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Preservation of bifurcations under Runge-Kutta methods

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Abstract. We prove that fold, cusp and Bogdanov-Takens bifurcations of N-dimensional, continuous-time systems persist under Runge-Kutta methods. Compact formulae for the computation of the discretized normal form coefficients and critical generalized eigenvectors are derived.

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1. Introduction

Consider a continuous-time dynamical system depending on parameters

$$\dot{x}(t) = f(x(t), \alpha), \tag{1.1}$$

where $f \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$ with open sets $0 \in \Omega \subset \mathbb{R}^N$, $0 \in \Lambda \subset \mathbb{R}^p$, $k \geq 1$ sufficiently large, $N \geq 1$, $p = 1$ or 2. A common task in mathematical analysis is to understand the dynamics generated by the vector field (1.1). To accomplish this, we can appeal to one-step methods, which approximate the evolution operator by a discrete-time system (at previously fixed step-size)

$$x \mapsto g(x, \alpha), \tag{1.2}$$

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with $g \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$. This means that the dynamics of the original system (1.1) is explored in terms of the dynamics of the discrete system (1.2). Therefore, it is important to determine how "well" a one-step method describes the evolution generated by a vector field, and furthermore how the dynamics of a vector field is represented by a discretization method, under variation of the parameters.

In this article we assume that system (1.1) undergoes one of the following singularities: fold, cusp or Bogdanov-Takens bifurcations. Furthermore, we consider a quite general type of one-step methods, i.e. Runge-Kutta methods. From this setting two important questions arise.

The first one can be formulated as follows: suppose we are given a continuous-time system that undergoes one of the listed singularities, then, we are interested to know whether Runge-Kutta methods reproduce this singularity, and if the singularity is reproduced by the one-step method, we may also ask whether the singularity is shifted by the method.

The second main question has to do with normal form analysis, more precisely, we want to know—provided that the underlying bifurcation persists under the one-step method—how the normal form coefficients and the generalized, critical eigenvectors of the one-step method are related to their continuous counterparts. Our results will provide compact formulae that relate these objects, which are also suitable for numerical implementations. Moreover, as a by-product of our analysis, we will be able to make conclusions about the local behavior of the center manifolds. Since critical eigenvectors span the tangent space of the center manifolds at the bifurcation, we will determine how the center manifold of the continuous system and its discretized counterpart intersect at the singularity.

In this direction we can find several contributions. Discretization of systems with e.g. Hopf points has been addressed to a large extent (cf. [4, 11]). It has been proven that Hopf points are $O(h^p)$-shifted and turned into Neimark-Sacker points by general one-step methods of order $p \geq 1$. Moreover, elementary bifurcations have been also considered, see e.g. [11] for fold, pitchfork and transcritical bifurcations, and also [8] for an analysis of these cases with respect to conjugacies. The present article summarizes the analysis of bifurcating dynamical systems under Runge-Kutta methods that appears in [8, 9].

It is worth pointing out that in the present work we treat all the codimension one and two bifurcations that can be preserved by one-step methods without shifting the singularity. The remaining singularities involve eigenvalues on the imaginary axis, which are, in general, turned into eigenvalues on the unit circle by one-step methods. The singularity is shifted in these cases (see [10] for fold-Hopf bifurcations).

2. Basic setup

Consider a one-step discretization method applied to (1.1) of the form

$$x \mapsto \psi^h(x, \alpha) := x + h\Phi(h, x, \alpha), \tag{2.1}$$

with step-size $h > 0$ and $\psi, \Phi : \mathbb{R}^+ \times \Omega \times \Lambda \to \mathbb{R}^N$ sufficiently smooth. The one-step method (2.1) is referred to as an *s-stage Runge-Kutta method*, $s \geq 1$, if

$$\Phi(h, x, \alpha) := \sum_{i=1}^{s} \gamma_i k_i(h, x, \alpha), \tag{2.2}$$

where the function $(k_i)_{i=1,\ldots,s}$ is a solution of the system

$$k_i(h, x, \alpha) = f(W_i(h, x, \alpha), \alpha), \quad i = 1, \ldots, s, \tag{2.3}$$

with

$$W_i(h, x, \alpha) := x + h \sum_{j=1}^{s} \tau_{ij} k_j(h, x, \alpha), \quad i = 1, \ldots, s, \tag{2.4}$$

and $\gamma_i, \tau_{ij},\ i, j = 1, \ldots, s$ are given real constants that determine the order of the method. This scheme may represent implicit methods, but also explicit (ERK), diagonal implicit (DIRK), and singly diagonal implicit (SDIRK) Runge-Kutta methods, depending on the values of τ_{ij} (cf. [3]), so the results we present here are valid for all these methods. We assume that

$$\sum_{i=1}^{s} \gamma_i = 1, \tag{2.5}$$

which is a necessary condition for the method to be of order at least one. (We remark however that the weaker assumption $\sum_{i=1}^{s} \gamma_i \neq 0$ would also be sufficient in all the proofs below.) In what follows, we suppose that system (1.2) is obtained by fixing a sufficiently small $h > 0$ in system (2.1).

