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Application of difference equations in insurance mathematics and process engineering

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Dedicated to Professor István Győri on the occasion on his 65th birthday

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Abstract. In this paper difference equations arising from economical and technical applications are presented and investigated. Starting from a stochastic model we introduce the z-transform of the sequence of the appropriate probabilities. We derive a difference equation for by the z-transform and prove the existence and uniqueness of the solution. Characteristic properties of the solutions are presented, and in some special cases explicit forms of the solutions are provided.

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1. Introduction

In insurance mathematics and modeling of process engineering systems there are mathematical models in which the probability and expected time of ruin, as well as the probability of crossing some level in an intermediate storage and its expected time are investigated. These problems are in close connection with each other since the continuous time functions describing the appropriate probabilities of these events satisfy integral equations of similar form which, in turn, can be transformed into integro-differential or differential equations with delayed or advanced arguments [1,2,3] .

Here we treat these problems in a different way formulating those as discrete time stochastic models. We investigate the level-crossing probabilities as a function of the level, and, as a generalization, we introduce the z-transform of the corresponding

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sequence, and we derive a difference equation for it. We investigate the limit properties of the solution and derive explicit solutions in some cases. The level-crossing probabilities and the expected time of crossing the given level can be derived directly from these solutions and from their derivatives.

The structure of the paper is as follows. First, we present the problems and their models introducing some notations as well. Then, using the methods of probability theory, we derive a difference equation for the z-transform of the first level-crossing time. A relationship between the sequence of level-crossing probabilities and the z-transform will be presented, while in the next part we prove the existence and uniqueness of the solution, and show that the solution tends to zero. Finally, we provide some explicit formulas for the solutions.

2. The model

The Sparre Andersen model is an often investigated model in insurance mathematics [1,4]. In this model, premiums are received continuously at a constant rate c, and the aggregate claims constitute a compound stochastic process. Let $t_0 = 0$, and let the interclaim times between the consecutive claims be $t_k \quad (k = 1, 2, 3, \ldots)$, which are nonnegative, independent, identically distributed random variables. The counting process $\{N(t) : t \geq 0\}$ denotes the number of claims up to time t, and is defined by

$$N(t) = \begin{cases} 0, & \text{if} \quad t_1 > t, \\ \max\left\{ l : \sum_{k=1}^{l} t_k \leq t \right\}, & \text{if} \quad t_1 \leq t. \end{cases}$$

The size of the kth claim is denoted by Y_k, $k = 1, 2, ...$, where variables Y_k, $k = 1, 2, ...$, are also nonnegative, independent and identically distributed random variables. Further, we assume that processes $N(t)$ and $\{Y_k\}_{k=1}^{\infty}$ are independent. Notice that when $N(t)$ is a Poisson process the Sparre Andersen model is reduced to the classical risk process called compound Poisson risk model.

Under these conditions, the surplus of the insurer can be expressed by

$$U(t) = u + ct - \sum_{k=1}^{N(t)} Y_k, \qquad t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the initial surplus.

In the case of continuous processes, the ruin probability is formulated as

$$\psi_1(u) = P\Big(U(t) < 0 \quad \text{for some} \quad t \geq 0\Big) = P\Big(U(t) \leq 0 \quad \text{for some} \quad t \geq 0\Big).$$

Furthermore, the probability of crossing some level ν of surplus, i.e. $U(t) \geq \nu$ for some $t \geq 0$, or $U(t) \geq \nu - u$, if the initial surplus equals 0.

Next we use the notation

$$\psi_2(\nu) = P\left(ct - \sum_{k=1}^{N(t)} Y_k \geq \nu \quad \text{for some} \quad t \geq 0\right).$$

In process engineering, intermediate storage systems collect material from the input processing units and the output units withdraw material from these at constant volumetric rate. Here, $Y_k, \quad k = 1, 2, ...$ denote the amounts of material filled into the storages, and the input process corresponds to the claim process in insurance mathematics. If the rate of the withdraw is c and the initial amount of material is z, then the process

$$V(t) = z + \sum_{k=1}^{N(t)} Y_k - ct \tag{2.2}$$

describes the amount of the material in the intermediate storage at time t. Running out of material means that $V(t) \leq 0$ for some $t \geq 0$. This corresponds to the level-crossing problem in the ruin model. When, however, we would like to determine the necessary size of the storage not to overflow during the process then we should investigate the inequality

$$z + \sum_{k=1}^{N(t)} Y_k - ct \leq z_s$$

where z_s denotes the storage capacity. Now,

$$P\left(z + \sum_{k=1}^{N(t)} Y_k - ct \geq z_s\right)$$

expresses the overflow probability and it corresponds to $\psi_1(u)$ with $u = z_s - z$. An integral equation for $\psi_1(u)$ for Poisson input process was determined in [5].

