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Lyapunov pairs for semilinear evolutions*

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Abstract. This paper deals with Lyapunov pairs for a semilinear evolution system in Banach spaces. Applications on a priori estimates are given.

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1. Introduction

This paper is concerned with the study of Lyapunov functions for the infinite dimensional system

$$\begin{cases} y'(t) = Ay(t) + F(y(t)) \\ y(0) = \xi, \end{cases} \tag{1.1}$$

where A is the generator of a C_0-semigroup on the Banach space X and $F : X \to X$ is a function. Recall that a mild solution $y : [0, T) \to X$ of (1.1) is called noncontinuable if there is no other solution $\widetilde{y} : [0, \widetilde{T}) \to X$ with $T < \widetilde{T}$ and $y(t) = \widetilde{y}(t)$ on $[0, T)$.

The aim of this paper is to characterize a Lyapunov pair for (1.1) by means of an inequality involving the contingent derivative related to the operator A. Recall that the functions V, $g : X \to (-\infty, +\infty]$ form a Lyapunov pair for problem (1.1) if for every $\xi \in \mathrm{dom}(V)$ there exist $T > 0$ and a solution $y : [0, T) \to X$ of (1.1) such that $t \longmapsto g(y(t))$ is locally integrable on $[0, T)$ and

$$V(y(t)) + \int_0^t g(y(s))\, ds \le V(\xi)$$

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for all $t \in [0, T)$.

A characterization of a Lyapunov pair (V, g) in the case where X is a Hilbert space, F is Lipschitz continuous, while the lower semicontinuous functions V and g satisfy

$$V(x), g(x) \geq -C\left(1 + \|x\|^2\right) \tag{1.2}$$

for all $x \in X$, with $C > 0$, was obtained by Kocan and Soravia [6] in case A is multi-valued using the viscosity solutions. In [2], Cârjă and Motreanu established a different characterization of a Lyapunov pair on a Hilbert space assuming the Lipschitz condition on F, by means of a suitable contingent derivative without using viscosity solutions. We give here the definition of the contingent derivative associated to the operator A, introduced in [2].

Definition 1.1. Let A be the generator of a the C_0-semigroup $S(t)$ on a Banach space X and $V : X \to (-\infty, +\infty]$ a proper function. The A-contingent derivative $\underline{D}^A V(\xi)(u)$ of V at $\xi \in \mathrm{dom}(V)$ in the direction $u \in X$ is defined by

$$\underline{D}^A V(\xi)(u) = \lim \inf_{\substack{t \downarrow 0 \\ w \to 0}} \frac{1}{t}[V(S(t)\xi + t(u + w)) - V(\xi)].$$

In our paper we study the case where F is a continuous function and the operator A generates a compact semigroup and we give a characterization of a Lyapunov pair using the contingent derivative associated to the operator. As application, we give a priori estimates for a mild solution of (1.1).

We introduce the basic concepts that will be used in the paper. First, we recall the tangency concept related to the operator A, which goes back to Pavel [7].

Definition 1.2. Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ be the generator of a C_0-semigroup $S(t)$, K a nonempty subset in X and $x \in K$. The set of all $\mathcal{A}$-tangent elements to K at x is

$$\mathcal{T}_K^{\mathcal{A}}(x) = \{v \in \mathcal{X};\ \liminf_{h \downarrow 0} \frac{1}{h} \mathrm{dist}(S(h)x + hv; K) = 0\}.$$

We have the following characterization. The tangent set $\mathcal{T}_K^{\mathcal{A}}(x)$ to K at the point $x \in K$ is the set of all $v \in X$ such that there exist $t_n \to 0^+$, $p_n \to 0$, and such that

$$\mathcal{S}(t_n)x + t_n(v + p_n) \in K, \quad \forall n \in \mathbb{N}.$$

We now give the definition of a viable set.

Definition 1.3. Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ the generator of a C_0-semigroup $S(t)$, K a nonempty subset in X and $\mathcal{F} : K \to \mathcal{X}$ a function. We say that K is viable with respect to $\mathcal{A} + \mathcal{F}$ if for each $x \in K$ there exist $T > 0$ and a mild solution $w : [0, T] \to K$ of

$$\begin{cases} w'(t) = \mathcal{A}w(t) + \mathcal{F}(w(t)) \\ w(0) = x. \end{cases}$$

In the end of this section, we present a viability result from [4, Theorem 8.2.3].