Let us denote by λ_i, μ_i, $i = 1, \ldots, N$ the eigenvalues of f_x^0, g_x^0, respectively, where subscript $_x$ denotes partial differentiation, further, superscript ^{0} denotes evaluation of functions at the origin $(x, \alpha) = (0, 0)$. Let us denote by $B_f(\cdot,\cdot)$, $B_g(\cdot,\cdot)$, $C_f(\cdot,\cdot,\cdot)$ and $C_g(\cdot,\cdot,\cdot)$ the multilinear forms given by

$$\begin{array}{rclrcl} B_f(v, w) & := & f_{xx}^0[v, w], & B_g(v, w) & := & g_{xx}^0[v, w], \\ C_f(v, w, z) & := & f_{xxx}^0[v, w, z], & C_g(v, w, z) & := & g_{xxx}^0[v, w, z], \end{array}$$

where $v, w, z \in \mathbb{R}^N$, and $f_{xx}^0[v, w] := \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 f(0,0)}{\partial x_j \partial x_i} v_i w_j$, and so on. As usual, I_N will denote the $N \times N$ identity matrix, superscript ^{T} the transpose and null the nullspace of a matrix. Where general indices, partial derivatives and evaluations occur together, we will use symbols such as k_{ix}^0 and f_x^{0T}, to be understood, of course, as $((k_i)_x)^0$ and $((f_x)^0)^T$.

The singularities we will work with in this article are listed below (cf. [5]).

Fold bifurcation (continuous case). The matrix f_x^0 has a simple eigenvalue $\lambda_1 = 0$ and no other critical eigenvalues. Then, there exist vectors $v_0, p_0 \in \mathbb{R}^N$ such that

$$f_x^0 v_0 = 0, \qquad f_x^{0T} p_0 = 0,$$

with $v_0^T p_0 = 1$. If the coefficient

$$a := \frac{1}{2} p_0^T B_f(v_0, v_0) \neq 0,$$

then the bifurcation is called nondegenerate, and furthermore the restriction of (1.1) (at $\alpha = 0$) to the corresponding center manifold is locally equivalent to

$$\dot{w} = aw^2 + O(w^3).$$

If, additionally, $p_0^T f_\alpha^0 \neq 0$, then the fold bifurcation is called generic.

Cusp bifurcation (continuous case). The matrix f_x^0 has a simple eigenvalue $\lambda_1 = 0$ and no other critical eigenvalues, and the coefficient

$$a := \frac{1}{2} p_0^T B_f(v_0, v_0)$$

vanishes, where $v_0, p_0 \in \mathbb{R}^N$ are defined as in the fold case. If the coefficient

$$c := \frac{1}{6} p_0^T (C_f(v_0, v_0, v_0) + 3B_f(v_0, q)) \neq 0,$$

then the bifurcation is called nondegenerate. The vector $q \in \mathbb{R}^N$ is any solution of the singular system $f_x^0 q = -B_f(v_0, v_0)$. For numerical purposes, q can be computed from the following nonsingular, bordered system

$$\begin{pmatrix} f_x^0 & v_0 \\ p_0^T & 0 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} -B_f(v_0, v_0) \\ 0 \end{pmatrix}. \tag{2.6}$$

The restriction of (1.1) (at $\alpha = 0$) to the corresponding center manifold is locally equivalent to

$$\dot{w} = cw^3 + O(w^4).$$

Bogdanov-Takens (continuous case). The matrix f_x^0 has a double, defective eigenvalue $\lambda_{1,2} = 0$ and no other critical eigenvalues. Then, there exist vectors $v_0, p_0, v_1, p_1 \in \mathbb{R}^N$ such that

$$\begin{array}{rclcrcl} f_x^0 v_0 & = & 0, & \qquad & f_x^0 v_1 & = & v_0, \\ f_x^{0T} p_0 & = & 0, & \qquad & f_x^{0T} p_1 & = & p_0, \end{array} \tag{2.7}$$

with $v_0^T p_1 = v_1^T p_0 = 1$ and $v_0^T p_0 = v_1^T p_1 = 0$ (biorthogonality). If the coefficients

$$\begin{array}{rcl} a & := & \frac{1}{2} p_0^T B_f(v_0, v_0) \neq 0, \\ b & := & p_1^T B_f(v_0, v_0) + p_0^T B_f(v_0, v_1) \neq 0, \end{array}$$

then the bifurcation is called nondegenerate, and furthermore the restriction of (1.1) (at $\alpha = 0$) to the corresponding center manifold is locally equivalent to

$$\begin{cases} \dot{w}_1 = w_2, \\ \dot{w}_2 = aw_1^2 + bw_1 w_2 + O(||w||^3). \end{cases}$$

