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Oscillatory Properties of Almost Periodic Solutions to Second Order Abstract Evolution Equations*

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Abstract. A survey of well- and less-known properties of almost periodic functions is given. A class of abstract evolution equations is shown for which the use of classical Fourier method is justified, all solutions are almost periodic in the energy space and under an additional assumption all one-dimensional non-zero projections are uniformly oscillatory: "observation" of the solution is an oscillatory function with a uniform oscillatory time.

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1. Introduction

Any linear combination of periodic functions with the same period is again a periodic function. On the other hand (see [8]), the sum of two continuous periodic functions with different periods is a periodic function if and only if their primitive periods are commensurable (linearly dependent over $\mathbb{Z}$, $\mathbb{Z}$ is the set of all integers). (The primitive period is the smallest positive period – an arbitrary non-constant continuous periodic function has its primitive period, but this is not true in general for discontinuous functions.) Hence, the set of all continuous periodic functions with non-prescribed periods is not a linear space. This great inconvenience is one of reasons which motivated mathematicians to introduce almost periodic functions (see [7], [12]). Inspired by earlier results of P. Bohl and E. Esclangon Harald Bohr developed (in the twenties of the last century) the theory of almost periodic functions as a generalization of

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pure periodicity. He ingeniously generalized two concepts: the period to the so-called almost period and the regular distribution of periods to the so-called relative density of almost periods.

A subset $E \subset \mathbb{R}$ is called *relatively dense* if there exists a number $\ell > 0$, the so-called *including length* (of the relative density), such that for any $a \in \mathbb{R}$ it holds $E \cap [a, a + \ell] \neq \emptyset$. (For example, an increasing bi-infinite sequence $\{s_m\}_{m\in\mathbb{Z}}$ is relatively dense if and only if $\{s_{m+1} - s_m\}_{m\in\mathbb{Z}}$ is bounded. For instance, for any $T \neq 0$ the set $\mathbb{Z}T = \{\, mT \mid m \in \mathbb{Z}\,\}$ is relatively dense.)

Let $t \mapsto u(t)$ be a real function defined on $\mathbb{R}$, $u\colon \mathbb{R} \to \mathbb{R}$, and $\varepsilon > 0$. A real number τ is called *ε–almost period* of the function u if the inequality $|u(t + \tau) - u(t)| \leq \varepsilon$ holds for any $t \in \mathbb{R}$.

A continuous function $u\colon \mathbb{R} \to \mathbb{R}$ is said to be *almost periodic* (due to Bohr) if for any $\varepsilon > 0$ the set $T(u, \varepsilon)$ of all ε–almost periods of the function u is relatively dense.

Solomon Bochner generalized quite another property of continuous periodic functions on $\mathbb{R}$ and obtained surprisingly the same class of functions. (Let u be a continuous periodic function with a positive period T. If $\{s_n\}$ is an arbitrary sequence of numbers then to any $n \in \mathbb{N}$ there exists an integer m_n such that $s_n + m_nT \in [0, T]$. Since any bounded sequence contains a convergent subsequence there exists a subsequence $\{s_{n_k} + m_{n_k}T\}$ which converges to $s_0 \in [0, T]$. Using now the uniform continuity of the function u on $\mathbb{R}$ (this follows from the uniform continuity on the compact interval $[0, T]$ and the periodicity) we get easily that the sequence of functions $\{u(t + s_n)\}$, the so-called s_n–shifts of the function u, contains a subsequence which converges uniformly on $\mathbb{R}$. This property of continuous periodic functions became starting point for Bochner to define the so-called normal functions. Later these functions were shown to be just the almost periodic functions in the sense of Bohr.) The original definition due to Bochner of almost periodic functions has been adapted for abstract functions with values in a Banach space. Let $\mathbb{X}$ be a Banach space with the norm $|\cdot|$. The symbol $CB(\mathbb{X})$ denotes the Banach space of all continuous and bounded functions $u\colon \mathbb{R} \to \mathbb{X}$, $t \mapsto u(t)$, with the norm $||u|| = \sup_{t\in\mathbb{R}} |u(t)|$.

By the *Bochner transform of a function* $u\colon \mathbb{R} \to \mathbb{X}$ we mean the function $\tilde{u}$ that to each $s \in \mathbb{R}$ assigns s–shift of the function u, i. e.

$$s \mapsto \tilde{u}(s) = u(\cdot + s).$$

In particular: if $u \in \mathbb{Y} = CB(\mathbb{X})$ then $\tilde{u} \in CB(\mathbb{Y})$.