Theorem 1.1. *Let $\mathcal{X}$ be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ the generator of a compact C_0-semigroup $S(t)$, let K be a nonempty and locally closed subset in X and $\mathcal{F} : K \to \mathcal{X}$ a continuous function. Then a necessary and sufficient condition in order that K be viable with respect to $\mathcal{A} + \mathcal{F}$ is that*

$$\mathcal{F}(x) \in \mathcal{T}_K^{\mathcal{A}}(x),$$

for each $x \in K$.

2. Results

Let us state the main abstract result of the paper.

Theorem 2.1. *Let $S(t)$ be a compact C_0-semigroup, let $g : X \to \mathbb{R}$ be a continuous function and let $V : X \to (-\infty, +\infty]$ be a proper function with locally closed epigraph. Then V and g form a Lyapunov pair for problem* (1.1) *if and only if*

$$\underline{D}^A V(\xi) F(\xi) + g(\xi) \leq 0, \tag{2.1}$$

for all $\xi \in \operatorname{dom}(V)$.

Proof. Consider the problem

$$\begin{cases} w'(t) = \mathcal{A}(w(t)) + \mathcal{F}(w(t)) \\ w(0) = (\xi, V(\xi)), \end{cases} \tag{2.2}$$

where $\mathcal{A} : D(A) \times \mathbb{R} \subseteq X \times \mathbb{R} \to X \times \mathbb{R}$ is defined by

$$\mathcal{A}(x, z) = A(x) \times \{0\}, \text{ for all } (x, z) \in D(A) \times \mathbb{R},$$

and $\mathcal{F} : X \times \mathbb{R} \to X \times \mathbb{R}$ is given by

$$\mathcal{F}(y, z) = (F(y), -g(y)).$$

If A generates a compact semigroup on X then $\mathcal{A}$ generates a compact semigroup on $X \times \mathbb{R}$. The proof is based on Theorem 1.1 applied on $X \times \mathbb{R}$ with $K = \operatorname{epi}(V)$, and the following equality proved in [3]

$$\operatorname{epi}\left(\underline{D}^A V(\xi)\right) = \cap_{\mu \geq V(\xi)} \mathcal{T}_{\operatorname{epi}(V)}^{\mathcal{A}}(\xi, \mu) = \mathcal{T}_{\operatorname{epi}(V)}^{\mathcal{A}}(\xi, V(\xi)), \tag{2.3}$$

for all $\xi \in \operatorname{dom}(V)$. □

The next result gives a priori estimates for a mild solution of problem (1.1). It is related to [5, Theorem 6.8.3] where F is locally Lipschitz and $S(t)$ is a general semigroup. We denote by $J(\cdot)$ the duality mapping, i.e. $J(x)$ is the set of all $x^* \in X^*$ such that $\|x^*\|^2 = \|x\|^2 = \langle x^*, x \rangle$, and $\langle \cdot, \cdot \rangle$ is the duality pairing of X and X^*.

Theorem 2.2. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be the generator of a compact C_0-semigroup such that $\|S(t)x\| \le e^{\omega t}\|x\|$ for all $x \in X$. Let $F : X \to X$ be a continuous function, mapping bounded subsets in X into bounded subsets in X and which, for some constant $c > 0$, verifies*

$$\langle x^*, F(x)\rangle \le c\left(1 + \|x\|^2\right), \tag{2.4}$$

for every $x \in X$ and $x^* \in J(x)$. Let $\xi \in X$ with $\xi \ne 0$ and $\theta = \omega + c\left(1 + \|\xi\|^{-2}\right)$. Then there exists a solution $y : [0, +\infty) \to X$ of (1.1) which satisfies

$$\|y(t)\| \le e^{\theta t}\|\xi\|, \tag{2.5}$$

for all $t \ge 0$.

First, we state the following result which can be proved using the Zorn Lemma or the Brezis–Browder Ordering Principle (see [1]; see also [4, Theorem 2.1.1]).

Lemma 2.1. *Suppose that for each $\xi \in X$ with $\xi \ne 0$ and $\alpha > \theta$ there exists a solution $y : [0, a) \to X$ of (1.1) such that*

$$\|y(t)\| \le e^{\alpha t}\|\xi\|, \tag{2.6}$$

for all $t \in [0, a)$. Then, for any $\alpha > \theta$, there exists a solution $y : [0, +\infty) \to X$ of (1.1) which satisfies (2.6) for all $t \ge 0$.

