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Delay-Dependent Robust Stability and Control of Uncertain Discrete Singular Systems with State-Delay

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Abstract: A new delay-dependent criterion is developed for robust stability of linear discrete-time singular systems with state-delay and parametric uncertainties. The time-delay is varying between known bounds and the uncertainties are assumed to be time-invariant and norm-bounded appearing in the state and delay matrices. The new criterion is achieved by an expanded state-space system representation. A solution to delay-dependent stabilization is attained based on guaranteed cost and H_∞ control approaches. All the developed results are conveniently cast in the format of linear matrix inequalities (LMIs) and numerical examples are presented.

Keywords: Singular systems, Discrete delay systems, Robust stability, Robust control, LMIs.

1. INTRODUCTION

For more than two decades, there has been numerous research studies related to singular systems or alternative designations such as descriptor systems, implicit systems [1], generalized state-space systems [6], differential-algebraic systems [4] or semistate systems. There are several applications of singular systems including large-scale systems, power systems, economic systems, to name a few [10, 11]. Recently, robust stability and robust stabilization problems of singular systems have been under investigation [5, 16, 17, 18]. From these results, it becomes clear that the robust stability problem for singular systems is more involved than the counterpart in state-space systems. Unlike ordinary state-space systems, singular systems require, in addition stability robustness, consideration of regularity and absence of impulses (case of continuous systems) or causality (case of discrete systems) simultaneously [5,7].

On another research front, it becomes quite evident that delays occur in physical and man-made systems due to various reasons including finite capabilities of information processing among different parts of the system, inherent phenomena like mass transport flow and recycling and/or by product of computational delays [2, 13]. Considerable discussions on delays and their stabilization/destabilization effects in control systems have attracted the interests of numerous investigators in recent years, see [13] and their references. Recent related results on discrete delay systems are presented in [3, 12, 14].

The class of discrete-time singular has been examined for robust stabilization in [17,18]. From the literature, it appears that the stabilization problem for discrete-time singular and state-delay and bounded-but-unknown parametric uncertainties is not fully investigated and most of the existing results are established under special conditions. In this paper, we examine the stabilization problem using guaranteed cost and H_∞ control approaches. In this paper, a new expanded state-space representation is developed which converts the singular time-delay system into an equivalent singular system in which all the original system matrices are grouped into the new

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system matrix. The benefit gained is that we do not require overbounding of the quantities involved. These advantages simplify the derivation of new delay-dependent stability and state-feedback stabilization results. All the results are formulated as linear matrix inequalities. A numerical example is worked out toillustrate the theoretical developments.

Notations: In the sequel, the Euclidean norm is used for vectors. We use W^t, W^{-1}, $\lambda(W)$ and $\|W\|$ to denote, respectively, the transpose, the inverse, the eigenvalues and the induced norm of any square matrix W and $W > 0$ ($W < 0$) stands for a symmetrical and positive- (negative-) definite matrix W. The n-dimensional Euclidean space and the space of bounded sequences are denoted by $\mathrm{I\,R}^{n\times n}$ and ℓ_2, respectively. The symbol • will be used in some matrix expressions to induce a symmetric structure, that is if given matrices $L = L^t$ and $R = R^t$ of appropriate dimensions, then

$$\begin{bmatrix} L & \bullet \\ N & R \end{bmatrix} = \begin{bmatrix} L & N^t \\ N & R \end{bmatrix}.$$

Sometimes, the arguments of a function will be omitted when no confusion can arise.

Fact 1: Given a scalar $\varepsilon > 0$ and matrices Σ_1, Σ_2 and Φ such that $\Phi^t\ \Phi \le I$, then

$$\Sigma_1 \Phi \Sigma_2 + \Sigma_2^t \Phi^t \Sigma_1^t \ \le\ \varepsilon^{-1}\, \Sigma_1 \Sigma_1^t + \varepsilon\, \Sigma_2^t \Sigma_2.$$

2. PROBLEM STATEMENT AND DEFINITIONS

We consider the following class of discrete-time singular systems with state-delay and parametric uncertainties:

$$E\ x_{k+1} = A_{\Delta o} x_k + A_{\Delta d} x_{k-d} + B_o u_k + \Gamma w_k, \quad x_0 = \psi_0$$

$$z_k = C_o x_k + D_o u_k \tag{2.1}$$

where $x_k \in IR^n$ is the state, $u_k \in IR^p$ is the control input, $w_k \in IR^r$ is the external disturbance, $z_k \in IR^q$ is the observed output and $\underline{d} \le d \le \overline{d}$ is an unknown integer representing the delay and $\underline{d}, \overline{d}$ are known bounds. The matrix $E \in IR^{n\times n}$ may be singular; we assume that rank $E = r \le n$. The matrices $A_{\Delta o} \in IR^{n\times n}$, $A_{\Delta d} \in IR^{n\times n}$ and $B_{\Delta o} \in IR^{n\times p}$ are represented by

$$[A_{\Delta o} \quad A_{\Delta d}] = [A_o \quad A_d] + M \Delta_k [N_a \quad N_d] \tag{2.2}$$

where $A_o \in IR^{n\times n}$, $B_o \in IR^{n\times p}$, $A_d \in IR^{n\times n}$, $M \in IR^{n\times n_m}$, $\Gamma \in IR^{n\times r}$, $N_a \in IR^{n_n\times n}$, $N_d \in IR^{n_n\times n}$ and $N_b \in IR^{n_n\times p}$, are real and known constant matrices with Δ_k is a bounded matrix of uncertainties satisfying $\Delta_k^t \Delta_k < I$. The uncertainties that satisfy (2.2) are referred to as admissible uncertainties.

For the time being we set $\Delta_k \equiv 0, u_k \equiv 0, A_{\Delta d} \equiv 0, x_{k-d} \equiv 0$ to yield the free nominal singular system

$$E\,x_{k+1} = A_o x_k. \tag{2.3}$$

For system (2.3), we recall the following definitions and results:

Definition 1. [10,16,18]:

1. System (2.3) is said to be regular if $det(zE - A_o)$ is not identically zero.

2. System (2.3) is said to be causal if it is regular and $deg(det(zE - A_o)) = rank(E)$.

3. System (2.3) is said to be stable if all the roots of $det(zE - A_o)$ lies inside the unit disk with center at the origin.

4. System (2.3) is said to be admissible if it is regular, causal and stable.

Next we consider the free nominal singular delay system

$$E\,x_{k+1} = A_o x_k + A_d x_{k-d}. \tag{2.4}$$

Extending on Definition (1), we provide the following

Definition 2. *System (2.4) is said to be regular and causal if the pair* (E, A_o) *is regular and causal. System (2.4) is said to be admissible if it is regular, causal and asymptotically stable.*

The objective of this paper is to develop delay-dependent methodologies for robust stability and stabilization for the class of uncertain, discrete-time singular delay systems of the type (2.1). This will be accomplished in Section 3 (delay-dependent stability) and Sections 4 (delay-dependent Control) based on guaranteed cost control and H_∞ control. This is made possible through the establishment of a new expanded state-space representation in which converts the singular time-delay system into an equivalent singular system in which the system matrix contains all the matrices of the original and the delay state has simple, certain and fixed matrix even if the original delay matrix is uncertain.

