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Stability Regions for Linear Delay Differential Equations with Four Parameters*

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Dedicated to Professor István Győri on the occasion of his 65th birthday

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Abstract. The purpose of this work is to give stability regions in the set of parameters for a linear delay differential equation $x'(t) = -Ax(t) - bx(t-\tau)$, where A is a 2×2 real constant matrix, b is a real number and τ is a positive number.

AMS Subject Classifications: 34K20, 34K25

Keywords: Asymptotic stability; Stability criterion; Delay; Characteristic equation.

1. Introduction

We consider a linear delay differential equation with two coefficients and one delay

$$x'(t) = -Ax(t) - Bx(t-\tau), \tag{1.1}$$

where A and B are 2×2 real constant matrices and τ is a positive number. The stability problem of Eq. (1.1) depends on the location of the roots of its associated characteristic equation

$$F(\lambda) \equiv \det(\lambda I + A + Be^{-\lambda\tau}) = 0, \tag{1.2}$$

where I is the 2×2 identity matrix. It is well-known (see, e.g., [4] or [8]) that the zero solution of Eq. (1.1) is asymptotically stable if and only if all the roots of Eq. (1.2) have negative real parts.

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In the scalar case, Eq. (1.1) is expressed as

$$x'(t) = -ax(t) - bx(t-\tau), \tag{1.3}$$

where a and b are real numbers. Then the following two results are known.

Theorem A. ([5]) *The zero solution of Eq.* (1.3) *is asymptotically stable if and only if*

$$a + b > 0 \quad \textit{and} \quad a > \varphi(b),$$

where the curve $a = \varphi(b)$ *is given parametrically by the equation*

$$a = -\frac{\omega}{\tan \omega\tau}, \quad b = \frac{\omega}{\sin \omega\tau}, \quad 0 < \omega < \frac{\pi}{\tau}.$$

Theorem B. ([1], [2]) *The zero solution of Eq.* (1.3) *is asymptotically stable if and only if either*

$$a + b > 0 \quad \textit{and} \quad b^2 - a^2 \le 0$$

or

$$a + b > 0, \quad b^2 - a^2 > 0 \quad \textit{and} \quad \tau < \frac{1}{\sqrt{b^2 - a^2}} \arccos\left(-\frac{a}{b}\right).$$

Theorem A presents the stability region which means the set of all (a, b) in which the zero solution of Eq. (1.3) is asymptotically stable, see Figure 1. On the other hand, Theorem B gives delay-dependent and delay-independent stability criteria for Eq. (1.3).

In case $A = aI$, where a is a real number, Eq. (1.1) is expressed as

$$x'(t) = -ax(t) - Bx(t-\tau). \tag{1.4}$$

Then the following two results corresponding to Theorems A and B are obtained.

Theorem C. ([7]) *Let* $be^{\pm i\theta}$ *be eigenvalues of* B *where* b *and* θ *are real numbers with* $0 < |\theta| \le \pi/2$. *Then the zero solution of Eq.* (1.4) *is asymptotically stable if and only if*

$$a > \varphi(b),$$

where the curve $a = \varphi(b)$ *is given parametrically by the equation*

$$a = -\frac{\omega}{\tan(\omega\tau - |\theta|)}, \quad b = \frac{\omega}{\sin(\omega\tau - |\theta|)}, \quad -\frac{\pi - |\theta|}{\tau} < \omega < \frac{|\theta|}{\tau}.$$

Theorem D. ([6]) *Let* $be^{\pm i\theta}$ *be eigenvalues of* B *where* b *and* θ *are real numbers with* $0 < |\theta| \le \pi/2$. *Then the zero solution of Eq.* (1.4) *is asymptotically stable if and only if either*

$$a + b\cos\theta > 0 \quad \textit{and} \quad b^2 - a^2 \le 0$$

or

$$a + b\cos\theta > 0, \quad b^2 - a^2 > 0 \quad \textit{and} \quad \tau < \frac{\operatorname{sgn}(b)}{\sqrt{b^2 - a^2}} \left\{\arccos\left(-\frac{a}{b}\right) - |\theta|\right\}.$$

