 | 0.6884846 | 0.6881282 |
| 0.0075 | | | 0.6860101 | 0.6860100 | 0.6856750 |
| 0.0100 | | | 0.6835296 | 0.6835296 | 0.6831968 |
| 0.0200 | | | | 0.6736693 | 0.6733446 |
| 0.1000 | 0.5981045 | 0.5981045 | 0.5981045 | 0.5981045 | 0.5978370 |
| 0.2000 | 0.5110353 | 0.5110353 | 0.5110353 | 0.5110353 | 0.5108256 |
| 0.3000 | 0.4309460 | 0.4309460 | 0.4309460 | 0.4309460 | 0.4307829 |
| 0.4000 | 0.3568004 | 0.3568004 | 0.3568004 | 0.3568004 | 0.3566749 |
| 0.5000 | 0.2877767 | 0.2877767 | 0.2877767 | 0.2877767 | 0.2876821 |
| 0.6000 | 0.2232126 | 0.2232126 | 0.2232126 | 0.2232126 | 0.2231435 |
| 0.7000 | 0.1625663 | 0.1625663 | 0.1625663 | 0.1625663 | 0.1625189 |
| 0.8000 | 0.1053897 | 0.1053897 | 0.1053897 | 0.1053897 | 0.1053605 |
| 0.9000 | 0.0513068 | 0.0513068 | 0.0513068 | 0.0513068 | 0.0512933 |
| 1.0000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

**Table 4(b)**
**Numerical Results of Example 2.2.1., ε = $10^{-4}$**

| *x* | *y(x)* $t_p=1$ | *y(x)* $t_p=5$ | *y(x)* $t_p=10$ | *y(x)* $t_p=20$ | *Exact Sol.* |
|---|---|---|---|---|---|
| 0.00000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.00005 | 0.5197571 | 0.4620008 | 0.4619929 | 0.4619935 | 0.4381026 |
| 0.00010 | 0.6929820 | 0.6159754 | 0.6159650 | 0.6159657 | 0.5992399 |
| 0.00025 | | 0.6899907 | 0.6899790 | 0.6899798 | 0.6882268 |
| 0.00050 | | 0.6925821 | 0.6925703 | 0.6925712 | 0.6926158 |
| 0.00075 | | | 0.6923324 | 0.6923332 | 0.6923972 |
| 0.00100 | | | 0.6920826 | 0.6920835 | 0.6921477 |
| 0.00200 | | | | 0.6910845 | 0.6911492 |
| 0.10000 | 0.5977872 | 0.5977872 | 0.5977872 | 0.5977872 | 0.5978370 |
| 0.20000 | 0.5107903 | 0.5107903 | 0.5107903 | 0.5107903 | 0.5108256 |
| 0.30000 | 0.4307584 | 0.4307584 | 0.4307584 | 0.4307584 | 0.4307829 |
| 0.40000 | 0.3566589 | 0.3566589 | 0.3566589 | 0.3566589 | 0.3566750 |
| 0.50000 | 0.2876732 | 0.2876732 | 0.2876732 | 0.2876732 | 0.2876821 |
| 0.60000 | 0.2231391 | 0.2231391 | 0.2231391 | 0.2231391 | 0.2231436 |
| 0.70000 | 0.1625173 | 0.1625173 | 0.1625173 | 0.1625173 | 0.1625189 |
| 0.80000 | 0.1053605 | 0.1053605 | 0.1053605 | 0.1053605 | 0.1053605 |
| 0.90000 | 0.0512940 | 0.0512940 | 0.0512940 | 0.0512940 | 0.0512933 |
| 1.00000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