Fold bifurcation (discrete case). The matrix g_x^0 has a simple eigenvalue $\mu_1 = 1$ and no other critical eigenvalues. Then, there exist vectors $\tilde{v}_0, \tilde{p}_0 \in \mathbb{R}^N$ such that

$$(g_x^0 - I_N)\tilde{v}_0 = 0, \qquad (g_x^0 - I_N)^T \tilde{p}_0 = 0,$$

with $\tilde{v}_0^T \tilde{p}_0 = 1$. If the coefficient

$$\tilde{a} := \frac{1}{2}\tilde{p}_0^T B_g(\tilde{v}_0, \tilde{v}_0) \neq 0,$$

then the bifurcation is called nondegenerate, and furthermore the restriction of (1.2) (at $\alpha = 0$) to the corresponding center manifold is locally equivalent to

$$w \mapsto w + \tilde{a}w^2 + O(w^3).$$

If, additionally, $\tilde{p}_0^T g_\alpha^0 \neq 0$, then the fold bifurcation is called generic.

Cusp bifurcation (discrete case). The matrix g_x^0 has a simple eigenvalue $\mu_1 = 1$ and no other critical eigenvalues, and the coefficient

$$\tilde{a} := \frac{1}{2}\tilde{p}_0^T B_g(\tilde{v}_0, \tilde{v}_0)$$

vanishes, where $\tilde{v}_0, \tilde{p}_0 \in \mathbb{R}^N$ are defined as in the discrete fold case. If the coefficient

$$\tilde{c} := \frac{1}{6}\tilde{p}_0^T (C_g(\tilde{v}_0, \tilde{v}_0, \tilde{v}_0) + 3B_g(\tilde{v}_0, \tilde{q})) \neq 0,$$

then the bifurcation is called nondegenerate. The vector $\tilde{q} \in \mathbb{R}^N$ is any solution of the singular system $(g_x^0 - I_N)\tilde{q} = -B_g(\tilde{v}_0, \tilde{v}_0)$. As in the continuous case, $\tilde{q}$ can be computed from the following nonsingular, bordered system

$$\begin{pmatrix} g_x^0 - I_N & \tilde{v}_0 \\ \tilde{p}_0^T & 0 \end{pmatrix} \begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix} = \begin{pmatrix} -B_g(\tilde{v}_0, \tilde{v}_0) \\ 0 \end{pmatrix}. \tag{2.8}$$

The restriction of (1.2) (at $\alpha = 0$) to the corresponding center manifold is locally equivalent to

$$w \mapsto w + \tilde{c}w^3 + O(w^4).$$

1 : 1 resonance (discrete case). The matrix g_x^0 has a double, defective eigenvalue $\mu_{1,2} = 1$ and no other critical eigenvalues. Then, there exist vectors $\tilde{v}_0, \tilde{p}_0, \tilde{v}_1, \tilde{p}_1 \in \mathbb{R}^N$ such that

$$\begin{aligned} (g_x^0 - I_N)\tilde{v}_0 &= 0, & (g_x^0 - I_N)\tilde{v}_1 &= \tilde{v}_0, \\ (g_x^0 - I_N)^T\tilde{p}_0 &= 0, & (g_x^0 - I_N)^T\tilde{p}_1 &= \tilde{p}_0, \end{aligned} \tag{2.9}$$

with $\tilde{v}_0^T\tilde{p}_1 = \tilde{v}_1^T\tilde{p}_0 = 1$ and $\tilde{v}_0^T\tilde{p}_0 = \tilde{v}_1^T\tilde{p}_1 = 0$. If the coefficients

$$\begin{aligned}\tilde{a} &:= \frac{1}{2}\tilde{p}_0^T B_g(\tilde{v}_0,\tilde{v}_0) \neq 0,\\ \tilde{b} &:= \tilde{p}_1^T B_g(\tilde{v}_0,\tilde{v}_0) + \tilde{p}_0^T B_g(\tilde{v}_0,\tilde{v}_1) \neq 0,\end{aligned}$$

then the bifurcation is called nondegenerate, and furthermore the restriction of (1.2) (at $\alpha = 0$) to the corresponding center manifold is locally equivalent to

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} w_1 + w_2 \\ w_2 + \tilde{a}w_1^2 + \tilde{b}w_1w_2 \end{pmatrix} + O(||w||^3).$$

3. The fold bifurcation

First we formulate some lemmata which will be used in the rest of the article.