A function $u \in CB(\mathbb{X})$ is said to be *almost periodic* (due to Bochner) if the range of its Bochner transform

$$\tilde{u}(\mathbb{R}) = \bigcup_{s\in\mathbb{R}} \{\tilde{u}(s)\} \quad \left(= \bigcup_{s\in\mathbb{R}} \{u(\cdot + s)\}\right)$$

is relatively compact in $CB(\mathbb{X})$, i. e. any sequence $\{s_n\} \subset \mathbb{R}$ contains a subsequence $\{s_{n_k}\}$ such that the sequence of shifts $\{u(\cdot + s_{n_k})\}$ is uniformly convergent on $\mathbb{R}$.

The Bohr definition can of course be made with any abstract continuous function $u\colon \mathbb{R} \to \mathbb{X}$ as well but the Bochner definition is preferable in applications. Alain Haraux [20] proved that in the Bochner definition instead of sequences of arbitrary

s–shifts it is sufficient to take sequences of such s–shifts where s belongs only to a relatively dense set (in the sense of Bohr) E. As a consequence the following assertion (which is a generalization of a criterion in [1], p. 10) can be obtained and its direct use is convenient in applications (see e. g. [20] for an extraordinarily simple proof of almost periodicity of bounded solutions of the autonomous system of ordinary equations $\dot{u} + Au = 0$).

Let $E \subset \mathbb{R}$ be a relatively dense set and $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, $\varphi(0) = 0$. Let $u \in CB(\mathbb{X})$. Then u is almost periodic provided that the following assumptions are fulfilled:

(i) $u(E) = \bigcup_{t\in E} \{u(t)\}$ *is relatively compact in* $\mathbb{X}$ *and*

(ii) $\sup_{t\in\mathbb{R}} |u(t+\tau) - u(t+\sigma)| \le \varphi(|u(\tau) - u(\sigma)|)$ *for all* $(\tau, \sigma) \in E \times E$.

2. Properties of almost periodic functions

The set of all almost periodic functions $u\colon \mathbb{R} \to \mathbb{X}$ contains the set of all continuous periodic functions, moreover, it is a linear space (the smallest with such a property) and it is closed with respect to the uniform convergence on $\mathbb{R}$. This space is usually denoted by the symbol $AP(\mathbb{X})$, it is a linear closed subspace in the Banach space $CB(\mathbb{X})$.

Any $u \in AP(\mathbb{X})$ is uniformly continuous on $\mathbb{R}$ and its range $u(\mathbb{R})$ is relatively compact in $\mathbb{X}$. An interesting question is when an almost periodic function is a periodic one. Any continuous periodic function has a compact range, however, the compactness of the range of an almost periodic function does not ensure the periodicity in general (examples are given in [13], [19]), but the affirmative case happens for some classes of functions (such as trigonometric polynomials of the second degree, see [10]). It is a remarkable fact that continuous periodic functions are characterized among almost periodic functions as functions for which the Bochner transform has a compact range (see [5], [19]). Another necessary and sufficient condition for an almost periodic function to be periodic is that its primitive including length is finite (see [9] for more details).

A function $u\colon \mathbb{R} \to \mathbb{X}$ is called *weakly almost periodic* if for any w from the dual space to $\mathbb{X}$ the real-valued function $t \mapsto \langle w, u(t)\rangle$ is almost periodic. Any weakly almost periodic function is weakly continuous and its range is bounded. An almost periodic function is weakly almost periodic and a weakly almost periodic function is almost periodic if and only if its range is relatively compact in $\mathbb{X}$ (see [15]). In contrast to the uniform continuity of any almost periodic function there exists a weakly almost periodic function with discontinuities of the second kind at all rational points in $\mathbb{R}$ (see [11]).

Another interesting question is the possibility of the uniform approximation on $\mathbb{R}$ of an almost periodic function by continuous periodic functions. It turns out (see [6]) that with arbitrary accuracy only a very narrow class of almost periodic functions can be approximated (the so-called functions with one-point basis). A fundamental result on approximation (essentially due to Bochner) says that if $\mathbb{X}$ is a real Hilbert space then to any almost periodic function there corresponds a sequence of trigonometric polynomials (the so-called Bochner-Féjér polynomials) uniformly convergent on $\mathbb{R}$ to the function. Thus, almost periodic functions are precisely those functions that can be uniformly on $\mathbb{R}$ approximated by trigonometric polynomials.