3. DELAY-DEPENDENT STABILITY

In the sequel, we employ the difference operator $\mathcal{D}_k \overset{\Delta}{=} x_{k+1} - x_k$ to rewrite system (2.1):

$$\begin{aligned} E\,x_{k+1} &= A_{\Delta o} x_k + A_{\Delta d} x_{k-d} + B_o u_k \\ &= (A_{\Delta o} + A_{\Delta d}) x_k - A_{\Delta d} \sum_{j=k-d}^{k-1} \mathcal{D}_j + B_o u_k. \end{aligned}$$

Together with the definition of $\mathcal{D}_k$, we get

$$0 = (A_{\Delta o} + A_{\Delta d} - E) x_k - E\,\mathcal{D}_k - A_{\Delta d} \sum_{j=k-d}^{k-1} \mathcal{D}_j + B_o u_k. \tag{3.1}$$

Define $\sigma_k = \sum_{j=k-d}^{k-1} D_j$, then it follows that

$$\sigma_{k+1} = \sigma_k + D_k - D_{k-d}$$

Introducing

$$\xi_k = [x_k^t \quad D_k^t \quad \sigma_k^t]^t, \quad A_{\Delta od} = A_{\Delta o} + A_{\Delta d}$$

we readily obtain the new expanded state-space system

$$(\Sigma_2): \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \xi_{k+1} = \begin{bmatrix} I & I & 0 \\ A_{\Delta od} - E & -E & -A_{\Delta d} \\ 0 & I & I \end{bmatrix} \xi_k + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \xi_{k-d} + \begin{bmatrix} 0 \\ B_o \\ 0 \end{bmatrix} u_k$$

$$U\ \xi_{k+1} = \overline{A}_{\Delta\xi}\ \xi_k + \overline{A}_{\xi d}\ \xi_{k-d} + \overline{B}_o u_k \tag{3.2}$$

where the initial conditions are characterized by

$$\xi_0 = \begin{bmatrix} x_0 \\ D_0 \\ \sigma_0 \end{bmatrix} = \begin{bmatrix} \psi_0 \\ (A_o - E)\psi_0 - A_d \psi_{-d_0} \\ \sum_{j=-\overline{d}}^{-1} D_{x_j} \end{bmatrix} \tag{3.3}$$

Remark 1. *In short, if x_k is a solution of uncertain delay system (2.1) with $\Delta_k \equiv 0$ and $u_k \equiv 0$, then ξ_k is a solution of the new expanded state-space system (3.10) subject to (3.15) and the reverse is true. This is the essence of descriptor transformation [8,9]. It is significant to observe that in system (3.10) the delay matrix has a simple, certain and fixed matrix even though the original delaymatrix $A_{\Delta d}$ is uncertain. In addition, all the matrices of the original singular system are grouped into the new system matrices and henceforth we call it the "Compact Form (CF)".*

We rewrite the CF matrix

$$\overline{\mathcal{A}}_{\Delta\xi} = \overline{\mathcal{A}}_{\xi o} + \overline{M}\ \Delta_k\ \overline{N} \tag{3.4}$$

with

$$\overline{\mathcal{A}}_{\xi o} = \begin{bmatrix} I & I & 0 \\ A_{od} - E & -E & -A_d \\ 0 & I & I \end{bmatrix}, \quad \overline{M} = \begin{bmatrix} 0 \\ M \\ 0 \end{bmatrix},$$

$$\overline{N} = [N_{ad} \quad 0 \quad -N_d], \ N_{ad} = N_a + N_d, \ A_{od} = A_o + A_d.$$

To derive tractable conditions for stability, we introduce the following Lyapunov functional

$$V(\xi_k) = V_a(\xi_k) + V_b(\xi_k) + V_c(\xi_k) + V_d(\xi_k) \tag{3.5}$$

with

$$V_a(\xi_k) = \xi_k^t U^t P U \xi_k, \quad 0 < P^t = P \in IR^{3n\times 3n}$$

$$V_b(\xi_k) = \sum_{j=k-d}^{k-1} \xi_j^t \bar{I}^t \mathcal{W} \bar{I} \xi_j, \; 0 < \mathcal{W}^t = \mathcal{W} \in \mathrm{IR}^{n\times n}, \; \bar{I} = [I \;\; 0 \;\; 0]$$

$$V_c(\xi_k) = \sum_{p=-\bar{d}+2}^{-\underline{d}+1} \; \sum_{j=k+p-1}^{k-1} \xi_j^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_j, \; 0 < \mathcal{Q}^t = \mathcal{Q} \in \mathrm{IR}^{n\times n}, \; \tilde{I} = [0 \;\; I \;\; 0]$$

$$V_d(\xi_k) = \sum_{p=-\bar{d}+1}^{-\underline{d}} \; \sum_{j=k+p}^{k-1} [(j-p-k+1)\xi_j^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_j] \tag{3.6}$$

where

$$\mathcal{P} \overset{\Delta}{=} \begin{bmatrix} \mathcal{P}_x & \mathcal{P}_f & 0 \\ \bullet & \mathcal{P}_d & 0 \\ \bullet & \bullet & \mathcal{P}_s \end{bmatrix} \in \mathrm{IR}^{3n\times 3n}, \; X \overset{\Delta}{=} \mathcal{P}^{-1} = \begin{bmatrix} \mathcal{X}_x & \mathcal{X}_f & 0 \\ \bullet & \mathcal{X}_d & 0 \\ \bullet & \bullet & X_s \end{bmatrix} \in \mathrm{IR}^{3n\times 3n},$$

$$\begin{aligned} &0 < \mathcal{P}_x = \mathcal{P}_x^t \in IR^{n\times n}, 0 < \mathcal{P}_d = \mathcal{P}_d^t \in IR^{n\times n}, 0 < \mathcal{P}_s = \mathcal{P}_s^t \in IR^{n\times n}, \mathcal{P}_f \in IR^{n\times n}, \\ &0 < \mathcal{X}_x = \mathcal{X}_x^t \in IR^{n\times n}, 0 < \mathcal{X}_d = \mathcal{X}_d^t \in IR^{n\times n}, 0 < \mathcal{X}_s = \mathcal{X}_s^t \in IR^{n\times n}, \mathcal{X}_f \in IR^{n\times n}. \end{aligned} \tag{3.7}$$