Let us return to the proof of Theorem 2.2.
Proof. Let $\alpha > 0$ be such that

$$G_\alpha = \left\{\xi \in X\setminus\{0\};\ \omega + c\left(1 + \|\xi\|^{-2}\right) < \alpha\right\}$$

is nonempty. We define $V : X \to (-\infty, +\infty]$ by $V(x) = \|x\|$ for $x \in G_\alpha$ and $V(x) = +\infty$ otherwise and $g : X \to \mathbb{R}$ by $g(x) = -\alpha\|x\|$. It is easy to see that V has locally closed epigraph. We show that $\underline{D}^{A-\omega I}V(x)(F(x) + \omega x) < \alpha\|x\|$, for $x \in \operatorname{dom}(V) = G_\alpha$ and applying Theorem 2.1 for $A - \omega I$ and $\omega I + F$ instead of A and F, respectively we obtain that V and g form a Lyapunov pair, so we have that, for all $\xi \in G_\alpha$ there exists a solution $y(\cdot)$ of (1.1) on some interval $[0, a)$ such that

$$\|y(t)\| - \alpha\int_0^t \|y(s)\|\,ds \le \|\xi\|, \text{ for all } t \in [0, a). \tag{2.7}$$

By Gronwall's inequality we get $\|y(t)\| \le e^{\alpha t}\|\xi\|$, for all $t \in [0, a)$. Since each $\xi \ne 0$ belongs to G_α for $\alpha > \omega + c\left(1 + \|\xi\|^{-2}\right)$, we apply Lemma 2.1 to achieve that, for any $\alpha > \theta$, there exists $y : [0, +\infty) \to X$ solution of (1.1) which satisfies (2.6) for all $t \ge 0$.

Now, let $\alpha_n \downarrow \theta$. Then, for $n = 1, 2, \ldots$ there exist $y_n : [0, +\infty) \to X$ solutions of (1.1) such that

$$\|y_n(t)\| \le e^{\alpha_n t}\|\xi\|, \text{ for all } t \ge 0. \tag{2.8}$$

Denote $f_n(t) = F(y_n(t))$, for $n = 1, 2, ...$ and $t \geq 0$, and consider the problem

$$\begin{cases} y'(t) = Ay(t) + f_n(t), & n = 1, 2, ... \\ y(0) = \xi. \end{cases} \tag{2.9}$$

Let $T > 0$ and let $\mathcal{K}$ be the set of all the functions $f_n : [0, T] \to X$, $n = 1, 2, ...$ defined as above. As F maps bounded subsets in X into bounded subsets in X and by (2.8) we obtain that $\mathcal{K}$ is bounded in $L^\infty(0, T; X)$ and, moreover, we get that $\mathcal{K}$ is uniformly integrable in $L^1(0, T; X)$. Then, by [4, Theorem 1.6.5], it follows that the set of all the solutions of problem (2.9), $\{y(\cdot, f_n) = y_n(\cdot);\ n = 1, 2, ...\}$, is relatively compact in $C([0, T]; X)$. Hence, there exists $\widetilde{y} \in C([0, T]; X)$ such that, at least for a subsequence, we have

$$\lim_{n\to\infty} y_n(t) = \widetilde{y}(t),$$

uniformly for $t \in [0, T]$. Further, since F is continuous and $f_n(t) = F(y_n(t))$ for $n = 1, 2, ...$ and $t \in [0, T]$, we have that

$$\lim_{n\to\infty} f_n(t) = F(\widetilde{y}(t)),$$

uniformly for $t \in [0, T]$. Moreover, it follows that

$$\lim_{n\to\infty} y(t, f_n) = y(t, F(\widetilde{y}(t))),$$

uniformly for $t \in [0, T]$. Thus, $\widetilde{y}(\cdot) = y(\cdot, F(\widetilde{y}(\cdot)))$ and, therefore it is a solution of (1.1) on $[0, T]$. Moreover, by (2.8), we have that $\|\widetilde{y}(t)\| \leq e^{\theta t}\|\xi\|$ for all $t \in [0, T]$.

Again by the Zorn Lemma or by the Brezis–Browder Ordering Principle we can prove that the solution is global. This completes the proof. □

Remark 2.1. Similar techniques could be applied to the fully nonlinear case, that is, when A is a multivalued m-dissipative operator. We shall give details in a forthcoming work.

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