It follows easily from the Bochner definition that any almost periodic function is uniformly recurrent. Recall that a continuous function $u\colon \mathbb{R} \to \mathbb{X}$ is *uniformly recurrent* if there exists a sequence $\tau_n \to +\infty$ such that the sequence $\{u(t+\tau_n)\}$ is uniformly convergent on $\mathbb{R}$ to $u(t)$. As a consequence we obtain the so-called Poisson stability of an almost periodic function: for any $t \in \mathbb{R}$ we have $\lim_{n\to+\infty} u(t+\tau_n) = \lim_{n\to+\infty} u(t-\tau_n) = u(t)$. A further consequence: any almost periodic function which has a limit for $t \to +\infty$ or $-\infty$ is necessarily a constant function.

For any $u \in AP(\mathbb{X})$ there exists $M(u) \in \mathbb{X}$, the so-called *mean value of* u, such that

$$\lim_{T\to+\infty} \sup_{s\in\mathbb{R}} \left| \frac{1}{T} \int_s^{s+T} u(t)\,dt - M(u) \right| = 0.$$

Any real-valued non-constant almost periodic function oscillates about its mean value. In particular, if $u \in AP(\mathbb{R})$ is non-constant with the mean value zero then u is an *oscillatory function* (*u oscillates about zero*), i. e. there exists a positive constant Θ, the so-called *oscillatory time,* such that the function u assumes both positive and negative values in any interval $J \subset \mathbb{R}$ the length of which is greater than Θ (for such functions in differential equations see e. g. [24] – [35]). In fact, a more precise result holds true (see [14]): there exist two positive constants δ and Θ such that in any interval $[a,\, a+\Theta]$ there are two disjoint closed subintervals J_a^+ and J_a^- of length 2δ such that

$$u(t) \geq \frac{||u^+||}{2}, \quad t \in J_a^+, \qquad u(s) \leq -\frac{||u^-||}{2}, \quad s \in J_a^-,$$

where

$$u^+(t) = \frac{|u(t)| + u(t)}{2}, \quad u^-(t) = \frac{|u(t)| - u(t)}{2}, \quad t \in \mathbb{R}.$$

Hence, classical oscillatory properties of continuous periodic functions with the mean value zero remain valid for real-valued almost periodic functions with the mean value zero. Several other properties of continuous periodic functions are shared by almost periodic ones: even though almost periodicity is a typical *global* property, an almost periodic function is known, roughly speaking, whenever we identify this function on an interval of *finite* (including) length. This is a similar property, but some properties can be quite different, for example, there exists a vector almost periodic function $u \in AP(\mathbb{C}^4)$, ($\mathbb{C}$ is the set of complex numbers), which is one-to-one (see [6]).

Once we know that a function is almost periodic we have a great deal of properties at our disposal. The theory of abstract almost periodic functions turns out to be an efficient tool for the study of differential equations both ordinary and partial (see [1], [4], [34], a series of papers of Haraux and many others). The importance of almost periodic solutions is stressed by the fact that they often appear as "asymptotic parts" of solutions in rather very general situations (see e. g. [3], [18]).

3. Selfadjoint extensions of symmetric operators

Let H be a real Hilbert space with the scalar product $\langle\cdot,\,\cdot\rangle$ and the norm $|\cdot|$.

Let $L\colon \mathcal{D}_L \subset H \to H$ be a linear densely defined operator which is symmetric (the adjoint operator L^* extends L) and bounded below (by a constant γ), i. e. it holds

$$\begin{aligned}\langle Lu,\, v\rangle &= \langle u,\, Lv\rangle, \quad u,\, v \in \mathcal{D}_L,\\ \langle Lu,\, u\rangle &\ge \gamma |u|^2, \quad u \in \mathcal{D}_L.\end{aligned}$$

If $\gamma = 0$ ($\gamma > 0$) the operator L is called non-negative (positive definite). To the operator L one associates the so-called *energy space* $\mathcal{H}_L$ that is defined by the completion of $\mathcal{D}_L$ in the so-called energy norm $|\cdot|_L$:

$$\langle u,\, v\rangle_L = \langle Lu,\, v\rangle + \mu\langle u,\, v\rangle, \quad |u|_L^2 = \langle u,\, u\rangle_L, \quad u,\, v \in \mathcal{D}_L,$$

where μ is chosen such that $\mu + \gamma > 0$.