Lemma 3.1. *Let system* (1.1) *undergo a fold ($p = 1$), cusp ($p = 2$) or a Bogdanov-Takens ($p = 2$) bifurcation at the origin $(x,\alpha) = (0,0)$. Consider a general s-stage Runge-Kutta method with step-size $h > 0$, given by* (2.1)–(2.5)*. Then there exists a positive constant ρ_1 such that the origin is an equilibrium of* (2.1) *for all $h \in (0,\rho_1)$.*

Proof. Let $L > 0$ be a local Lipschitz constant of f. Then, by [3, Theorem 7.2], it follows that system (2.3) has an unique, smooth solution $(k_i)_{i=1,\ldots,s}$ defined in some small neighborhood of the origin, provided that $0 < h < \rho_1$ with $\rho_1 := \min\left(\left(L\max_{i=1,\ldots,s}\sum_{j=1}^s|\tau_{ij}|\right)^{-1}, 1\right)$. In particular, we have that for $(x,\alpha) = (0,0)$ system (2.3) has the solution $k_i^0(h) := k_i(h,0,0) = 0$, $i = 1,\ldots,s$, thereby obtaining for all $h \in (0,\rho_1)$ that $\psi^0(h) := \psi^h(0,0) = h\sum_{i=1}^s \gamma_i k_i^0(h) = 0$. □

Lemma 3.2. *Let assumptions of Lemma 3.1 be fulfilled. Then, there exists a positive constant ρ_2 such that* $\mathrm{null}(f_x^0) = \mathrm{null}(\psi_x^0(h) - I_N)$ *for all $h \in (0,\rho_2)$.*

Proof. Choose any $0 \neq v \in \mathrm{null}(f_x^0)$. We will first show that $v \in \mathrm{null}(k_{ix}^0(h))$, $i = 1,\ldots,s$, and for all h in some interval. Consider $h \in (0,\rho_1)$, ρ_1 given by Lemma 3.1. Then, by differentiating (2.3) with respect to x, we obtain that

$$k_{ix}^0(h) = f_x^0\left(I_N + h\sum_{j=1}^s \tau_{ij}k_{jx}^0(h)\right), \quad i = 1,\ldots,s. \tag{3.1}$$

Define $z_i(h) := k_{ix}^0(h)v$, $i = 1,\ldots,s$. By multiplying both sides of (3.1) by v, we obtain the following system

$$z_i(h) - h\sum_{j=1}^s \tau_{ij}f_x^0 z_j(h) = 0, \quad i = 1,\ldots,s,$$

which can be represented by the matrix equation

$$(I_{sN} - h\tau \otimes f_x^0) \begin{pmatrix} z_1(h) \\ \vdots \\ z_s(h) \end{pmatrix} = 0 \in \mathbb{R}^{sN},$$

where $\otimes$ stands for the Kronecker product of matrices (see [7]) and $\tau := (\tau_{ij})_{i,j=1,\dots,s} \in \mathbb{R}^{s,s}$. For all $h \in (0, \rho_2')$, $\rho_2' := \min\left(\rho_1, \|\tau \otimes f_x^0\|^{-1}\right)$, the Banach Lemma (cf. [7]) guarantees the invertibility of $I_{sN} - h\tau \otimes f_x^0$, therefore we have that $z_i(h) = k_{ix}^0(h)v = 0$, so $v \in \mathrm{null}(k_{ix}^0(h))$, $i = 1, \dots, s$, and for all $h \in (0, \rho_2')$. This allows us to conclude that $v \in \mathrm{null}(\psi_x^0(h) - I_N)$, because

$$(\psi_x^0(h) - I_N)v = \left(I_N + h\sum_{i=1}^{s} \gamma_i k_{ix}^0(h) - I_N\right)v = h\sum_{i=1}^{s} \gamma_i k_{ix}^0(h)v = 0.$$

Conversely, for any $h \in (0, \rho_2')$ choose an arbitrary $0 \neq w \in \mathrm{null}(\psi_x^0(h) - I_N)$. We will show that $w \in \mathrm{null}(f_x^0)$. By (3.1) and (2.5), we can write $\psi_x^0(h) - I_N$ as

$$\psi_x^0(h) - I_N = hf_x^0 A(h), \tag{3.2}$$

where $A(h) := I_N + h\sum_{i=1}^{s}\sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h)$. Since $w \in \mathrm{null}(\psi_x^0(h) - I_N)$, we have that $hf_x^0 A(h)w = 0$, so $A(h)w \in \mathrm{null}(f_x^0) \subseteq \mathrm{null}(k_{ix}^0(h))$, $i = 1, \dots, s$. This implies that

$$A(h)A(h)w = A(h)w + h\sum_{i=1}^{s}\sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h)A(h)w = A(h)w. \tag{3.3}$$

Now set $\rho_2 := \min\left(\rho_2', \left(\sup_{h\in(0,\rho_2')} \|\sum_{i=1}^{s}\sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h)\|\right)^{-1}\right)$ and take $h \in (0, \rho_2)$, so the Banach Lemma ensures the invertibility of $A(h)$, thus from (3.3) we can deduce that $A(h)w = w$, which implies that $w \in \mathrm{null}(f_x^0)$, and hence $\mathrm{null}(f_x^0) = \mathrm{null}(\psi_x^0(h) - I_N)$. □

Lemma 3.3. *Let assumptions of Lemma 3.1 be fulfilled. Then for all* $h \in (0, \rho_2)$ $\mathrm{null}(f_x^{0T}) = \mathrm{null}((\psi_x^0(h) - I_N)^T)$.