Based on (3.23), we define

$$\mathcal{X} \bar{I}^t \mathcal{W} \bar{I} \mathcal{X} \overset{\Delta}{=} \mathcal{M} = \begin{bmatrix} \mathcal{M}_x & \mathcal{M}_f & 0 \\ \bullet & \mathcal{M}_d & 0 \\ \bullet & \bullet & 0 \end{bmatrix},$$

$$0 < \mathcal{M}_x^t = \mathcal{M}_x \in IR^{n\times n}, \mathcal{M}_f \in IR^{n\times n}, 0 < \mathcal{M}_d^t = \mathcal{M}_d \in IR^{n\times n}$$

$$d^+ \, \mathcal{X} \tilde{I}^t \mathcal{Q} \tilde{I} \mathcal{X} \overset{\Delta}{=} d^+ \, \mathcal{N} = d^+ \begin{bmatrix} \mathcal{N}_x & \mathcal{N}_f & 0 \\ \bullet & \mathcal{N}_d & 0 \\ \bullet & \bullet & 0 \end{bmatrix},$$

$$0 < \mathcal{N}_x^t = \mathcal{N}_x \in IR^{n\times n}, \mathcal{N}_f \in IR^{n\times n}, 0 < \mathcal{N}_d^t = \mathcal{N}_d \in IR^{n\times n},$$

$$\mathcal{Z} = \mathcal{X}U^t\mathcal{P}U\mathcal{X} \overset{\Delta}{=} \begin{bmatrix} \mathcal{Z}_x & \mathcal{Z}_f & 0 \\ \bullet & \mathcal{Z}_d & 0 \\ \bullet & \bullet & \mathcal{Z}_s \end{bmatrix},$$

$$0 < \mathcal{Z}_x^t = \mathcal{Z}_x \in \mathrm{IR}^{n\times n}, \mathcal{Z}_f \in IR^{n\times n}, 0 < \mathcal{Z}_d^t = \mathcal{Z}_d \in IR^{n\times n}, 0 < \mathcal{Z}_s^t = \mathcal{Z}_s \in IR^{n\times n},$$

$$d^+ = \overline{d} + \frac{1}{2}(\overline{d} - \underline{d})\,(\overline{d} + \underline{d} - 1). \tag{3.8}$$

The following theorem establishes LMI-based sufficient conditions for dealy-dependent robust stability of system (Σ_2).

Theorem 1 . *System* (Σ_2) *with* $u_k \equiv 0$ *is delay-dependent robustly stable if there exist matrices* $0<\mathcal{X}_x = \mathcal{X}_x^t \in IR^{n\times n}$, $0<\mathcal{X}_d = \mathcal{X}_d^t \in IR^{n\times n}, 0<\mathcal{X}_s = \mathcal{X}_s^t \in IR^{n\times n}$, $\mathcal{X}_f \in IR^{n\times n}, 0<\mathcal{Z}_x = \mathcal{Z}_x^t \in IR^{n\times n}$, $0<\mathcal{Z}_d = \mathcal{Z}_d^t \in IR^{n\times n}$, $\mathcal{Z}_f \in IR^{n\times n}, 0<\mathcal{Z}_s = \mathcal{Z}_s^t \in IR^{n\times n}$, $0<\mathcal{M}_x = \mathcal{M}_x^t \in \mathcal{M}_x^t \in IR^{n\times n}$, $0<\mathcal{M}_d = \mathcal{M}_d^t \in IR^{n\times n}$, $0<\mathcal{N}_x = \mathcal{N}_x^t \in IR^{n\times n}$, $0<\mathcal{N}_d = \mathcal{N}_d^t \in IR^{n\times n}$, $\mathcal{M}_f \in IR^{n\times n}, \mathcal{N}_f \in IR^{n\times n}$ *and a scalar* $\delta > 0$ *such that the following inequality holds for all admissible uncertainties.*

$$\begin{bmatrix} -Z + \mathcal{M} + d^+\mathcal{N} & 0 & \Pi_a & \Pi_n \\ \bullet & -\overline{I}^t\mathcal{W}\overline{I} & \overline{A}_{\xi d}^t & 0 \\ \bullet & \bullet & -\mathcal{X} + \delta\overline{M}\,\overline{M}^t & 0 \\ \bullet & \bullet & \bullet & -\delta I \end{bmatrix} < 0 \tag{3.9}$$

where

$$\Pi_a = \begin{bmatrix} \mathcal{X}_x + \mathcal{X}_f & \mathcal{X}_x A_{od}^t - \mathcal{X}_x - \mathcal{X}_f & \mathcal{X}_f \\ \mathcal{X}_f^t + \mathcal{X}_d & \mathcal{X}_f^t A_{od}^t - \mathcal{X}_f^t - \mathcal{X}_d & \mathcal{X}_d \\ 0 & -\mathcal{X}_s A_d^t & \mathcal{X}_s \end{bmatrix}, \quad \Pi_n = \begin{bmatrix} \mathcal{X}_x N_{ad}^t \\ \mathcal{X}_f^t N_{ad}^t \\ -\mathcal{X}_s N_d^t \end{bmatrix}. \tag{3.10}$$

Proof: We consider V_k and evaluate the first difference of the functionals V_a, V_b, V_c and V_d. For V_a, we have

$$\begin{aligned} V_a(\xi_{k+1}) - V_a(\xi_k) &= \xi_{k+1}^t U^t P U \xi_{k+1}^t - \xi_k^t U^t P U \xi_k^t \\ &= [\overline{A}_{\Delta\xi}\xi_k + \overline{A}_{\xi d}\xi_{k-d}]^t \mathcal{P}[\overline{A}_{\Delta\xi}\xi_k + \overline{A}_{\xi d}\xi_{k-d}] - \xi_k^t U^t \mathcal{P} U \xi_k^t \\ &= \xi_k^t[\overline{A}_{\Delta\xi}^t \mathcal{P}\overline{A}_{\Delta\xi} - U^t\mathcal{P}U]\xi_k^t + \xi_{k-d}^t\overline{A}_{\xi d}^t\mathcal{P}\overline{A}_{\xi d}\xi_{k-d}^t + 2\xi_k^t\overline{A}_{\Delta\xi}^t\mathcal{P}\overline{A}_{\xi d}\xi_{k-d}. \end{aligned} \tag{3.11}$$

For V_b, we have

$$V_b(\xi_{k+1}) - V_b(\xi_k) = \sum_{j=k+1-d}^{k} \xi_j^t \overline{I}^t \mathcal{W} \overline{I} \xi_j - \sum_{j=k-d}^{k-1} \xi_j^t \overline{I}^t \mathcal{W} \overline{I} \xi_j$$

$$= \xi_k^t \overline{I}^t \mathcal{W} \overline{I} \xi_k - \xi_{k-d}^t \overline{I}^t \mathcal{W} \overline{I} \xi_{k-d}. \tag{3.12}$$

For V_c, we have

$$V_c(\xi_{k+1}) - V_c(\xi_k) = \sum_{p=-\overline{d}+2}^{-\underline{d}+1} \sum_{j=k+p}^{k} \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j - \sum_{p=-\overline{d}+2}^{-\underline{d}+1} \sum_{j=k+p-1}^{k-1} \xi_j^t deI^t Q \tilde{I} \xi_j$$

$$= \sum_{p=-\overline{d}+2}^{-\underline{d}+1} [\xi_k^t \tilde{I}^t Q \tilde{I} \xi_k + \sum_{j=k+p}^{k-1} \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j - \xi_{k+p-1}^t \tilde{I}^t Q \tilde{I} \xi_{k+p-1}$$

$$- \sum_{j=k+p-1}^{k-1} \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j]$$

$$= \sum_{p=-\overline{d}+2}^{-\underline{d}+1} \xi_k^t \tilde{I}^t Q \tilde{I} \xi_k - \sum_{p=-\overline{d}+2}^{-\underline{d}+1} \xi_{k+p-1}^t \tilde{I}^t Q \tilde{I} \xi_{k+p-1}$$

$$= (\overline{d} - \underline{d})\, \xi_k^t \tilde{I}^t Q \tilde{I} \xi_k + \sum_{j=k-\overline{d}+1}^{k-\underline{d}} \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j. \tag{3.13}$$