The properties of the energy norm make it possible to identify the energy space $\mathcal{H}_L$ with a subspace of H and also to define $\mathcal{H}_L$ equivalently as $\mathcal{H}_L = \{u \in H \mid \exists\, u_n \in \mathcal{D}_L,\, u_n \to u \text{ in } H,\, |u_n - u_m|_L \to 0\}$. The energy space $\mathcal{H}_L$ is a Hilbert space with the scalar product given by the formula $\langle u,\, v\rangle_L = \lim\langle u_n,\, v_n\rangle_L$, $u,\, v \in \mathcal{H}_L$, with the evident meaning of u_n and v_n. (Various μ satisfying $\mu + \gamma > 0$ give rise to the same space $\mathcal{H}_L$ with equivalent norms.)

In general, a symmetric operator cannot be extended to a selfadjoint operator. In case of bounded below operators such an extension is possible, though. In this case the extension is not determined uniquely in general and the most important among them is the operator defined by the formula

$$\mathcal{D}_{\hat{L}} = \mathcal{H}_L \cap \mathcal{D}_{L^*}, \quad \hat{L}u = L^*u.$$

The operator $\hat{L}$ is selfadjoint and it is called *the Friedrichs extension* of the operator L. Two properties are significant:

- $\hat{L}$ is the unique selfadjoint extension the domain of which is contained in $\mathcal{H}_L$;
- $\hat{L}$ is bounded below (by the same constant as L).

The operators L and $\hat{L}$ have the same energy spaces, $\mathcal{H}_L = \mathcal{H}_{\hat{L}}$.

Let $L\colon \mathcal{D}_L \subset H \to H$ be a linear densely defined operator. Let us assume that

$$L \text{ is symmetric}; \tag{3.1}$$

$$L \text{ is non-negative}; \tag{3.2}$$

the energy space $\mathcal{H}_L$ of the operator L is compactly embedded in H, (3.3)
(i. e. the unit ball in $\mathcal{H}_L$ is relatively compact in H);

$$\text{the null-space of the operator } L^* \text{ is trivial, } \mathcal{N}_{L^*} = \{0\}, \tag{3.4}$$
$$(\text{i. e. if } v \in H \text{ and } \langle Lu, v \rangle = 0 \text{ for all } u \in \mathcal{D}_L, \text{ then } v = 0).$$

Then the operator L is positive definite. Moreover, L is essentially selfadjoint, i. e. its closure $\bar{L}$ is selfadjoint (cf. [26]) and since any essentially selfadjoint operator has only one selfadjoint extension we have $\hat{L} = \bar{L}$.

(Note, that the assumption $\mathcal{N}_{L^*} = \{0\}$ cannot be replaced by $\mathcal{N}_L = \{0\}$.)

For selfadjoint bounded below operators the following equivalence holds true: $\bar{L}$ has a compact resolvent if and only if the energy space $\mathcal{H}_{\bar{L}}$ is compactly embedded in H. Thus, $\bar{L}^{-1}$ is a self-adjoint compact operator in H and using elementary spectral theory of self-adjoint compact operators in a Hilbert space (of non-finite dimension) we get that *there exists a complete orthonormal set in H of eigenvectors $\{v_k\}_{k=1}^{\infty}$ of the operator $\bar{L}$ with the corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$* (listed according to their multiplicities),

$$0 < \inf_{\substack{v \in \mathcal{D}_L \\ v \neq 0}} \frac{\langle Lv, v \rangle}{|v|^2} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \quad \text{and } \lambda_k \to \infty \text{ as } k \to \infty.$$

If we identify H with its dual H^* and H^* with a subspace of the dual $\mathcal{H}_L^*$ of the energy space $\mathcal{H}_L$ we get $\mathcal{H}_L \hookrightarrow H \hookrightarrow \mathcal{H}_L^*$, both embeddings are continuous and dense and it is correct to denote the duality pairing on $\mathcal{H}_L^* \times \mathcal{H}_L$ by the same symbol $\langle \cdot, \cdot \rangle$ as the scalar product in H. In addition to be an unbounded operator in H the operator $\bar{L}$ is an isomorphism of $\mathcal{H}_L$ on $\mathcal{H}_L^*$.

4. Oscillatory properties of energy solutions

Under the assumptions (3.1) – (3.4) we have (by [1]): *for any $\varphi \in \mathcal{H}_L$ and $\psi \in H$ there exists a unique function $u\colon \mathbb{R} \to \mathcal{H}_L$* (energy solution) *such that*

$$u \in C(\mathbb{R}; \mathcal{H}_L) \cap C^1(\mathbb{R}; H) \cap C^2(\mathbb{R}; \mathcal{H}_L^*),$$

$$\ddot{u} + \bar{L}u = 0, \quad t \in \mathbb{R}, \qquad u(0) = \varphi, \; \dot{u}(0) = \psi.$$