Actually, we focus our attention to the level-crossing problem in insurance mathematics which corresponds to the problem of running out of material in processing systems. In case of compound Poisson process integral equations for $R_2(z) = 1 - \psi_2(z)$ were presented in [2] and [3].

In the mathematical literature, difference equations are used to be constructed as analogues of the integro-differential equations appearing in continuous time models, and methods of solutions of continuous cases are adopted [6,7]. Furthermore, since in practice the amounts of material or claim sizes are really discrete, we formulate these problems as discrete models.

Let us assume that both the time intervals between the consecutive claims and the claim sizes have discrete distributions with integer values and let $P(t_k = j) = f(j)$, $j = 0, 1, 2...$, and $P(Y_k = i) = g(i)$, $i = 0, 1, 2...,$, and $c = 1$. Naturally, $f(j) \geq 0$, $g(i) \geq 0$, $\sum_{j=0}^{\infty} f(j) = 1$ and $\sum_{i=0}^{\infty} g(i) = 1$. Further, we assume that the expectations of these random variables are finite, that is $\mu_f = \sum_{j=0}^{\infty} j f(j) < \infty$ and $\mu_g = \sum_{i=0}^{\infty} i g(i) < \infty$. In this case, $x(n)$ $n = 0, 1, ...$ is defined by

$$x(n) = P\left(m - \sum_{k=1}^{N(m)} Y_k \geq n \quad \text{for some} \quad m = 0, 1, ...\right), \tag{2.3}$$

while

$$p_m^{(n)} = P\left(m - \sum_{k=1}^{N(m)} Y_k \geq n \quad \text{and} \quad s - \sum_{k=1}^{N(s)} Y_k < n, \qquad 0 \leq s < m\right) \tag{2.4}$$

expresses the probability that the first time of crossing the level n is the mth moment. We note that $p_0^{(0)} = 1$ and $p_m^{(0)} = 0$ if $m > 0$.

When investigating the ruin probability in continuous case, usually the Gerber-Shiu discounted penalty function is applied which, in principle, is the Laplace-transform of the density function of the first level-crossing time [8]. In correspondence with that, we introduce the z-transform of the sequence $p_m^{(n)}$ for all fixed values of n. The z-transform of the sequence $(p_m^{(n)})_{m\geq 0}$ is the function

$$T^{(n)}(z) = \sum_{m=0}^{\infty} p_m^{(n)} z^{-m}, \tag{2.5}$$

where z is a complex variable for which the series (2.5) converges absolutely. We will restrict ourselves to real values of z and $z \geq 1$. It is clear, that $0 \leq T^{(n)}(z) \leq 1$ for any $z \geq 1$ and $T^{(n)}(1) = \sum_{m=0}^{\infty} p_m^{(n)} = x(n)$ for any $n \in N$. Furthermore, $-\frac{\partial T^{(n)}(z)}{\partial z}\big|_{z=1}$ equals to the expectation of time of the first crossing level n. Consequently, we focus our attention on the function $T^{(n)}(z)$.

3. Difference equation for function $T^{(n)}(z)$

Now, applying the theorem of renewal theory we set up a difference equation for the sequence of functions $T^{(n)}(z)$ $(n = 0, 1, 2\dots)$ defined by (2.5). This means that for any fixed value of $z \geq 1$ we have a difference equation for the sequence of $T^{(n)}(z)$.