**Table 5 (a)**

**Numerical Results of Example 2.2.2., $1 = 10^{-3}$**

| *x* | *y(x)* $t_p=1$ | *y(x)* $t_p=5$ | *y(x)* $t_p=10$ | *y(x)* $t_p=20$ | *Exact Sol.* |
|---|---|---|---|---|---|
| 0.0000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.0005 | 2.5001290 | 2.4288060 | 2.4289330 | 2.4294390 | 1.1484590 |
| 0.0010 | 3.0005760 | 2.9190640 | 2.9192090 | 2.9197880 | 2.4569400 |
| 0.0025 | | 3.0018900 | 3.0020380 | 3.0026290 | 2.9953620 |
| 0.0050 | | 3.0046260 | 3.0047740 | 3.0053650 | 3.0044960 |
| 0.0075 | | | 3.0072740 | 3.0078650 | 3.0070000 |
| 0.0100 | | | 3.0097740 | 3.0103650 | 3.0095000 |
| 0.0200 | | | | 3.0203650 | 3.0195000 |
| 0.1000 | 3.0995650 | 3.0995650 | 3.0995650 | 3.0995650 | 3.0995000 |
| 0.2000 | 3.1995580 | 3.1995580 | 3.1995580 | 3.1995580 | 3.1995000 |
| 0.3000 | 3.2995510 | 3.2995510 | 3.2995510 | 3.2995510 | 3.2995000 |
| 0.4000 | 3.3995440 | 3.3995440 | 3.3995440 | 3.3995440 | 3.3995000 |
| 0.5000 | 3.4995360 | 3.4995360 | 3.4995360 | 3.4995360 | 3.4995000 |
| 0.6000 | 3.5995290 | 3.5995290 | 3.5995290 | 3.5995290 | 3.5995000 |
| 0.7000 | 3.6995220 | 3.6995220 | 3.6995220 | 3.6995220 | 3.6995000 |
| 0.8000 | 3.7995150 | 3.7995150 | 3.7995150 | 3.7995150 | 3.7995000 |
| 0.9000 | 3.8995070 | 3.8995070 | 3.8995070 | 3.8995070 | 3.8995000 |
| 1.0000 | 3.9995000 | 3.9995000 | 3.9995000 | 3.9995000 | 3.9995000 |

**Table 5(b)**

**Numerical Results of Example 2.2.2., $\varepsilon = 10^{-4}$**

| *x* | *y(x)* $t_p=1$ | *y(x)* $t_p=5$ | *y(x)* $t_p=10$ | *y(x)* $t_p=20$ | *Exact Sol.* |
|---|---|---|---|---|---|
| 0.00000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.00005 | 2.5002850 | 2.4287840 | 2.4287840 | 2.4287870 | 1.1480090 |
| 0.00010 | 3.0006260 | 2.9189050 | 2.9189050 | 2.9189080 | 2.4560400 |
| 0.00025 | | 3.0005370 | 3.0005370 | 3.0005400 | 2.9931120 |
| 0.00050 | | 3.0010260 | 3.0010260 | 3.0010290 | 2.9999960 |
| 0.00075 | | | 3.0012760 | 3.0012800 | 3.0002500 |
| 0.00100 | | | 3.0015270 | 3.0015300 | 3.0005000 |
| 0.00200 | | | | 3.0025310 | 3.0015000 |
| 0.10000 | 3.1004240 | 3.1004240 | 3.1004240 | 3.1004240 | 3.0995000 |
| 0.20000 | 3.2003210 | 3.2003210 | 3.2003210 | 3.2003210 | 3.1995000 |
| 0.30000 | 3.3002180 | 3.3002180 | 3.3002180 | 3.3002180 | 3.2995000 |
| 0.40000 | 3.4001160 | 3.4001160 | 3.4001160 | 3.4001160 | 3.3995000 |
| 0.50000 | 3.5000130 | 3.5000130 | 3.5000130 | 3.5000130 | 3.4995000 |
| 0.60000 | 3.5999100 | 3.5999100 | 3.5999100 | 3.5999100 | 3.5995000 |
| 0.70000 | 3.6998080 | 3.6998080 | 3.6998080 | 3.6998080 | 3.6995000 |
| 0.80000 | 3.7997050 | 3.7997050 | 3.7997050 | 3.7997050 | 3.7995000 |
| 0.90000 | 3.8996030 | 3.8996030 | 3.8996030 | 3.8996030 | 3.8995000 |
| 1.00000 | 3.9995000 | 3.9995000 | 3.9995000 | 3.9995000 | 3.9995000 |

**Table 6(a)**
**Numerical Results of Example 2.2.3., $\varepsilon = 10^{-3}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| -1.0000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| -0.9995 | -0.6249622 | -0.4024044 | -0.3999731 | -0.3999106 | -0.2449296 |
| -0.9990 | -0.9999394 | -0.6438470 | -0.6399569 | -0.6398570 | -0.4621121 |
| -0.9975 | | -0.9277834 | -0.9221778 | -0.9220338 | -0.8482833 |
| -0.9950 | | -0.9999278 | -0.9938864 | -0.9937312 | -0.9866142 |
| -0.9925 | | | -0.9994625 | -0.9993064 | -0.9988945 |
| -0.9900 | | | -0.9998962 | -0.9997400 | -0.9999092 |
| -0.9800 | | | | -0.9997768 | -1.0000000 |
| 0.1000 | -0.9999464 | -0.9999464 | -0.9999464 | -0.9999464 | -1.0000000 |
| 0.2000 | -0.9999523 | -0.9999523 | -0.9999523 | -0.9999523 | -1.0000000 |
| 0.3000 | -0.9999583 | -0.9999583 | -0.9999583 | -0.9999583 | -1.0000000 |
| 0.4000 | -0.9999642 | -0.9999642 | -0.9999642 | -0.9999642 | -1.0000000 |
| 0.5000 | -0.9999702 | -0.9999702 | -0.9999702 | -0.9999702 | -1.0000000 |
| 0.6000 | -0.9999762 | -0.9999762 | -0.9999762 | -0.9999762 | -1.0000000 |
| 0.7000 | -0.9999821 | -0.9999821 | -0.9999821 | -0.9999821 | -1.0000000 |
| 0.8000 | -0.9999881 | -0.9999881 | -0.9999881 | -0.9999881 | -1.0000000 |
| 0.9000 | -0.9999940 | -0.9999940 | -0.9999940 | -0.9999940 | -1.0000000 |
| 1.0000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |

**Table 6(b)**
**Numerical Results of Example 2.2.3., $\varepsilon = 10^{-4}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| -1.00000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| -0.99995 | -0.6250000 | -0.4024334 | -0.4000144 | -0.3999992 | -0.2449577 |
| -0.99990 | -1.0000000 | -0.6438934 | -0.6400231 | -0.6399987 | -0.4621824 |
| -0.99975 | | -0.9278503 | -0.9222732 | -0.9222381 | -0.8482583 |
| -0.99950 | | -0.9999999 | -0.9939891 | -0.9939513 | -0.9866174 |
| -0.99925 | | | -0.9995658 | -0.9995278 | -0.9988945 |
| -0.99900 | | | -0.9999995 | -0.9999616 | -0.9999092 |
| -0.99800 | | | | -0.9999984 | -1.0000000 |
| 0.10000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.20000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.30000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.40000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.50000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.60000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.70000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.80000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 0.90000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |
| 1.00000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 | -1.0000000 |

**Table 7(a)**
**Numerical Results of Example 3.1.1., $\varepsilon = 10^{-3}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.0000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1000 | 0.9999881 | 0.9999881 | 0.9999881 | 0.9999881 | 1.0000000 |
| 0.2000 | 0.9999762 | 0.9999762 | 0.9999762 | 0.9999762 | 1.0000000 |
| 0.3000 | 0.9999642 | 0.9999642 | 0.9999642 | 0.9999642 | 1.0000000 |
| 0.4000 | 0.9999523 | 0.9999523 | 0.9999523 | 0.9999523 | 1.0000000 |
| 0.5000 | 0.9999404 | 0.9999404 | 0.9999404 | 0.9999404 | 1.0000000 |
| 0.6000 | 0.9999285 | 0.9999285 | 0.9999285 | 0.9999285 | 1.0000000 |
| 0.7000 | 0.9999166 | 0.9999166 | 0.9999166 | 0.9999166 | 1.0000000 |
| 0.8000 | 0.9999046 | 0.9999046 | 0.9999046 | 0.9999046 | 1.0000000 |
| 0.9000 | 0.9998927 | 0.9998927 | 0.9998927 | 0.9998927 | 1.0000000 |
| 0.9800 | | | | 0.9964869 | 1.0000000 |
| 0.9900 | | | 0.9875388 | 0.9964502 | 0.9999546 |
| 0.9925 | | | 0.9871105 | 0.9960179 | 0.9994469 |
| 0.9950 | | 0.9587915 | 0.9816034 | 0.9904611 | 0.9932620 |
| 0.9975 | | 0.8896151 | 0.9107810 | 0.9189996 | 0.9179148 |
| 0.9990 | 0.6665881 | 0.6173596 | 0.6320480 | 0.6377513 | 0.6321158 |
| 0.9995 | 0.4166176 | 0.3858497 | 0.3950300 | 0.3985946 | 0.3934835 |
| 1.0000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

**Table 7(b)**
**Numerical Results of Example 3.1.1., $\varepsilon = 10^{-4}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.00000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.10000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.20000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.30000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.40000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.50000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.60000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.70000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.80000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.90000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.99800 | | | | 0.9966024 | 1.0000000 |
| 0.99900 | | | 0.9876543 | 0.9965657 | 0.9999546 |
| 0.99925 | | | 0.9872259 | 0.9961334 | 0.9994469 |
| 0.99950 | | 0.9589041 | 0.9817181 | 0.9905760 | 0.9932636 |
| 0.99975 | | 0.8897195 | 0.9108875 | 0.9191063 | 0.9179001 |
| 0.99990 | 0.6666666 | 0.6174321 | 0.6321218 | 0.6378254 | 0.6321816 |
| 0.99995 | 0.4166666 | 0.3858950 | 0.3950761 | 0.3986409 | 0.3935197 |
| 1.00000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