Proof. Take any $h \in (0, \rho_2)$. Choose any $0 \neq v \in \mathrm{null}(f_x^{0T})$, then it follows by (3.2) that $v^T(\psi_x^0(h) - I_N) = hv^T f_x^0 A(h) = 0$, so $v \in \mathrm{null}((\psi_x^0(h) - I_N)^T)$. Conversely, take $0 \neq w \in \mathrm{null}((\psi_x^0(h) - I_N)^T)$, hence

$$w^T f_x^0 = w^T(hf_x^0 A(h))(hA(h))^{-1} = w^T(\psi_x^0(h) - I_N)(hA(h))^{-1} = 0,$$

thereby $w \in \mathrm{null}(f_x^{0T})$. □

Lemma 3.4. *Let assumptions of Lemma 3.1 be fulfilled. Then for all $h \in (0, \rho_2)$*

$$(\kappa p_0)^T B_\psi(v_0, v_0) = h\kappa p_0^T B_f(v_0, v_0),$$

where κ is a nonzero constant.

Proof. First we compute $k_{jxx}(h, x, \alpha)[v, w]$, for $v, w \in \mathbb{R}^N$, $h \in (0, \rho_2)$, (x, α) in a small neighborhood of the origin and $j = 1, \ldots, s$. We obtain

$$\begin{aligned} k_{jxx}(h, x, \alpha)[v, w] &= (f(W_j(h, x, \alpha), \alpha))_{xx}[v, w] \\ &= f_{xx}(W_j(h, x, \alpha), \alpha)[W_{jx}(h, x, \alpha)v, W_{jx}(h, x, \alpha)w] \\ &\quad + f_x(W_j(h, x, \alpha), \alpha)W_{jxx}(h, x, \alpha)[v, w]. \end{aligned} \tag{3.4}$$

By evaluating the above expression at $(x, \alpha) = (0, 0)$, $v = w = v_0$, using Lemma 3.1 and recalling from the proof of Lemma 3.2 that $v_0 \in \text{null}(k_{jx}^0(h))$, $j = 1, \ldots, s$, we arrive at

$$k_{jxx}^0(h)[v_0, v_0] = f_{xx}^0[v_0, v_0] + f_x^0 W_{jxx}^0(h)[v_0, v_0]. \tag{3.5}$$

Finally, (3.5) and (2.5) yield that

$$(\kappa p_0)^T \psi_{xx}^0(h)[v_0, v_0] = h\kappa p_0^T \sum_{i=1}^{s} \gamma_i (f_{xx}^0[v_0, v_0] + f_x^0 W_{ixx}^0(h)[v_0, v_0]) = h\kappa p_0^T B_f(v_0, v_0).$$

□

Theorem 3.1. *Let system* (1.1) *(with $p = 1$) undergo a nondegenerate, generic fold bifurcation at the origin. Consider a general s-stage Runge-Kutta method with stepsize $h > 0$, given by* (2.1)–(2.5). *Then there exists a positive constant ρ such that* (2.1) *has a nondegenerate, generic fold bifurcation at the origin for all $h \in (0, \rho)$.*

Proof. Set $\rho := \min(\rho_1, \rho_2)$. Then Lemma 3.1 proves that the origin is an equilibrium of (2.1), further, Lemma 3.2 and Lemma 3.3 imply that $\tilde{v}_0(h) = v_0$ and $\tilde{p}_0(h) = p_0$ are appropriate choices in the definition of the discrete fold bifurcation. This also means that $\psi_x^0(h)$ has an eigenvalue equal to 1, with geometric multiplicity 1, for all $h \in (0, \rho)$. Consequently, there exists only one Jordan block associated to this eigenvalue. In order to check that the eigenvalue is simple, we show that there does not exist any generalized eigenvectors.

Indeed, suppose to the contrary that for some $h \in (0, \rho)$, $(\psi_x^0(h) - I_N)\tilde{w}(h) = v_0$ holds with some $\tilde{w}(h) \in \mathbb{R}^N$. Then by (3.2) we would get

$$1 = p_0^T v_0 = p_0^T (\psi_x^0(h) - I_N)\tilde{w}(h) = p_0^T h f_x^0 A(h)\tilde{w}(h) = h p_0^T f_x^0 A(h)\tilde{w}(h) = 0,$$

due to the definition of p_0.