Theorem 3.1. *For any $z \geq 1$ the function $T^{(n)}(z)$ satisfies the following difference equation:*

$$T^{(n)}(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} T^{(n+i-j)}(z) f(j) g(i) z^{-j} + \sum_{j=n}^{\infty} f(j) z^{-n}, \qquad n = 0, 1, 2, \ldots \tag{3.1}$$

Proof. Let

$$t^{(n)} = \begin{cases} \infty, & \text{if } l - \sum_{k=1}^{N(l)} Y_k < n \text{ for all } l = 0, 1, \ldots \\ \min\left\{ l : l - \sum_{k=1}^{N(l)} Y_k \geq n \right\}, & \text{if } l - \sum_{k=1}^{N(l)} Y_k \geq n \text{ for some } l = 0, 1, \ldots. \end{cases}$$

Now, applying the theorem of total probability we obtain

$$T^{(n)}(z) = E\left(z^{-t^{(n)}} 1_{t^{(n)}<\infty}\right) = E\left(E\left(z^{-t^{(n)}} 1_{t^{(n)}<\infty} | t_1, Y_1\right)\right).$$

Here, two cases can be distinguished.

If $t_1 < n$, then the first level-crossing cannot happen and the process will be renewed but the time to the level-crossing will be increased by the time t_1. The level to be reached will change also from n to $n - j + i$ supposing $t_1 = j$ and $Y_1 = i$.

If $t_1 \geq n$, then the process reaches the level n at time n.

Consequently

$$\begin{aligned} T^{(n)}(z) &= \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} E\left(z^{-(t^{(n)}+j)} 1_{t^{(n)}<\infty} \mid t_1 = j, Y_1 = i\right) P(t_1 = j) P(Y_1 = i) \\ &\quad + \sum_{i=0}^{\infty} E\left(z^{-n} \mid t_1 \geq n, Y_1 = i\right) P(Y_1 = i) P\left(t_1 \geq n\right) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} f(j) g(i) z^{-j} T^{(n-j+i)}(z) + z^{-n} \sum_{i=0}^{\infty}\sum_{j=n}^{\infty} f(j) g(i). \end{aligned}$$

Taking into account the equality $\sum_{i=0}^{\infty} g(i) = 1$, we obtain (3.1). □

Using a similar argument it can be proved that for the sequence $x(n)$ the following equation holds

$$x(n) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} x(n - j + i) f(j) g(i) + \sum_{j=n}^{\infty} f(j), \tag{3.2}$$

which can be obtained also from Eq.(3.1) substituting $z = 1$.

Returning to the storage problem of process engineering, when intending to solve the problem of initial amount of material necessary to start the process without running out of material, we have to solve Eq.(3.2). However, although Eq.(3.1) has a unique bounded solution for any $z > 1$, the bounded solution of Eq. (3.2) is not unique.

We note that $T^{(0)}(z) = 1$ for any values $z \geq 1$ what is concluded from Eqs (2.5) and (3.1) substituting $n = 0$.

4. Existence and uniqueness of the solution of the difference equation

In this section we show that Eq.(3.1) has a unique solution in the set of bounded sequences if $z > 1$ and $f(0) < 1$. In order to prove that, we start with a more general theorem.

Theorem 4.1. *Let z be a fixed value, and let H and a be functions for which $c(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| < 1$ holds and $a(n,z)$ is bounded. Then the difference equation of the form*

$$w(n,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} w(n+i-j,z)H(j,i,z) + a(n,z) \tag{4.1}$$

has a unique solution in the set of bounded sequences.

Proof. Let us introduce operator K_z by the following definition:

$$K_z(w)(n) := \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} w(n+i-j)H(j,i,z) + a(n,z).$$

This operator generates a bounded sequence if its argument is a bounded sequence. Furthermore, K_z is a contraction since

$$|K_z(w_1)(n) - K_z(w_2)(n)| \leq \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} |w_1(n+i-j) - w_2(n+i-j)| \cdot |H(j,i,z)|,$$

and hence

$$\|K_z(w_1) - K_z(w_2)\|_{\infty} \leq \|w_1 - w_2\|_{\infty} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| = c(z) \|w_1 - w_2\|_{\infty}.$$

Since the space of bounded sequences is complete with the supremum norm, there is a unique solution of the equation $w = K_z(w)$ in the set of bounded sequences. Equivalently, Eq.(4.1) has a solution in the set of bounded sequences and this solution is unique.