**Table 8(a)**
**Numerical Results of Example 3.1.2., $\varepsilon = 10^{-3}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.0000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1000 | 0.9048354 | 0.9048354 | 0.9048354 | 0.9048354 | 0.9048374 |
| 0.2000 | 0.8187271 | 0.8187271 | 0.8187271 | 0.8187271 | 0.8187308 |
| 0.3000 | 0.7408130 | 0.7408130 | 0.7408130 | 0.7408130 | 0.7408182 |
| 0.4000 | 0.6703135 | 0.6703135 | 0.6703135 | 0.6703135 | 0.6703200 |
| 0.5000 | 0.6065235 | 0.6065235 | 0.6065235 | 0.6065235 | 0.6065307 |
| 0.6000 | 0.5488038 | 0.5488038 | 0.5488038 | 0.5488038 | 0.5488116 |
| 0.7000 | 0.4965771 | 0.4965771 | 0.4965771 | 0.4965771 | 0.4965853 |
| 0.8000 | 0.4493205 | 0.4493205 | 0.4493205 | 0.4493205 | 0.4493290 |
| 0.9000 | 0.4065610 | 0.4065610 | 0.4065610 | 0.4065610 | 0.4065697 |
| 0.9800 | | | | 0.3789507 | 0.3753111 |
| 0.9900 | | | 0.3841280 | 0.3752148 | 0.3716216 |
| 0.9925 | | | 0.3835937 | 0.3747066 | 0.3711978 |
| 0.9950 | | 0.4109517 | 0.3881178 | 0.3793018 | 0.3764278 |
| 0.9975 | | 0.4789672 | 0.4578260 | 0.4496637 | 0.4506805 |
| 0.9990 | 0.7013589 | 0.7504933 | 0.7358339 | 0.7301740 | 0.7357640 |
| 0.9995 | 0.9511850 | 0.9818902 | 0.9727293 | 0.9691922 | 0.9742767 |
| 1.0000 | 1.3678790 | 1.3678790 | 1.3678790 | 1.3678790 | 1.3678790 |

**Table 8(b)**
**Numerical Results of Example 3.1.2., $\varepsilon = 10^{-4}$**

| $x$ | $y(x)$ $t_p=1$ | $y(x)$ $t_p=5$ | $y(x)$ $t_p=10$ | $y(x)$ $t_p=20$ | Exact Sol. |
|---|---|---|---|---|---|
| 0.00000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.10000 | 0.9048275 | 0.9048275 | 0.9048275 | 0.9048275 | 0.9048374 |
| 0.20000 | 0.8187125 | 0.8187125 | 0.8187125 | 0.8187125 | 0.8187308 |
| 0.30000 | 0.7407931 | 0.7407931 | 0.7407931 | 0.7407931 | 0.7408182 |
| 0.40000 | 0.6702896 | 0.6702896 | 0.6702896 | 0.6702896 | 0.6703200 |
| 0.50000 | 0.6064963 | 0.6064963 | 0.6064963 | 0.6064963 | 0.6065307 |
| 0.60000 | 0.5487744 | 0.5487744 | 0.5487744 | 0.5487744 | 0.5488116 |
| 0.70000 | 0.4965458 | 0.4965458 | 0.4965458 | 0.4965458 | 0.4965853 |
| 0.80000 | 0.4492880 | 0.4492880 | 0.4492880 | 0.4492880 | 0.4493290 |
| 0.90000 | 0.4065278 | 0.4065278 | 0.4065278 | 0.4065278 | 0.4065697 |
| 0.99800 | | | | 0.3719963 | 0.3686159 |
| 0.99900 | | | 0.3805730 | 0.3716602 | 0.3682929 |
| 0.99925 | | | 0.3809058 | 0.3719991 | 0.3687081 |
| 0.99950 | | 0.4091330 | 0.3863159 | 0.3774611 | 0.3747965 |
| 0.99975 | | 0.4782037 | 0.4570374 | 0.4488232 | 0.4500508 |
| 0.99990 | 0.7011999 | 0.7504264 | 0.7357389 | 0.7300389 | 0.7356979 |
| 0.99995 | 0.9511929 | 0.9819589 | 0.9727794 | 0.9692169 | 0.9743478 |
| 1.00000 | 1.3678790 | 1.3678790 | 1.3678790 | 1.3678790 | 1.3678790 |