Lemma 3.4 with $\kappa = 1$ shows that $\tilde{a}(h) = ha \neq 0$ by definition, thus the discrete fold bifurcation is also nondegenerate.

Finally, to show genericity, observe that

$$\tilde{p}_0^T \psi_\alpha^0(h) = p_0^T h \sum_{i=1}^{s} \gamma_i \left(f_x^0 \left(h \sum_{j=1}^{s} \tau_{ij} k_{j\alpha}^0(h) \right) + f_\alpha^0 \right) = h p_0^T f_\alpha^0 \neq 0,$$

by using (2.5) and the genericity of the fold bifurcation of the continuous system. □

It is seen from the analysis above that the one-dimensional center manifolds of systems (1.1) and (2.1) intersect tangentially at the origin, further the critical coefficients and eigenvectors are related by

$$\tilde{v}_0(h) = v_0, \qquad \tilde{p}_0(h) = p_0,$$

and

$$\tilde{a}(h) = ha \neq 0.$$

4. The cusp bifurcation

The following lemma will also be useful in the Bogdanov-Takens case.

Lemma 4.1. *Let assumptions of Lemma 3.1 be fulfilled. Then for all $h \in (0, \rho_2)$*

(i) $p_0^T \sum_{i=1}^{s} \gamma_i W_{ixx}^0(h)[v_0, v_0] = h\omega p_0^T B_f(v_0, v_0)$, *where* $\omega := \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij}$,

(ii) $p_0^T k_{ix}^0(h) = 0$, *for all* $i = 1, \ldots, s$.

Proof. By using (3.4), (3.5) and (2.5), we get that

$$\begin{aligned} p_0^T \sum_{i=1}^{s} \gamma_i W_{ixx}^0(h)[v_0, v_0] &= h p_0^T \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij} (f_{xx}^0[v_0, v_0] + f_x^0 W_{jxx}^0(h)[v_0, v_0]) \\ &= h\omega p_0^T f_{xx}^0[v_0, v_0]. \end{aligned}$$

To prove (ii), simply notice that that by (3.1)

$$p_0^T k_{ix}^0(h) = p_0^T f_x^0 \left(I_N + h \sum_{j=1}^{s} \tau_{ij} k_{jx}^0(h) \right) = 0.$$

□

Theorem 4.1. *Let system* (1.1) *(with* $p = 2$*) undergo a nondegenerate cusp bifurcation at the origin. Consider a general s-stage Runge-Kutta method with step-size* $h > 0$*, given by* (2.1)–(2.5)*. Then there exists a positive constant* ρ *such that* (2.1) *has a nondegenerate cusp bifurcation at the origin for all* $h \in (0, \rho)$*.*

Proof. The proof of Theorem 3.1 shows that $\rho := \min(\rho_1, \rho_2)$, $\tilde{v}_0(h) = v_0$ and $\tilde{p}_0(h) = p_0$ are again appropriate choices, further, that eigenvalue $\mu_1 = 1$ is simple. Now $\tilde{a}(h)$ vanishes, since this time $\tilde{a}(h) = ha = 0$.

We will show that $\tilde{c}(h) = hc$, which completes the proof, since $c \neq 0$ by definition.

For 0 is a simple eigenvalue of both f_x^0 and $(\psi_x^0(h) - I_N)$, further, v_0 and p_0 have been fixed, we know (see [5]) that the bordered matrices in (2.6) and (2.8) are nonsingular, hence q and $\tilde{q}(h)$ are uniquely determined. It is also true that $r = \tilde{r} = 0 \in \mathbb{R}$. First we claim that q and $\tilde{q}(h)$ are related by

$$q = \tilde{q}(h) + \sum_{i=1}^{s} \gamma_i W_{ixx}^0(h)[v_0, v_0] + h \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h) \tilde{q}(h). \tag{4.1}$$

Due to uniqueness, (4.1) is verified if this q and $r = 0$ solves (2.6). Indeed, by (3.2), (2.8), (3.5) and (2.5) we have that

$$f_x^0 q + r v_0 = f_x^0 \tilde{q}(h) + \sum_{i=1}^{s} \gamma_i f_x^0 W_{ixx}^0(h)[v_0, v_0] + h f_x^0 \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h) \tilde{q}(h) =$$

$$f_x^0 A(h) \tilde{q}(h) + \sum_{i=1}^{s} \gamma_i f_x^0 W_{ixx}^0(h)[v_0, v_0] =$$

$$\frac{1}{h} \left(\left(\psi_x^0(h) - I_N \right) \tilde{q}(h) + \tilde{r} v_0 \right) + \sum_{i=1}^{s} \gamma_i f_x^0 W_{ixx}^0(h)[v_0, v_0] =$$