For V_d, we have

$$V_d(\xi_{k+1}) - V_d(\xi_k) = \sum_{p=-\overline{d}+1}^{-\underline{d}} \sum_{j=k+p+1}^{k} [(j-p-k) \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j]$$

$$- \sum_{p=-\overline{d}+1}^{-\underline{d}} \sum_{j=k+p}^{k-1} [(j-p-k+1) \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j]$$

$$= - \sum_{p=-\overline{d}+1}^{-\underline{d}} p\, \xi_k^t \tilde{I}^t Q \tilde{I} \xi_k - \sum_{p=-\overline{d}+1}^{-\underline{d}} \xi_{k+p}^t \tilde{I}^t Q \tilde{I} \xi_{k+p}$$

$$= - \sum_{j=k-\overline{d}+1}^{k-\underline{d}} \xi_j^t \tilde{I}^t Q \tilde{I} \xi_j + \frac{1}{2} (\overline{d} + \underline{d}) (\underline{d} - \overline{d} + 1)\, \xi_k^t \tilde{I}^t Q \tilde{I} \xi_k. \tag{3.14}$$

It follows from (3.17) and (3.40)-(3.43) that

$$V_{k+1} - V_k = \xi_k^t [\overline{A}_{\Delta\xi}^t \mathcal{P} \overline{A}_{\Delta\xi} - U^t \mathcal{P} U] \xi_k^t + \xi_{k-d}^t \overline{A}_{\xi d}^t \mathcal{P} \overline{A}_{\xi d} \xi_{k-d}^t + 2 \xi_k^t \overline{A}_{\Delta\xi}^t \mathcal{P} \overline{A}_{\xi d} \xi_{k-d}$$

$$+ \xi_k^t \overline{I}^t \mathcal{W} \overline{I} \xi_k - \xi_{k-d}^t \overline{I}^t \mathcal{W} \overline{I} \xi_{k-d}$$

$$+ (\overline{d} - \underline{d})\, \xi_k^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_k + \sum_{j=k-\overline{d}+1}^{k-\underline{d}} \xi_j^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_j$$

$$- \sum_{j=k-\overline{d}+1}^{k-\underline{d}} \xi_j^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_j + \frac{1}{2}(\overline{d}+\underline{d})(\underline{d}-\overline{d}+1)\, \xi_k^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_k$$

$$= \xi_k^t [\overline{A}_{\Delta\xi}^t \mathcal{P} \overline{A}_{\Delta\xi} - U^t \mathcal{P} U] \xi_k^t + \xi_{k-d}^t \overline{A}_{\xi d}^t \mathcal{P} \overline{A}_{\xi d} \xi_{k-d}^t + 2\xi_k^t \overline{A}_{\Delta\xi}^t \mathcal{P} \overline{A}_{\xi d} \xi_{k-d}$$

$$+ \xi_k^t \overline{I}^t \mathcal{W} \overline{I} \xi_k - \xi_{k-d}^t \overline{I}^t \mathcal{W} \overline{I} \xi_{k-d}$$

$$+ [\overline{d} + \frac{1}{2}(\overline{d}-\underline{d})(\overline{d}+\underline{d}-1)]\, \xi_k^t \tilde{I}^t \mathcal{Q} \tilde{I} \xi_k$$

$$= \begin{bmatrix} \xi_k \\ \xi_{k-d} \end{bmatrix}^t \begin{bmatrix} \overline{A}_{\Delta\xi}^t \mathcal{P} \overline{A}_{\Delta\xi} - U^t \mathcal{P} U + \\ \overline{I}^t \mathcal{W} \overline{I} + d^+ \tilde{I}^t \mathcal{Q} \tilde{I} & \overline{A}_{\Delta\xi}^t \mathcal{P} \overline{A}_{\xi d} \\ \bullet & -\overline{I}^t \mathcal{W} \overline{I} + \overline{A}_{\xi d}^t \mathcal{P} \overline{A}_{\xi d} \end{bmatrix} \begin{bmatrix} \xi_k \\ \xi_{k-d} \end{bmatrix}$$

$$= \begin{bmatrix} \xi_k \\ \xi_{k-d} \end{bmatrix}^t \Upsilon(d^+) \begin{bmatrix} \xi_k \\ \xi_{k-d} \end{bmatrix}. \tag{3.15}$$

By Laypunov theory, asymptotic stability ($V_{k+1} - V_k < 0, \forall \xi_k \neq 0$) implies that $\Upsilon(d^+) < 0$ which by Schur complement is equivalent to

$$\begin{bmatrix} -U^t \mathcal{P} U + \overline{I}^t \mathcal{W} \overline{I} + d^+ \tilde{I}^t \mathcal{Q} \tilde{I} & 0 & \overline{A}_{\Delta\xi}^t P \\ \bullet & -\overline{I}^t \mathcal{W} \overline{I} & \overline{A}_{\xi d}^t \mathcal{P} \\ \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0. \tag{3.16}$$

In terms of (3.7) we use the congruence transformation $diag[X \quad I \quad X]$ and invoking the linearizations (3.8) then inequality (3.16) becomes

$$\begin{bmatrix} -\mathcal{Z} + \mathcal{M} + d^+ \mathcal{N} & 0 & X \overline{A}_{\Delta\xi}^t \\ \bullet & -\overline{I}^t \mathcal{W} \overline{I} & \overline{A}_{\xi d}^t \\ \bullet & \bullet & -\mathcal{X} \end{bmatrix} < 0. \tag{3.17}$$

Further simple Schur complements with arrangements bring inequality (3.17) to (3.9) and the proof is completed.