□

Remark 4.1. If $c(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| < \infty$ and $a(n,z)$ are bounded, then $K_z(w)$ is bounded provided w is bounded too.

Corollary 4.1. *If $z > 1$ and $f(0) < 1$, then Eq.(3.1) has a unique solution in the set of bounded sequences for any fixed z.*

Proof. Indeed, in this case

$$a(n,z) = \sum_{j=n}^{\infty} f(j) z^{-n} < 1$$

and

$$c(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(j)g(i)z^{-j} < 1.$$

Consequently, the solution of Eq.(3.1) is unique. □

Now we prove a qualitative property of the solution of Eq.(3.1) in a more general form.

Theorem 4.2. *For any fixed z for which $c(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| < 1$ and $a(n,z)$ is bounded, the bounded solution $w(n,z)$ of Eq. (4.1) satisfies the inequality*

$$\limsup_{n\to\infty} |w(n,z)| \leq \frac{\limsup_{n\to\infty} |a(n,z)|}{1-c(z)}.$$

Proof. Let us assume that $N \geq 1, m \geq 2$ and $n \geq mN-1$, and denote $\overline{w}_K(z) = \sup_{n\geq K} |w(n,z)|$ and $\overline{a}_K(z) = \sup_{n\geq K} |a(n,z)|$. Then

$$|w(n,z)| \leq \overline{w}_{(m-1)N}(z) \sum_{i=0}^{\infty}\sum_{j=0}^{N-1} |H(j,i,z)| + \overline{w}_1(z) \sum_{i=0}^{\infty}\sum_{j=N}^{n-1} |H(j,i,z)| + |a(n,z)|\,.$$

Taking the supremum over all $n \geq mN-1$, we obtain

$$|\overline{w}_{mN-1}(z)| \leq \overline{w}_{(m-1)N}(z) \sum_{i=0}^{\infty}\sum_{j=0}^{N-1} |H(j,i,z)| + \overline{w}_1(z) \sum_{i=0}^{\infty}\sum_{j=N}^{\infty} |H(j,i,z)| + \overline{a}_{mN-1}.$$

Since

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} |H(j,i,z)|$$

is finite,

$$\sum_{i=0}^{\infty}\sum_{j=N}^{\infty} |H(j,i,z)| = \sum_{j=N}^{\infty}\sum_{i=0}^{\infty} |H(j,i,z)| \to 0, \qquad \text{as} \qquad N \to \infty\ .$$

Consequently, the relation

$$\limsup_{n\to\infty} |w(n,z)| \leq \limsup_{n\to\infty} |w(n,z)| \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| + \limsup_{n\to\infty} |a(n,z)|\,,$$

holds which proves the conclusion of the theorem. □

Corollary 4.2. *If $z > 1$ and $f(0) < 1$, then* $\lim_{n\to\infty} T^{(n)}(z) = 0$.

Proof. Indeed,

$$c(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |H(j,i,z)| = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(j)g(i)z^{-j} < 1$$

and

$$a(n,z) = \sum_{j=n}^{\infty} f(j)z^{-n} < 1.$$

Hence

$$\limsup_{n\to\infty} |a(n,z)| = 0.$$

Consequently

$$\lim_{n\to\infty} T^{(n)}(z) = \lim_{n\to\infty} \left|T^{(n)}(z)\right| = 0.$$

□

We have proved that the bounded solution of Eq.(3.1) tends to zero if $f(0) < 1$. This latter is a natural condition both in the storage problems of process engineering and in insurance mathematics.

5. Special cases

Now let us assume that the discrete random variable f is given by the form $f(j) = (1-\overline{f})\overline{f}^{j}$ $j = 0, 1, 2, 3, ...$, supposing that $0 < f(0) = 1 - \overline{f} < 1$. This provides a discrete analogue of the Poisson process so that the discrete Sparre Andersen model reduces to the discrete classical risk process similarly to the case of continuous risk processes [5, 9, 10]. This section is devoted to the study of this process.