$$-\frac{1}{h} B_\psi(v_0, v_0) + \sum_{i=1}^{s} \gamma_i f_x^0 W_{ixx}^0(h)[v_0, v_0] =$$

$$-\sum_{i=1}^{s} \gamma_i \left(f_{xx}^0[v_0, v_0] + f_x^0 W_{ixx}^0(h)[v_0, v_0] \right) + \sum_{i=1}^{s} \gamma_i f_x^0 W_{ixx}^0(h)[v_0, v_0] =$$

$$-f_{xx}^0[v_0, v_0] = -B_f(v_0, v_0).$$

Let us now verify the second equation of (2.6). By using (2.8) and (i) and (ii) of Lemma 4.1, we get that

$$p_0^T q = p_0^T \tilde{q}(h) + p_0^T \sum_{i=1}^{s} \gamma_i W_{ixx}^0(h)[v_0, v_0] + p_0^T h \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h) \tilde{q}(h) =$$

$$0 + h\omega p_0^T B_f(v_0, v_0) + h \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij} p_0^T k_{jx}^0(h) \tilde{q}(h) = 2h\omega a + 0 = 0,$$

because $a = 0$ in the continuous cusp case. These prove (4.1).

Let us compute now $C_\psi(v_0, v_0, v_0)$. By evaluations also used in (3.4) and (3.5), we have that

$$C_\psi(v_0, v_0, v_0) = \psi_{xxx}^0(h)[v_0, v_0, v_0] = h \sum_{i=1}^{s} \gamma_i \left(f_{xxx}^0[W_{ix}^0(h) v_0, W_{ix}^0(h) v_0, W_{ix}^0(h) v_0] + \right.$$

$$3f^0_{xx}[W^0_{ixx}(h)[v_0,v_0],W^0_{ix}(h)v_0]+f^0_xW^0_{ixxx}[v_0,v_0,v_0]\big)=$$

$$h\sum_{i=1}^{s}\gamma_i\left(f^0_{xxx}[v_0,v_0,v_0]+3f^0_{xx}[v_0,W^0_{ixx}(h)[v_0,v_0]]+f^0_xW^0_{ixxx}[v_0,v_0,v_0]\right),$$

where symmetry of the bilinear form f^0_{xx} is also used. We can now prove $\tilde{c}(h)=hc$. In fact, since $p_0^Tf^0_x=0$, we get that

$$6\tilde{c}(h)-6hc=hp_0^T\sum_{i=1}^{s}\gamma_i\Big(f^0_{xxx}[v_0,v_0,v_0]+3f^0_{xx}[v_0,W^0_{ixx}(h)[v_0,v_0]]+$$

$$f^0_xW^0_{ixxx}[v_0,v_0,v_0]+3f^0_{xx}\big[v_0,\tilde{q}(h)+h\sum_{j=1}^{s}\tau_{ij}k^0_{jx}(h)\tilde{q}(h)\big]+3f^0_xW^0_{ixx}(h)[v_0,\tilde{q}(h)]\Big)-$$

$$hp_0^Tf^0_{xxx}[v_0,v_0,v_0]-3hp_0^Tf^0_{xx}[v_0,q]=$$

$$hp_0^T\sum_{i=1}^{s}\gamma_i\Big(3f^0_{xx}[v_0,W^0_{ixx}(h)[v_0,v_0]]+3f^0_{xx}\big[v_0,\tilde{q}(h)+h\sum_{j=1}^{s}\tau_{ij}k^0_{jx}(h)\tilde{q}(h)\big]\Big)-$$

$$3hp_0^Tf^0_{xx}[v_0,q]=$$

$$3hp_0^Tf^0_{xx}\Big[v_0,\sum_{i=1}^{s}\gamma_i\Big(W^0_{ixx}(h)[v_0,v_0]+\tilde{q}(h)+h\sum_{j=1}^{s}\tau_{ij}k^0_{jx}(h)\tilde{q}(h)\Big)-q\Big]=$$

$$3hp_0^Tf^0_{xx}\Big[v_0,\tilde{q}(h)+\sum_{i=1}^{s}\gamma_i\Big(W^0_{ixx}(h)[v_0,v_0]+h\sum_{j=1}^{s}\tau_{ij}k^0_{jx}(h)\tilde{q}(h)\Big)-q\Big]=$$

$$3hp_0^Tf^0_{xx}[v_0,0]=0,$$

by (2.5) and (4.1). □

As a conclusion, we see that the one-dimensional center manifolds of systems (1.1) and (2.1) intersect tangentially at the origin, further the critical coefficients and eigenvectors are related by

$$\begin{array}{rclrcl}\tilde{v}_0(h)&=&v_0,&\tilde{p}_0(h)&=&p_0,\\ \tilde{a}(h)&=&a=0,&\tilde{c}(h)&=&hc\neq 0.\end{array}$$

5. The Bogdanov-Takens bifurcation

Theorem 5.1. *Let system* (1.1) *(with $p=2$) undergo a nondegenerate Bogdanov-Takens bifurcation at the origin. Consider a general s-stage Runge-Kutta method with step-size $h>0$, given by* (2.1)–(2.5)*. Then there exists a positive constant ρ such that* (2.1) *has a nondegenerate* $1:1$ *resonance at the origin for all $h\in(0,\rho)$.*

For the proof of the above theorem, the following two lemmata will be useful.