Remark 3.2. *In contrast with the works in [8,9], it significant to observe that the use the developed CF representation has overcome the use of overbounding inequalitites to remove cross terms thereby leading to less conservative delay-dependent stability results.*

In the absence of uncertainties we get the following corollary

Corollary 3.1. *System* (Σ_2) *with* $u_k \equiv 0,\ M \equiv 0,\ N_a \equiv 0$ *and* $N_d \equiv 0$ *is delay-dependent quadratically stable if there exist matrices* $0<\mathcal{X}_x=\mathcal{X}_x^t \in IR^{n\times n}$, $0<\mathcal{X}_d=\mathcal{X}_d^t \in IR^{n\times n}$, $0<\mathcal{X}_s=\mathcal{X}_s^t \in IR^{n\times n}, \mathcal{X}_f \in IR^{n\times n}$, $0<\mathcal{Z}_x=\mathcal{Z}_x^t \in IR^{n\times n}, 0<\mathcal{Z}_d=\mathcal{Z}_d^t \in IR^{n\times n}, \mathcal{Z}_f \in IR^{n\times n}, 0<\mathcal{Z}_s= \mathcal{Z}_s^t \in IR^{n\times n}, 0<\mathcal{M}_x=\mathcal{M}_x^t \in IR^{n\times n}, 0<\mathcal{M}_d=\mathcal{M}_d^t \in IR^{n\times n}$, $0<\mathcal{N}_x=\mathcal{N}_x^t \in IR^{n\times n}, 0<\mathcal{N}_d=\mathcal{N}_d^t \in IR^{n\times n}, \mathcal{M}_f \in IR^{n\times n},\ \mathcal{N}_f \in IR^{n\times n}$ *satisfying the following inequality*

$$\begin{bmatrix} -\mathcal{Z}+\mathcal{M}+d^+\mathcal{N} & 0 & \Pi_a \\ \bullet & -\overline{I}^t\mathcal{W}\overline{I} & \overline{A}_{\xi d}^t \\ \bullet & \bullet & -\mathcal{X} \end{bmatrix} < 0. \tag{3.18}$$

4. DELAY-DEPENDENT CONTROL

Next, will derive expressions for gain matrix K_o using guaranteed cost and H_∞ control approaches. Let the state-feedback control be

$$u = K_o\, x_k = K_o\, \overline{I}\, \xi_k. \tag{4.1}$$

1. Guaranteed Cost Control Synthesis

The cost function associated with system (Σ_2) is:

$$\mathrm{J}_g = \sum_{k=0}^{\infty} \xi_k^t \overline{I}^t \mathcal{S} \overline{I}\, \xi_k + u_k^t R u_k \tag{4.2}$$

where $0<\mathcal{S}=\mathcal{S}^t \in IR^{n\times n}, 0<R=R^t \in IR^{m\times m}$ are weighting matrices and $\overline{I}=[I\ \ 0\ \ 0] \in IR^{n\times 3n}$. To proceed further, we start by the following definition [12]:

Definition 4.1. *Consider system* (Σ_2) *with cost function (4.2) and* $w_k \equiv 0$. *The state-feedback control law (4.1) is said to be a guaranteed cost control (GCC) with quadratic cost matrix* $\mathcal{P}>0$ *if given matrices* $0<\mathcal{S}=\mathcal{S}^t \in \mathrm{IR}^{n\times n}$ *and* $0<R=R^t \in IR^{m\times m}$, *the following LMI*

$$\begin{bmatrix} -U^t\mathcal{P}U+\mathcal{D} & 0 & \overline{\mathcal{A}}_{\Delta\xi}^t 0\mathcal{P} & \overline{I}^t K_o^t R \\ \bullet & -\overline{I}^t\mathcal{W}\overline{I} & \overline{\mathcal{A}}_{\xi d}^t\mathcal{P} & 0 \\ \bullet & \bullet & -\mathcal{P} & 0 \\ \bullet & et & \bullet & -R \end{bmatrix} < 0 \tag{4.3}$$

has a feasible solution with respect to $\mathcal{P}$ *for all admissible uncertainties* ΔA_k *satisfying (2.2).*

In the sequel we take

$$D = \overline{I}^t \mathcal{W} \overline{I} + d^+ \tilde{I}^t \mathcal{Q} \tilde{I} + \overline{I}^t \mathcal{S} \overline{I}.$$

Remark 4.1. *It is readily seen that inequality (4.6) with* $K_o \equiv 0, S \equiv 0$ *reduces to the results attained in [15] ensuring the delay-dependent asymptotic stability of the uncontrolled system (3.2).*

Introducing

$$\mathcal{X} \overline{I}^t \mathcal{S} \overline{I} \mathcal{X} \stackrel{\Delta}{=} \mathcal{V} = \begin{bmatrix} \mathcal{V}_x & \mathcal{V}_f & 0 \\ \bullet & \mathcal{V}_d & 0 \\ \bullet & \bullet & 0 \end{bmatrix} 0 < \mathcal{V}_x^t = \mathcal{V}_x \in IR^{n\times n}, \mathcal{V}_f \in IR^{n\times n}, 0 < \mathcal{V}_d^t = \mathcal{M}_d \in IR^{n\times n}.$$

The following theorem provides a necessary and sufficient condition for GCC.

Theorem 1. *Consider system* (Σ_2) *with cost function (4.2). There exists a GCC law (4.1) if and only if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}_x^t \in IR^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}_d^t \in IR^{n\times n}, 0 < \mathcal{X}_s = \mathcal{X}_s^t \in IR^{n\times n}$, $\mathcal{X}_f \in IR^{n\times n}$, $0 < \mathcal{Z}_x = \mathcal{Z}_x^t \in IR^{n\times n}$, $0 < \mathcal{Z}_d = \mathcal{Z}_d^t \in IR^{n\times n}$, $\mathcal{Z}_f \in IR^{n\times n}, 0 < \mathcal{Z}_s = \mathcal{Z}_s^t \in IR^{n\times n}$, $0 < \mathcal{M}_x = \mathcal{M}_x^t \in IR^{n\times n}$, $0 < \mathcal{M}_d = \mathcal{M}_d^t \in IR^{n\times n}$, $0 < \mathcal{N}_x = \mathcal{N}_x^t \in IR^{n\times n}$, $0 < \mathcal{N}_d = \mathcal{N}_d^t \in IR^{n\times n}$, $M_f \in IR^{n\times n}, \mathcal{N}_f \in IR^{n\times n}$, $\mathcal{Y}_f \in IR^{m\times n}$, $\mathcal{Y}_f \in IR^{m\times n}$, $0 < \mathcal{W} = \mathcal{W}^t \in IR^{n\times n}, 0 < \mathcal{V}_x = \mathcal{V}_x^t \in IR^{n\times n}$, $0 < \mathcal{V}_d = \mathcal{V}_d^t \in IR^{n\times n}$, $\mathcal{V}_f \in IR^{n\times n}$ *and a scalar* $\varepsilon > 0$ *such that the following LMI holds for all admissible uncertainties*

$$\begin{bmatrix} -\mathcal{Z} + \mathcal{V} + \mathcal{M} + d^+ \mathcal{N} & 0 & \Pi_a & \Pi_c & \Pi_n \\ \bullet & -\overline{I}^t \mathcal{W} \overline{I} & \overline{A}_{\xi d}^t & 0 & 0 \\ \bullet & \bullet & -\mathcal{X} + \varepsilon \overline{M}\, \overline{M}^t & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\varepsilon I \end{bmatrix} < 0 \tag{4.4}$$

where

$$\Pi_a = \begin{bmatrix} \mathcal{X}_x + \mathcal{X}_f & \mathcal{X}_x A_{od}^t - \mathcal{X}_x - \mathcal{X}_f & \mathcal{X}_f \\ \mathcal{X}_f^t + \mathcal{X}_d & \mathcal{X}_f^t A_{od}^t - \mathcal{X}_f^t - \mathcal{X}_d & \mathcal{X}_d \\ 0 & -\mathcal{X}_s A_d^t & \mathcal{X}_s \end{bmatrix}, \quad \Pi_n = \begin{bmatrix} \mathcal{X}_x N_{ad}^t \\ \mathcal{X}_f^t N_{ad}^t \\ -\mathcal{X}_s N_d^t \end{bmatrix}, \quad \Pi_c = \begin{bmatrix} \mathcal{Y}_x^t R \\ \mathcal{Y}_f^t R \\ 0 \end{bmatrix}. \tag{4.5}$$

The feedback gain is given by $K_o = \mathcal{Y}_x \mathcal{X}_x^{-1}$.