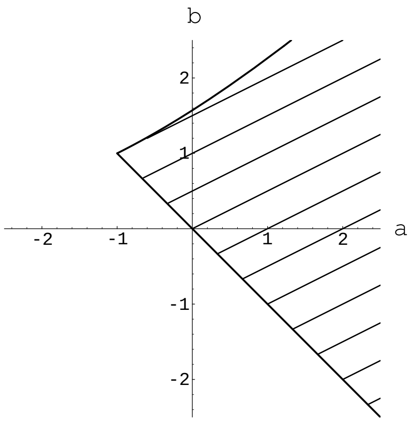

Fig. 1 Stability region for Eq. (1.3) with $\tau = 1$

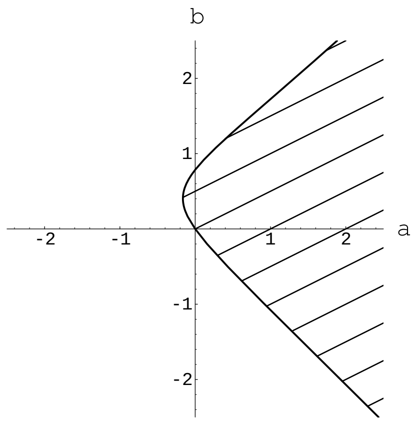

Fig. 2 Stability region for Eq. (1.4) with $\tau = 1$ and $\theta = \pi/4$

The purpose of this work is to give the exact stability region for Eq. (1.1) with $B = bI$, where b is a real number, that is,

$$x'(t) = -Ax(t) - bx(t - \tau). \tag{1.5}$$

Note that the asymptotic stability of Eq. (1.5) is invariant under a constant invertible linear transfomation. Throughout the work, we only consider Eq. (1.5) where the matrix A has complex eigenvalues, that is,

$$x'(t) = -a \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x(t) - bx(t - \tau), \tag{1.6}$$

where a and θ are real numbers with $0 < |\theta| \le \pi/2$. The following is our main result.

Theorem 1.1. *Let* $|\theta| = \pi/2$. *Then the zero solution of Eq.* (1.6) *is asymptotically stable if and only if either*

$$0 < b < \frac{\pi}{2\tau} \quad \textit{and} \quad |a| \in \{0\} \cup I_0(b) \cup I_1(b) \cup I_2(b) \cup \cdots \tag{1.7}$$

or

$$-\frac{\pi}{2\tau} < b < 0 \quad \textit{and} \quad |a| \in J_0(b) \cup J_1(b) \cup J_2(b) \cup \cdots, \tag{1.8}$$

where $I_k(b)$ *and* $J_k(b)$ *are intervals defined by*

$$I_0(b) = \left(0, \frac{\pi}{2\tau} - b\right), \quad I_k(b) = \left(\frac{(4k-1)\pi}{2\tau} + b, \frac{(4k+1)\pi}{2\tau} - b\right), \quad k = 1, 2, \ldots,$$

$$J_k(b) = \left(\frac{(4k+1)\pi}{2\tau} - b, \frac{(4k+3)\pi}{2\tau} + b\right), \quad k = 0, 1, 2, \ldots,$$

respectively.