Moreover, the law of conservation of energy holds:

$$|u(t)|_L^2 + |\dot{u}(t)|^2 = |\varphi|_L^2 + |\psi|^2, \quad t \in \mathbb{R},$$

and *the function $t \mapsto u(t)$ is almost periodic in $\mathcal{H}_L$ and $t \mapsto \dot{u}(t)$ is almost periodic in H. Both functions u and $\dot{u}$ have the mean value zero.* The solution can be expressed in the form of the Fourier series with respect to the complete orthonormal set $\{v_k\}_{k=1}^{\infty}$:

$$u(t) = \sum_{k=1}^{\infty} \Big[\varphi_k \cos(\sqrt{\lambda_k} t) + \frac{\psi_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \Big] v_k, \quad \varphi_k = \langle \varphi, v_k \rangle, \; \psi_k = \langle \psi, v_k \rangle,$$

the series converges in H uniformly in t and the same is true for the series arising by differentiating term-by-term with respect to t or multiplying term-by-term by $\sqrt{\lambda_k}$.

In fact, the assumptions (3.1) – (3.4) justify the use of the well-known Fourier method when solving initial-boundary value problems for classical partial differential equations. Notice the advantage of the approach: the selfadjointness and positive definiteness need not be verified, it is enough to assume (more easily verifiable) symmetricity and non-negativity of the operator (cf. [26]).

Any solution $u\colon \mathbb{R} \to \mathcal{H}_L$ is also weakly almost periodic (with the mean value zero) and consequently the non-zero functions $\langle w, u(t)\rangle$, $w \in \mathcal{H}_L^*$, are oscillatory, but no uniformity with respect to w (and u) of the oscillatory time is guaranteed. In fact, these one-dimensional projections are uniformly oscillatory under an additional assumption on the growth of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, which is introduced (following [22]) in the next theorem.

Theorem. *Let the assumptions* (3.1) – (3.4) *be satisfied and, moreover,*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} < +\infty\,. \tag{4.1}$$

Then for any $w \in \mathcal{H}_L^*$ either $t \mapsto \langle w, u(t)\rangle \equiv 0$ on $\mathbb{R}$ or $t \mapsto \langle w, u(t)\rangle$ is uniformly oscillatory with the uniform oscillatory time

$$\Theta = 2\pi \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}}\,.$$

The proof relies on the above representation of the solution in the form of a Fourier expansion and on results of Haraux-Komornik [23] modified in the sense suggested in Haraux [22]. In the first step of the proof we obtain the implication: $\langle w, u(t)\rangle \geq 0$ on J, where $|J| > \Theta$, then $\langle w, u(t)\rangle \equiv 0$ on J. In the second step we get $\langle w, u(t)\rangle \equiv 0$ on $\mathbb{R}$.

(The theory does not provide results for the wave equation: the semilinear wave equation in one space dimension and the oscillating character of the function $t \mapsto u(t,x)$ for fixed x is dealt with in [2], [22].)

In conclusion we mention the application to the equation of simply supported Euler-Bernoulli beam (see [22], also for another example):

$$\partial_t^2 u + \partial_x^4 u + p(x)\, u = 0, \quad t \in \mathbb{R},\ x \in (0,\ell),$$

$$u(t,0) = \partial_x^2 u(t,0) = 0, \quad u(t,\ell) = \partial_x^2 u(t,\ell) = 0, \quad t \in \mathbb{R}.$$

Here $H = L_2(0,\ell)$, $Lv = v'''' + p(x)\, v$, $v(0) = v''(0) = 0$, $v(\ell) = v''(\ell) = 0$, $p \in L_\infty(0,\ell)$ non-negative, $\mathcal{H}_L = W_2^2(0,\ell) \cap \overset{\circ}{W}{}_2^1(0,\ell)$, $\Theta = \frac{1}{3}\pi\ell^3$ (namely, the eigenvalues λ_k are simple and by the Courant-Fischer principle $\lambda_k \geq \nu_k$, where $\nu_k = \left(\frac{k\pi}{\ell}\right)^4$ are eigenvalues of the operator L with $p \equiv 0$, and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.) The result can be applied to

$$\langle w, u(t)\rangle = \sum_{j=1}^{j_0} \alpha_j\, u(t, x_j) + \sum_{k=1}^{k_0} \beta_k\, u(t, y_k),$$

where $x_1, \ldots, x_{j_0}, y_1, \ldots, y_{k_0} \in (0, \ell)$ and $\alpha_1, \ldots, \alpha_{j_0}, \beta_1, \ldots, \beta_{k_0} \in \mathbb{R}$ are arbitrary.

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