Proof: By **Definition 4.1** and [14], it follows that inequality (4.3) holds if and only if the LMI

$$\begin{bmatrix} -U^t\mathcal{P}U+\bar{I}^t\mathcal{S}\bar{I}+\bar{I}^t\mathcal{W}\bar{I}+d^+\,\tilde{I}^t\mathcal{Q}\tilde{I} & 0 & \bar{A}^t_{\xi o}\mathcal{P} & \bar{I}^tK^t_oR \\ \bullet & -\bar{I}^t\mathcal{W}\bar{I} & \bar{A}^t_{\xi d}\mathcal{P} & 0 \\ \bullet & \bullet & -\mathcal{P} & 0 \\ \bullet & \bullet & \bullet & -R \end{bmatrix}$$

$$+\,\varepsilon\begin{bmatrix} 0 \\ 0 \\ P\bar{M} \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ P\bar{M} \\ 0 \end{bmatrix}^t+\varepsilon^{-1}\begin{bmatrix} N^t \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} N^t \\ 0 \\ 0 \\ 0 \end{bmatrix}^t$$

$$=\begin{bmatrix} \begin{matrix}-U^t\mathcal{P}U+\bar{I}^t\mathcal{S}\bar{I}+\\ \bar{I}^t\mathcal{W}\bar{I}+d^+\,\tilde{I}^t\mathcal{Q}\tilde{I}+\\ \varepsilon^{-1}N^tN\end{matrix} & 0 & \bar{\mathcal{A}}^t_{\xi o}\mathcal{P} & \bar{I}^tK^t_oR \\ \bullet & -\bar{I}^t\mathcal{W}\bar{I} & \bar{\mathcal{A}}^t_{\xi o}\mathcal{P} & 0 \\ \bullet & \bullet & -\mathcal{P}+\varepsilon\mathcal{P}\bar{M}\,\bar{M}^t\mathcal{P} & 0 \\ \bullet & \bullet & \bullet & -R \end{bmatrix}<0 \tag{4.6}$$

holds for a scalar $\varepsilon_1 > 0$. Applying the congruence transformation $[\mathcal{X}\ \ \mathcal{X}\ \ I\ \ I], \mathcal{X}=\mathcal{P}^{-1}$ and recalling $\mathcal{Z}$, $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{V}$, we get

$$\begin{bmatrix} \begin{matrix}-\mathcal{Z}+\mathcal{V}+\mathcal{M}+d^+\mathcal{N}\\ \varepsilon^{-1}\mathcal{X}N^tN\mathcal{X}\end{matrix} & 0 & X\,\bar{\mathcal{A}}^t_{\xi o} & \mathcal{X}\bar{I}^tK^t_oR \\ \bullet & -\bar{I}^t\mathcal{W}\bar{I} & \bar{\mathcal{A}}^t_{\xi o} & 0 \\ \bullet & \bullet & -\mathcal{X}+\varepsilon\bar{M}\,\bar{M}^t & 0 \\ (4.43) & & & \\ \bullet & \bullet & \bullet & -R \end{bmatrix}<0. \tag{4.7}$$

Using (3.4) and (3.8) into (4.7) with Schur complement we obtain LMI (4.4).

In the case of nominal system, **Theorem 4.1** reduces to:

Corollary 4.1. *Consider system* (Σ_2) *with* $M\equiv 0, N_a\equiv 0, N_d\equiv 0$ *and cost function (4.2). There exists a state feedback gain* K_o *such that the control law (4.1) is a GCC with a quadratic cost matrix* $\mathcal{P} > 0$ *given* *matrices* $0 < S = S^t$ *and* $0 < R = R^t$ *if and only if there exist matrices there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}_x^t \in IR^{n \times n}$, $0 < \mathcal{X}_d = \mathcal{X}_d^t \in IR^{n \times n}$, $0 < \mathcal{X}_s = \mathcal{X}_s^t \in IR^{n \times n}, \mathcal{X}_f \in IR^{n \times n}$, $0 < \mathcal{Z}_x = \mathcal{Z}_x^t \in IR^{n \times n}$, $0 < \mathcal{Z}_d = \mathcal{Z}_d^t \in IR^{n \times n}$, $\mathcal{Z}_f \in IR^{n \times n}$, $0 < \mathcal{Z}_s = \mathcal{Z}_s^t \in IR^{n \times n}$, $0 < \mathcal{M}_x = \mathcal{M}_x^t \in IR^{n \times n}$, $0 < \mathcal{M}_d = \mathcal{M}_d^t \in \mathcal{M}_d^t \in IR^{n \times n}$, $0 < \mathcal{N}_x = \mathcal{N}_x^t \in IR^{n \times n}$, $0 < \mathcal{N}_d = \mathcal{N}_d^t \in IR^{n \times n}$, $\mathcal{M}_f \in IR^{n \times n}, \mathcal{N}_f \in IR^{n \times n}$, $\mathcal{Y}_x \in IR^{m \times n}$, $\mathcal{Y}_f \in IR^{m \times n}$, $0 < \mathcal{W} = \mathcal{W}^t \in IR^{n \times n}$, $0 < \mathcal{V}_x = \mathcal{V}_x^t \in IR^{n \times n}$, $0 < \mathcal{V}_d = \mathcal{V}_d^t \in IR^{n \times n}, \mathcal{V}_f \in IR^{n \times n}$ *satisfying the following LMI for all admissible gain perturbations*

$$\begin{bmatrix} -\mathcal{Z} + \mathcal{V} + \mathcal{M} + d^+ \mathcal{N} & 0 & \Pi_a & \Pi_c \\ \bullet & -\overline{I}^t \mathcal{W} \overline{I} & \mathcal{A}_{\xi d}^t & 0 \\ \bullet & \bullet & -\mathcal{X} & 0 \\ \bullet & \bullet & \bullet & -R \end{bmatrix} < 0. \tag{4.8}$$

The GCC feedback gain is $K_o = \mathcal{Y}_x \mathcal{X}_x^{-1}$.