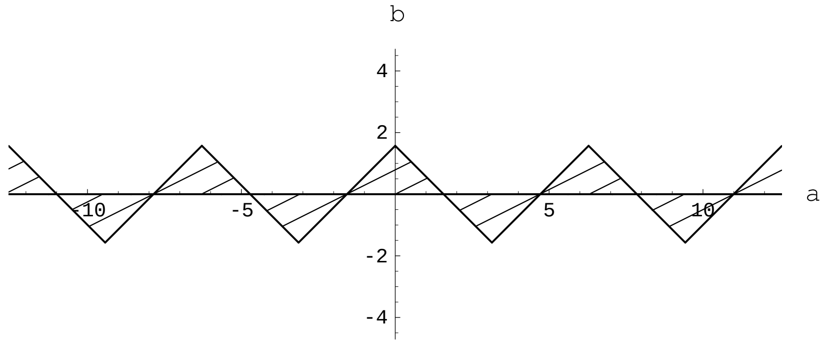

Fig. 3 Stability region for Eq. (1.6) with $\tau = 1$ and $\theta = \pi/2$

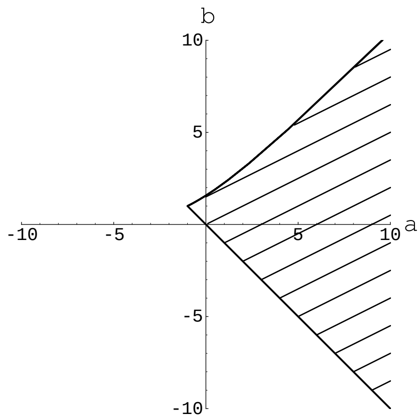

Fig. 4 Stability region for Eq. (1.6) with $\tau = 1$ and $\theta = 0$

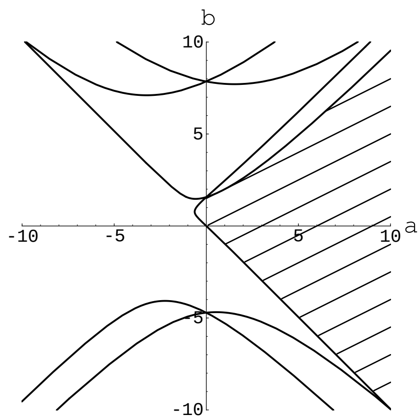

Fig. 5 Stability region for Eq. (1.6) with $\tau = 1$ and $\theta = \pi/10$

Theorem 1.1 shows that if $0 < |b| < \pi/(2\tau)$ and $|\theta| = \pi/2$, then stability switches in Eq. (1.6) appear as a varies monotonously, see Figure 3.

To our regret, we can not yet find the exact stability region for Eq. (1.6) with $0 < |\theta| < \pi/2$; however, we believe that Figures 4 through 7 illustrate the stability regions for Eq. (1.6). The boundaries of the regions are given parametrically by the equation

$$a = -\frac{\omega \cos \omega\tau}{\sin(\omega\tau + |\theta|)}, \quad b = \frac{\omega \cos \theta}{\sin(\omega\tau + |\theta|)}, \quad \omega \neq \frac{l\pi - |\theta|}{\tau},$$

where l is an integer, which are also found by D-decomposition method; see, e.g., [3].

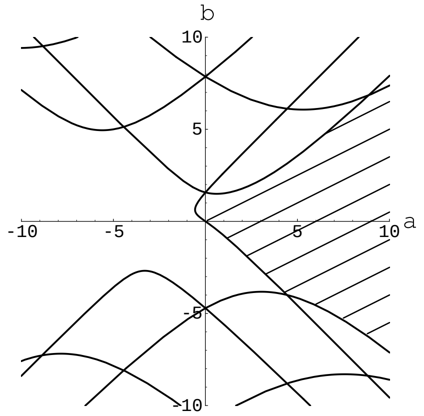

Fig. 6 Stability region for Eq. (1.6) with $\tau = 1$ and $\theta = \pi/4$

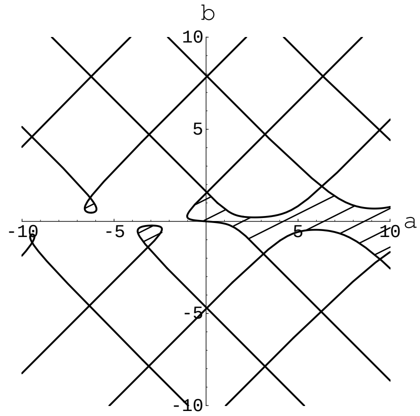

Fig. 7 Stability region for Eq. (1.6) with $\tau = 1$ and $\theta = \pi/(2.1)$

2. Proof of Main Result

The characteristic function $F(\lambda)$ for Eq. (1.6) becomes

$$\begin{aligned}F(\lambda) &= (\lambda + a\cos\theta + be^{-\lambda\tau})^2 + (a\sin\theta)^2 \\ &= (\lambda + a\cos|\theta| + be^{-\lambda\tau})^2 - (ia\sin|\theta|)^2 \\ &= (\lambda + a\cos|\theta| + be^{-\lambda\tau} + ia\sin|\theta|)(\lambda + a\cos|\theta| + be^{-\lambda\tau} - ia\sin|\theta|) \\ &= f(\lambda)\overline{f(\bar{\lambda})},\end{aligned}$$

where $f(\lambda)$ is a function defined by

$$f(\lambda) = \lambda + ae^{i|\theta|} + be^{-\lambda\tau}.$$

Taking note that $\overline{f(\bar{\lambda})} = 0$ implies $f(\bar{\lambda}) = 0$, one can easily see that the following proposition holds.