Theorem 5.1. *If $f(j) = (1-\overline{f})\overline{f}^{j}$, $j = 0, 1, 2, 3, ...$, and $0 < f(0) = 1 - \overline{f} < 1$, then for fixed $z > 1$ the solution of Eq.(3.1) is $T^{(n)}(z) = (\mu(z))^n$, where $0 < \mu(z) < 1$.*

Proof. We will seek the solution of Eq.(3.1) in the form

$$T^{(n)}(z) = (\mu(z))^n, 0 < \mu(z) < 1.$$

Substitution into Eq.(3.1) yields

$$(\mu(z))^n = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} (\mu(z))^{n+i-j}(1-\overline{f})\overline{f}^{j} g(i)z^{-j} + \sum_{j=n}^{\infty}(1-\overline{f})\overline{f}^{j} z^{-n}. \tag{5.1}$$

In Eq.(5.1), $\overline{f} \neq z\mu(z)$ which can be seen by elementary computations. Using this fact when summarizing and rearranging Eq.(5.1) we obtain

$$\left((\mu(z))^n z^n - \overline{f}^n\right)\left(z\mu(z) - \overline{f} - \sum_{i=0}^{\infty}(\mu(z))^i g(i)\left(1-\overline{f}\right)z\mu(z)\right) = 0 \qquad (5.2)$$

Since Eq.(5.2) holds for any values $n \in N$, the following characteristic equation has to be satisfied:

$$k_z(\mu) = \mu - \frac{\overline{f}}{z} - \sum_{i=0}^{\infty}\mu^{i+1}g(i)(1-\overline{f}) = 0. \qquad (5.3)$$

Since $k_z(\mu)$ is continuous, $k_z(0) < 0$, and $k_z(1) > 0$, by the Bolzano theorem we conclude that there exists a function $0 < \mu(z) < 1$, for which $k_z(\mu(z)) = 0$. □

Notice that if there exists such $i \geq 1$ for which $g(i) > 0$, then for any value $z > 1$ there exists $\mu(z) > 1$, for which $k_z(\mu(z)) = 0$. Then $(\mu(z))^n$ provides an unbounded solution of the difference equation (5.1). Now we point out that in this special case Eq.(3.1), in addition to its unique bounded solution, possesses also an unbounded solution for any $z > 1$.

Let us turn to the case $z = 1$.

Theorem 5.2. *Consider the sequence $f(j) = (1-\overline{f})\overline{f}^j$, $j = 1, 2, 3, \ldots$, supposing that $0 < f(0) = 1 - \overline{f} < 1$, and let us assume that $z = 1$. If $E(t_k) < E(Y_k)$, then there exists a solution of Eq.(3.1) of the form $T^{(n)}(1) = \mu^n$ for $0 < \mu = \mu(1) < 1$. When, however, $E(t_k) \geq E(Y_k)$, then there is no solution of Eq.(3.1) of the form $T^{(n)}(1) = \mu^n$ for $0 < \mu = \mu(1) < 1$.*

Proof. Again, in this case we have $\overline{f} \neq \mu(1)$ so that Eq.(5.2) can be satisfied if and only if

$$k_1(\mu) = \mu - \overline{f} - \sum_{i=0}^{\infty}\mu^{i+1}g(i)(1-\overline{f}) = 0.$$

In this case, it is easy to see that $k_1(0) < 1$, $k_1(1) = 0$, and

$$k_1'(\mu) = 1 - \sum_{i=0}^{\infty}(i+1)\mu^i g(i)(1-\overline{f}),$$

and $k_1'(\mu)$ is a monotone decreasing function of μ. Furthermore,

$$k_1'(1) = \overline{f} - \left(1-\overline{f}\right)\sum_{i=0}^{\infty} ig(i).$$

As $E(Y_k) = \sum_{i=0}^{\infty} ig(i)$ is supposed to be finite and

$$E(t_k) = \sum_{j=0}^{\infty} jf(j) = \frac{1}{1-\overline{f}} - 1 = \frac{\overline{f}}{1-\overline{f}} < \infty$$

we distinguish two different cases.

I. $E(t_k) < E(Y_k)$, that is, $k_1'(1) < 0$. Now, there exists $0 < \mu < 1$ for which $k_1(\mu) = 0$. This inequality expresses the condition that the expectation of the amount of material filled into the storage during a unit time interval is more than the material withdrawn from the storage during this time interval. In the field of actuarial research this fact means that the expectation of payments by the insurance company exceeds the income.