4.2 Example 1

Consider the discrete-time system with data as follows:

$$A_o = \begin{bmatrix} 1 & 0.6 \\ 0.4 & 0.5 \end{bmatrix}, \; A_d = \begin{bmatrix} 0.5 & -0.2 \\ 0.6 & 0.4 \end{bmatrix}, B_o = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \; M = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

$$N_a^t = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, N_d^t = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \underline{d} = 2, \; \overline{d} = 5, C_o = \begin{bmatrix} 2 & 1 \end{bmatrix}, \; D_o = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \; S = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Observe that the system is open-loop unstable. Solving LMI (4.10) we obtain a feasible solution yielding the control law

$$u_k = \begin{bmatrix} 4.6249 & -0.2935 \\ -6.1869 & 0.3778 \end{bmatrix} x_k$$

as the desired GCC controller with $J_g = 4.1568$.

5. H_∞ CONTROL SYNTHESIS

In the sequel, we extend the results attained in the forgoing section to the case of H_∞ control. Application of (4.1) to system (Σ_2) yields the closed-loop system

$$U\ \xi_{k+1} = \bar{A}_{\Delta\xi}\ \xi_k + \bar{A}_{\xi d}\ \xi_{k-d} + \bar{\Gamma} w_k$$

$$= [\bar{A}_{\Delta\xi o} + \bar{M}\Delta\bar{N}]\ \xi_k + \bar{A}_{\xi d}\ \xi_{k-d} + \bar{\Gamma} w_k$$

$$\bar{A}_{\xi ko} = \begin{bmatrix} I & I & 0(5.1) \\ A_{od} + B_o K_o - E & -E & -A_d(5.2) \\ 0 & I & I \end{bmatrix}$$

$$z_k = [C_o + D_o K_o]x_k = \mathcal{C}_o \bar{I}\ \xi_k \tag{5.3}$$

Let $\{z_k\}, \{w_k\}$ be the sequences of the observed output and external disturbances with respective norms $\| z_k \|, \| w_k \|$. Then it is required for a given $\gamma > 0$ to have

$$\mathrm{J}_h = \| z_k \|^2 - \gamma^2 \| w_k \|^2 < 0 \quad \forall\ 0 \neq \{w_k\} \in \ell_2, \psi_k = 0,\ -\bar{h} \le k \le 0$$

This means that given a prespecified disturbance attenuation level γ, it is required to develop conditions for the state-feedback controller (4.1) that render the closed-looped system (5.1) quadratically stable for all admissible uncertainties $\Delta_k^t \Delta_k \le I$. Following [3], we have the following definition:

Definition 5.1. *Consider system* (Σ_2)*. The state-feedback control law (4.1) is said to be a* H_∞ *with disturbance attenuation* $\gamma > 0$ *if there exists matrices* $0 < \mathcal{P}^t = \mathcal{P} \in IR^{n\times n}$ *and* $0 < \mathcal{W}^t = \mathcal{W} \in \mathrm{IR}^{n\times n}$ *such that the following LMI*

$$\begin{bmatrix} -U^t\mathcal{P}U + \bar{I}^t\mathcal{W}\bar{I} + d^+\ \tilde{I}^t\mathcal{Q}\tilde{I} & 0 & \bar{\mathcal{A}}^t_{\Delta\xi k}\mathcal{P} & \bar{I}^t\mathcal{C}^t_o & \bar{\mathcal{A}}^t_{\Delta\xi k}\mathcal{P} \\ \bullet & -\bar{I}^t\mathcal{W}\bar{I} & \bar{\mathcal{A}}^t_{\xi d}\mathcal{P} & 0 & 0 \\ \bullet & \bullet & -\mathcal{P} & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I + \Gamma^t\bar{I}\mathcal{P}\bar{I}^t\Gamma \end{bmatrix} < 0 \tag{5.2}$$

has a feasible solution with respect to $\mathcal{P}$ *for all admissible uncertainties* ΔA_k *satisfying (2.2)*

The following theorem summarizes the corresponding result.

Theorem 5.1. *Consider system* (Σ_2)*. There exist state feedback gain* K_o *such that the control law (4.1) is a* H_∞ *control with disturbance attention* $\gamma > 0$ if and only if there exist matrices $0 < \mathcal{X}_x = \mathcal{X}_x^t \in IR^{n\times n}$,

$0<\mathcal{X}_d=\mathcal{X}_d^t\in IR^{n\times n}, 0<\mathcal{X}_s=\mathcal{X}_s^t\in IR^{n\times n}, X_f\in IR^{n\times n}, \quad 0<\mathcal{Z}_x=\mathcal{Z}_x^t\in IR^{n\times n}, \quad 0<\mathcal{Z}_d=\mathcal{Z}_d^t\in IR^{n\times n},$
$\mathcal{Z}_f\in IR^{n\times n}, 0<\mathcal{Z}_s=\mathcal{Z}_s^t\in IR^{n\times n}, 0<\mathcal{M}_x=\mathcal{M}_x^t\in IR^{n\times n}, \quad 0<\mathcal{M}_d=\mathcal{M}_d^t\in IR^{n\times n},$
$0<\mathcal{N}_x=\mathcal{N}_x^t\in IR^{n\times n}, 0<\mathcal{N}_d=\mathcal{N}_d^t\in IR^{n\times n}, \quad \mathcal{N}_f\in IR^{n\times n}, \mathcal{Y}_x\in IR^{m\times n}, \mathcal{Y}_f\in IR^{m\times n}, \quad 0<\mathcal{W}^t=\mathcal{W}\in IR^{n\times n}$
and scalars $\varepsilon_1>0, \varepsilon_3>0$ *satisfying the following LMI for all admissible uncertainties*

$$\begin{bmatrix} -\mathcal{Z}+\mathcal{M}+d^+\mathcal{N} & 0 & \Pi_b & \Pi_c & \Pi_b & 0 & \Pi_n & \Pi_n \\ \bullet & -\bar{I}^t\mathcal{W}\bar{I} & \bar{A}_{\xi d}^t & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -\mathcal{X}+\varepsilon_1\bar{M}\,\bar{M}^t & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I+\varepsilon_3\bar{M}\,\bar{M}^t & \Gamma^t\bar{I} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_1 I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\varepsilon_3 I \end{bmatrix} < 0 \tag{5.3}$$

The feedback gain is $K_o=\mathcal{Y}_x\mathcal{X}_x^{-1}$, *where*

$$\Pi_b=\begin{bmatrix} \mathcal{X}_x+\mathcal{X}_f & \mathcal{X}_x A_{od}^t+\mathcal{Y}_x^t B_o^t-\mathcal{X}_x E^t-\mathcal{X}_f E^t & \mathcal{X}_f \\ \mathcal{X}_f^t+\mathcal{X}_d & \mathcal{X}_f^t A_{od}^t+\mathcal{Y}_f^t B_o^t-\mathcal{X}_f^t E^t-\mathcal{X}_d E^t & \mathcal{X}_d \\ 0 & -X_s A_d^t & \mathcal{X}_s \end{bmatrix} \tag{5.4}$$

Proof: A simple Schur complement converts (5.2) into

$$\begin{bmatrix} -U^t\mathcal{P}U+\bar{I}^t\mathcal{W}\bar{I}+d^+\,\tilde{I}^t\mathcal{Q}\tilde{I} & 0 & \bar{\mathcal{A}}_{\Delta\xi k}^t\mathcal{P} & \bar{I}^t\mathcal{C}_o^t & \bar{\mathcal{A}}_{\Delta\xi k}^t\mathcal{P} & 0 \\ et & -\bar{I}^t\mathcal{W}\bar{I} & \bar{\mathcal{A}}_{\xi k}^t\mathcal{P} & 0 & 0 & 0 \\ \bullet & \bullet & -\mathcal{P} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & \Gamma^t\bar{I}\mathcal{P} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{5.5}$$