Proposition 2.1. *The zero solution of Eq.* (1.6) *is asymptotically stable if and only if all the roots of* $f(\lambda) = 0$ *have negative real parts.*

To prove our main theorem, we deal with the case where $|\theta| = \pi/2$. Since $F(\lambda) = (\lambda + i|a| + be^{-\lambda\tau})(\lambda - i|a| + be^{-\lambda\tau})$, it suffices to investigate the location of roots of

$$\lambda + i|a| + be^{-\lambda\tau} = 0. \tag{2.1}$$

First, we will determine the value of b at which Eq. (2.1) has roots on the imaginary axis. Note that $\lambda = 0$ is not a root of Eq. (2.1) if $a \neq 0$. Substituting $\lambda = i\omega$ into Eq. (2.1), we have $i\omega + i|a| + be^{-i\omega\tau} = 0$, or, equivalently,

$$b\cos\omega\tau = 0 \quad \text{and} \quad b\sin\omega\tau = \omega + |a|. \tag{2.2}$$

For simplicity, let $\omega_n^\pm$ and $b_{j,m}^\pm$ ($j = 0, 1$; $n, m = 0, 1, 2, \ldots$) be real numbers defined by

$$\begin{aligned} \omega_n^+ &= \frac{(2n+1)\pi}{2\tau}, \quad b_{0,m}^+ = \frac{(4m+1)\pi}{2\tau} + a, \quad b_{1,m}^+ = -\frac{(4m+1)\pi}{2\tau} - a, \\ \omega_n^- &= -\frac{(2n+1)\pi}{2\tau}, \quad b_{0,m}^- = \frac{(4m+1)\pi}{2\tau} - a, \quad b_{1,m}^- = -\frac{(4m+3)\pi}{2\tau} + a, \end{aligned}$$

respectively. Then we obtain the following lemma by (2.2).

Lemma 2.1. *Suppose that $a > 0$ and $b \neq 0$. Let $\lambda = i\omega$ be a root of Eq.* (2.1) *where ω is a nonzero real number. Then the values of ω and b are expressed as*

$$\omega = \omega_n^+, \quad b = \begin{cases} b_{0,m}^+ & (\omega = \omega_{2m}^+) \\ b_{1,m}^+ & (\omega = \omega_{2m+1}^+) \end{cases} \qquad n, m = 0, 1, 2, \ldots$$

or

$$\omega = \omega_n^-, \quad b = \begin{cases} b_{0,m}^- & (\omega = \omega_{2m}^-) \\ b_{1,m}^- & (\omega = \omega_{2m+1}^-) \end{cases} \qquad n, m = 0, 1, 2, \ldots.$$

Conversely, if $b = b_{j,m}^\pm$ ($j = 0, 1$; $m = 0, 1, \ldots$), then $i\omega_{2m+j}^\pm$ are roots of Eq. (2.1).

Next, we will observe how the roots of Eq. (2.1) on the imaginary axis move as $|b|$ increases. Clearly, if $b = 0$, Eq. (2.1) has the only root $-i|a|$. Let I_k and J_k be intervals defined by $I_k = I_k(0)$ and $J_k = J_k(0)$, that is,

$$I_0 = \left(0, \frac{\pi}{2\tau}\right), \quad I_k = \left(\frac{(4k-1)\pi}{2\tau}, \frac{(4k+1)\pi}{2\tau}\right), \quad J_k = \left(\frac{(4k+1)\pi}{2\tau}, \frac{(4k+3)\pi}{2\tau}\right).$$

Lemma 2.2. *For $k = 0, 1, 2, \ldots$, the following three statements hold:*

(i) If $a \in I_k$, then the root $-ia$ moves in the left-half plane (resp. in the right-half plane) as b increases from 0 (resp. decreases from 0).