II. $E(t_k) \geq E(Y_k)$ that is $k_1'(1) \geq 0$. In this case, if $k_1(\mu) = 0$ for any $0 < \mu < 1$, then $k_1(\nu) \equiv 0$ for $\mu \leq \nu \leq 1$ which is a contradiction. Hence in this case there is no solution of Eq.(3.1) in (0,1). □

Remark 5.1. Let $f(j) = (1-\overline{f})\overline{f}^j$, $j = 0, 1, 2, 3, ...$, and assume $0 < f(0) = 1-\overline{f} < 1$ and $E(t_k) < E(Y_k)$. If $z = 1$, then there are infinitely many bounded solutions of Eq.(3.1), for the function $x(n) = c_1\mu^n + c_2$ satisfies Eq.(3.2) for any n and $c_1 + c_2 = 1$. This latter condition is induced by the equality $x(0) = 1$ which is based on Eq.(3.2). Here, it is natural to ask which of the possible solutions provides the solution of the original level-crossing problem and of the initial amount problem in processing systems. Assuming finite dispersions for the random variables Y_k and t_k, applying the methods of probability theory one can prove that the limit of the solution of the physical problem has to be 0. This results in equalities $c_1 = 1$, $c_2 = 0$.

Special case 1. Let us consider the sequence $f(j) = (1 - \overline{f})\overline{f}^j$, $j = 0, 1, 2, 3, ...$, assuming that $0 < f(0) = 1 - \overline{f} < 1$. If

$$g(i) = \begin{cases} 1 & if \quad i = 1 \\ 0 & if \quad i \neq 1 \end{cases},$$

which is valid for constant $Y_k \equiv 1$, then the characteristic equation takes the form

$$k_z(\mu) = \mu - \frac{\overline{f}}{z} - \mu^2(1 - \overline{f}) = 0.$$

This equation has two positive solutions which can be easily expressed by the parameters. It can be checked that, if $z > 1$, then one of them belongs to the interval (0,1), while the second one belongs to $(1, \infty)$. The value $\mu = 1$ is a root of the characteristic equation if and only if $z = 1$. If, however, $z = 1$ then the characteristic eqution has a root in the interval $(0, 1)$ if and only if the inequality $E(t_k) < E(Y_k)$ holds.

Special case 2. Let us consider the sequence $f(j) = (1 - \overline{f})\overline{f}^j$, $j = 0, 1, 2, 3...$, assuming that $0 < f(0) = 1 - \overline{f} < 1$, and let $g(i) = (1 - \overline{g})\overline{g}^i$, $i \geq 0$, with $0 < \overline{g} < 1$. Then the characteristic equation has the form

$$k_z(\mu) = \mu - \frac{\overline{f}}{z} - \mu\sum_{i=0}^{\infty}(\mu\overline{g})^i(1 - \overline{g})(1 - \overline{f}).$$

Since assumption $\mu\overline{g} = 1$ leads to a contradiction, we can make computations only for $\mu\overline{g} \neq 1$. Rearranging the characteristic equation we get

$$-\overline{g}\mu^2 + \left(1 + \frac{\overline{fg}}{z} - (1-\overline{g})\left(1-\overline{f}\right)\right)\mu - \frac{\overline{f}}{z} = 0.$$

Again, this equation has two positive solutions. If $z > 1$, one of those belongs to the interval $(0,1)$, while the second belongs to $(1,\infty)$. The value $\mu = 1$ is a root of the above characteristic equation if and only if $z = 1$. However, if $z = 1$, then this equation has a root in the interval $(0,1)$ if and only if the inequality $E(t_k) < E(Y_k)$ holds.

Remark 5.2. One can prove that each of the bounded solutions of special cases 1 and 2 has a form $x(n) = c_1\mu^n + c_2$ with $c_1 + c_2 = 1$. If we fix the limit of the solution, then assuming $E(t_k) < E(Y_k)$,we obtain a unique bounded solution to Eq.(3.2). We conjecture that the uniqueness of the bounded solution of Eq. (3.2) holds only if we prescribe the limit of the solution as a boundary condition.

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