By **Definition 4.1** and **Theorem 4.1** with simple Schur operations, it follows that inequality (5.5) holds if and only if the LMI

$$\begin{bmatrix} \begin{matrix}-U^t\mathcal{P}U + \\ \bar{I}^t\mathcal{W}\bar{I} + d^+\ \tilde{I}^t\mathcal{Q}\tilde{I}\end{matrix} & 0 & \bar{\mathcal{A}}^t_{\xi ok}\mathcal{P} & \bar{I}^t\mathcal{C}^t_o & \bar{A}^t_{\xi ok}\mathcal{P} & 0 \\ \bullet & -\bar{I}^t\mathcal{W}\bar{I} & \bar{A}^t_{\xi d}\mathcal{P} & 0 & 0 & 0 \\ \bullet & \bullet & -\mathcal{P} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & \Gamma^t\bar{I}P \\ \bullet & \bullet & \bullet & \bullet & \bullet & -P \end{bmatrix}$$

$$+\,\varepsilon_1\begin{bmatrix}0\\0\\\mathcal{P}\bar{M}\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\\mathcal{P}\bar{M}\\0\\0\\0\end{bmatrix}^t + \varepsilon_1^{-1}\begin{bmatrix}\bar{N}^t\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}\bar{N}^t\\0\\0\\0\\0\\0\end{bmatrix}^t$$

$$+\,\varepsilon_3\begin{bmatrix}0\\0\\0\\0\\\mathcal{P}\bar{M}\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\\mathcal{P}\bar{M}\\0\end{bmatrix}^t + \ \varepsilon_3^{-1}\begin{bmatrix}\bar{N}^t\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}\bar{N}^t\\0\\0\\0\\0\\0\end{bmatrix}^t < 0 \tag{5.6}$$

holds for some parameters $\varepsilon_1 > 0, \varepsilon_2 > 0$. Grouping like terms and by the congruence transformation $diag[\mathcal{X}\ I\ \mathcal{X}\ I\ \mathcal{X}\ \mathcal{X}]$ with $\mathcal{X} = \mathcal{P}^{-1}, \mathcal{Y}_x = K_o X_x$ and using Schur complement operations we obtain the LMI (5.16).

For the nominal case, the following results holds

Corollary 1. *Consider system* (Σo). *There exist state feedback gain* K_o *such that the control law (4.1) is a* $\mathcal{H}_\infty$ *control with disturbance attenuation* $\gamma > 0$ *if and only if there exist matrices if and only if there exist matrices* $0 < \mathcal{X}_x = \mathcal{X}_x^t \in IR^{n\times n}$, $0 < \mathcal{X}_d = \mathcal{X}_d^t \in IR^{n\times n}$, $0 < \mathcal{X}_s = \mathcal{X}_s^t \in IR^{n\times n}, \mathcal{X}_f \in IR^{n\times n}$,

$0<\mathcal{Z}_x=\mathcal{Z}_x^t\in\mathcal{Z}_x^t\in IR^{n\times n}$, $0<\mathcal{Z}_d=\mathcal{Z}_d^t=\mathcal{Z}_d^t\in IR^{n\times n}$, $\mathcal{Z}_f\in IR^{n\times n}, 0<\mathcal{Z}_s=\mathcal{Z}_s^t\in IR^{n\times n}$, $0<\mathcal{M}_x=\mathcal{M}_x^t\in IR^{n\times n}$, $0<\mathcal{M}_d=\mathcal{M}_d^t\in IR^{n\times n}, 0<\mathcal{N}_x=\mathcal{N}_x^t\in IR^{n\times n}$, $0<\mathcal{N}_d=\mathcal{N}_d^t\in IR^{n\times n}$, $\mathcal{M}_f\in IR^{n\times n}, \mathcal{N}_f\in IR^{n\times n}$, $\mathcal{Y}_x\in IR^{m\times n}, \mathcal{Y}_f\in IR^{m\times n}$, $0<\mathcal{W}^t=\mathcal{W}\in IR^{n\times n}$ *satisfying the following LMI for all admissible uncertainties*

$$\begin{bmatrix} -\mathcal{Z}+\mathcal{M}+d^+\mathcal{N} & 0 & \Pi_b & \Pi_c & \Pi_b & 0 \\ \bullet & -\overline{I}^t\mathcal{W}\overline{I} & \overline{A}_{\xi d}^t & 0 & 0 & 0 \\ \bullet & \bullet & -\mathcal{X} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & \Gamma^t\overline{I} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X} \end{bmatrix}<0. \tag{5.7}$$

The feedback gain is given by $K_o=\mathcal{Y}_x\mathcal{X}_x^{-1}$.

5.1 Example 2

Consider the discrete-time system with data as follows:

$$A_o=\begin{bmatrix}0.1 & 0 & 0\\ 0.2 & 0.5 & -0.1\\ 0 & 1 & 0.9\end{bmatrix},\ A_d=\begin{bmatrix}1 & -0.6 & 0\\ 0 & 0.5 & 0.6\\ 0 & 0 & 0.3\end{bmatrix}, B_o=\begin{bmatrix}1 & 0\\ 0 & 0.5\\ 0.1 & 0.1\end{bmatrix},\ M=\begin{bmatrix}0.2\\ 0.1\\ 0\end{bmatrix}$$

$$N_a^t=\begin{bmatrix}0.05\\ 0\\ 0.3\end{bmatrix}, N_d^t=\begin{bmatrix}0.02\\ 0.4\\ 0\end{bmatrix}, \underline{d}=2,\ \overline{d}=6$$

$$C_o=\begin{bmatrix}0.2 & 0.1 & 0\end{bmatrix},\ D_o=[0.3\ \ 0.1],\ E=\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

Observe that the system is open-loop unstable. A feasible solution of the LMI (5.16) yields the H_∞ control law

$$u_k=\begin{bmatrix}0.03 & -0.0914 & 1.12207\\ 0.0145 & -1.1879 & 3.3118\end{bmatrix}x_k,\quad \gamma=2.2157.$$

6. CONCLUSIONS

For linear discrete-time singular systems with state-delay and parametric uncertainties, this paper has established

(1) An expanded state-space representation to exhibit the delay-dependent dynamics while preserving the equivalence with the original system

(2) A new delay-dependent stability criteria in a systematic way and without relying on overbounding by using an appropriate Lyapunov-Krasovskii functional, and

(3) A new delay-dependent stabilization based on guaranteed cost and $\mathcal{H}_\infty$ control approaches.

All the developed results have been conveniently cast in the format of linear matrix inequalities (LMIs) and numerical examples are presented. Superiority over existing techniques have been illuminated. Numerical examples have been presented to illustratethe theoretical developments.

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