(ii) If $a \in J_k$, then the root $-ia$ moves in the right-half plane (resp. in the left-half plane) as b increases from 0 (resp. decreases from 0).

(iii) If $a = (2k+1)\pi/(2\tau)$, then the root $-ia$ moves in the right-half plane as $|b|$ increases from 0.

Proof. Taking the derivative of λ with respect to b on Eq. (2.1), we have

$$\frac{d\lambda}{db} = -\frac{e^{-\lambda\tau}}{1 - b\tau e^{-\lambda\tau}} = -\frac{e^{-\lambda\tau}}{1 + \tau(\lambda + ia)}, \tag{2.3}$$

which implies

$$\mathrm{Re}\, \frac{d\lambda}{db}\bigg|_{\lambda=-ia} = \mathrm{Re}\,(-e^{ia\tau}) = -\cos a\tau.$$

This shows the assertions (i) and (ii). To verify the assertion (iii), we need the sign of Re $\frac{d^2\lambda}{db^2}\big|_{\lambda=-ia}$. It follows from (2.3) that

$$\frac{d^2\lambda}{db^2} = -\frac{\tau e^{-2\lambda\tau}\{2+\tau(\lambda+ia)\}}{\{1+\tau(\lambda+ia)\}^3},$$

which yields

$$\frac{d^2\lambda}{db^2}\bigg|_{\lambda=-ia} = -2\tau e^{2ia\tau} = -2\tau(\cos 2a\tau + i\sin 2a\tau).$$

Hence, if $a=(2k+1)\pi/(2\tau)$, then Re $\frac{d\lambda}{db}\big|_{\lambda=-ia}=0$ and Re $\frac{d^2\lambda}{db^2}\big|_{\lambda=-ia}=2\tau>0$. This shows the assertion (iii). □

Lemma 2.3. *If $a>0$ and $b\neq 0$, then the roots $i\omega_n^{\pm}$ move in the right-half plane as $|b|$ increases.*

Proof. By (2.3) and the relation $e^{-\lambda\tau}=-(\lambda+ia)/b$, we have

$$\text{Re}\,\frac{d\lambda}{db}\bigg|_{\lambda=i\omega} = \text{Re}\,\frac{i(\omega+a)}{b\{1+i\tau(\omega+a)\}} = \frac{\tau(\omega+a)^2}{b\{1+\tau^2(\omega+a)^2\}},$$

which implies the assertion of this lemma. □

Now, we are in a position to present necessary and sufficient conditions for the roots of Eq. (2.1) have negative real parts.

Proposition 2.2. *Let $a>0$ and $b>0$. Then all the roots of Eq.* (2.1) *have negative real parts if and only if either*

$$\frac{2k\pi}{\tau} \le a < \frac{(4k+1)\pi}{2\tau} \quad \textit{and} \quad 0<b<\frac{(4k+1)\pi}{2\tau}-a \quad (k=0,1,2,\ldots) \tag{2.4}$$

or

$$\frac{(4k-1)\pi}{2\tau} < a < \frac{2k\pi}{\tau} \quad \textit{and} \quad 0<b<-\frac{(4k-1)\pi}{2\tau}+a \quad (k=0,1,2,\ldots). \tag{2.5}$$

Proof. For $b>0$, $\nu(b)$ denotes the number of the roots of Eq. (2.1) whose real parts are positive. Our argument is divided into two cases.

Case (I): $a\in I_k$ $(k=0,1,2,\ldots)$. Lemma 2.2-(i) shows $\nu(b)=0$ for $b>0$ sufficiently small by the continuity of the roots with respect to b. By Lemma 2.1, if the positive number b is given by

$$b_{0,m}^{+}\ (m=0,1,2,\ldots),\quad b_{0,m}^{-}\ (m=k,k+1,\ldots)\quad \text{or}\quad b_{1,m}^{-}\ (m=0,1,\ldots,k-1),$$

then Eq. (2.1) has roots on the imaginary axis.

Subcase (I-a): $2k\pi/\tau \le a < (4k+1)\pi/(2\tau)$ $(k = 0, 1, 2, \ldots)$. The number $b^-_{0,k}$ is the positive minimum value of $b^+_{0,m}$, $b^-_{0,m}$ and $b^-_{1,m}$ because

$$b^+_{0,0} - b^-_{0,k} = \frac{\pi}{2\tau} + a - \left\{ \frac{(4k+1)\pi}{2\tau} - a \right\} = -\frac{2k\pi}{\tau} + 2a > 0,$$
$$b^-_{1,k-1} - b^-_{0,k} = -\frac{(4k-1)\pi}{2\tau} + a - \left\{ \frac{(4k+1)\pi}{2\tau} - a \right\} = -\frac{4k\pi}{\tau} + 2a \ge 0.$$

This, together with Lemma 2.3, implies $\nu(b) = 0$ if $b \in (0, b^-_{0,k})$; $\nu(b) \ge 1$ if $b \in (b^-_{0,k}, \infty)$.

Subcase (I-b): $(4k-1)\pi/(2\tau) < a < 2k\pi/\tau$ $(k = 1, 2, \ldots)$. The number $b^-_{1,k-1}$ is the positive minimum value of $b^+_{0,m}$, $b^-_{0,m}$ and $b^-_{1,m}$ because

$$b^+_{0,0} - b^-_{1,k-1} = \frac{\pi}{2\tau} + a - \left\{ -\frac{(4k-1)\pi}{2\tau} + a \right\} = \frac{2k\pi}{\tau} > 0,$$
$$b^-_{0,k} - b^-_{1,k-1} = \frac{(4k+1)\pi}{2\tau} - a - \left\{ -\frac{(4k-1)\pi}{2\tau} + a \right\} = \frac{4k\pi}{\tau} - 2a > 0.$$

This, together with Lemma 2.3, yields $\nu(b) = 0$ if $b \in (0, b^-_{1,k-1})$; $\nu(b) \ge 1$ if $b \in (b^-_{1,k-1}, \infty)$.

Case (II): $a \in J_k$ or $a = (2k+1)\pi/(2\tau)$ $(k = 0, 1, 2, \ldots)$. Lemma 2.2-(ii) or 2.2-(iii) shows $\nu(b) = 1$ for $b > 0$ sufficiently small, which, together with Lemma 2.3, implies $\nu(b) \ge 1$ if $b \in (0, \infty)$.

By virtue of the preceding argument and Lemma 2.1, we therefore conclude that under $a > 0$ and $b > 0$, all the roots of Eq. (2.1) have negative real parts if and only if either the condition (2.4) or (2.5) holds. This completes the proof. □

The following result is analogous to Proposition 2.2. The proof is carried out in a similar way to that of Proposition 2.2 and will be omitted.

Proposition 2.3. *Let $a > 0$ and $b < 0$. Then all the roots of Eq.* (2.1) *have negative real parts if and only if either*

$$\frac{(4k+1)\pi}{2\tau} < a < \frac{(2k+1)\pi}{\tau} \quad \textit{and} \quad \frac{(4k+1)\pi}{2\tau} - a < b < 0 \quad (k = 0, 1, 2, \ldots) \tag{2.6}$$

or

$$\frac{(2k+1)\pi}{\tau} \le a < \frac{(4k+3)\pi}{2\tau} \quad \textit{and} \quad -\frac{(4k+3)\pi}{2\tau} + a < b < 0 \quad (k = 0, 1, 2, \ldots). \tag{2.7}$$

Finally, we will prove our main theorem.

Proof of Theorem 1.1. In case $a = 0$, Eq. (1.6) with $|\theta| = \pi/2$ is reduced to

$$x'(t) = -bx(t-\tau) \tag{2.8}$$

and the condition (1.7) becomes $0 < b < \pi/(2\tau)$, which coincides with the classical stability criterion for Eq. (2.8). In case $|a| > 0$, we may assume $a > 0$. Then it is easily seen that the condition (1.7) is equivalent to $b > 0$ and either the condition (2.4) or (2.5) as well as the condition (1.8) is equivalent to $b < 0$ and either the condition (2.6) or (2.7). These facts and Proposition 2.1 imply Theorem 1.1. □

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