Lemma 5.1. *Let assumptions of Theorem 5.1 be fulfilled. Then, for every $h \in (0, \rho_2)$ the following assertions hold:*

(i) $\exists \tilde{v}_1(h) \in \mathbb{R}^N : (\psi_x^0(h) - I_N)\tilde{v}_1(h) = v_0$,

(ii) $\nexists \tilde{v}_2(h) \in \mathbb{R}^N : (\psi_x^0(h) - I_N)\tilde{v}_2(h) = \tilde{v}_1(h)$.

Proof. Let us first prove (i). Define $\tilde{v}_1(h) := \frac{1}{h}A^{-1}(h)v_1$, $h \in (0, \rho_2)$. So by (2.7) and (3.2), it follows

$$(\psi_x^0(h) - I_N)\tilde{v}_1(h) = hf_x^0 A(h)\left(\frac{1}{h}A^{-1}(h)v_1\right) = f_x^0 v_1 = v_0.$$

As for (ii), suppose that for some $h \in (0, \rho_2)$ there exists a $\tilde{v}_2(h) \in \mathbb{R}^N$ such that $(\psi_x^0(h) - I_N)\tilde{v}_2(h) = \tilde{v}_1(h)$. We will see that this assumption leads us to a contradiction. First, by (ii) of Lemma 4.1 we get that

$$p_0^T A(h) = p_0^T\left(I_N + h\sum_{i=1}^{s}\sum_{j=1}^{s}\gamma_i\tau_{ij}k_{jx}^0(h)\right) = p_0^T, \tag{5.1}$$

and by the assumed existence of $\tilde{v}_2(h)$, we can express v_1 in terms of $\tilde{v}_2(h)$ as follows

$$v_1 = hA(h)\tilde{v}_1(h) = hA(h)(hf_x^0 A(h)\tilde{v}_2(h)) = h^2 A(h)f_x^0 A(h)\tilde{v}_2(h).$$

However, by (5.1) and the biorthogonality imposed on the vectors v_0, v_1, p_0, p_1, we would obtain

$$1 = p_0^T v_1 = h^2 p_0^T A(h)f_x^0 A(h)\tilde{v}_2(h) = h^2 p_0^T f_x^0 A(h)\tilde{v}_2(h) = 0. \qquad \square$$

Lemma 5.2. *Let assumptions of Theorem 5.1 be fulfilled. Then there exists a positive constant $\rho_3 \le \rho_2$ such that for all $h \in (0, \rho_3)$ the following assertions hold:*

(i) $\frac{1}{h}p_1^T(\psi_x^0(h) - I_N) = p_0^T$,

(ii) $\sum_{i=1}^{s}\gamma_i W_{ix}^0(h)\tilde{v}_1(h) = \frac{1}{h}v_1$,

(iii) $\lim_{h\to 0^+}\sigma(h) = 0$, $\sigma(h) := v_1^T(A^{-1}(h))^T p_1$,

(iv) $2(h\omega - \sigma(h))a + b \neq 0$.

Proof. Assume $h \in (0, \rho_2)$. Let us show (i). For this purpose, we use (2.7) and (5.1) in order to obtain

$$\frac{1}{h}p_1^T(\psi_x^0(h) - I_N) = \frac{1}{h}p_1^T(hf_x^0 A(h)) = p_0^T A(h) = p_0^T.$$

Next, we show (ii):

$$\begin{aligned}\sum_{i=1}^{s}\gamma_i W_{ix}^0(h)\tilde v_1(h) &= \sum_{i=1}^{s}\gamma_i\Big(I_N+h\sum_{j=1}^{s}\tau_{ij}k_{jx}^0(h)\Big)\Big(\frac{1}{h}A^{-1}(h)v_1\Big)\\ &= \frac{1}{h}A(h)A^{-1}(h)v_1=\frac{1}{h}v_1.\end{aligned}$$

Now we take up with (iii). By the Banach Lemma, we can write $A^{-1}(h)$ as follows

$$\begin{aligned}A^{-1}(h) &= \sum_{l=0}^{\infty}(-1)^l\Big(h\sum_{i=1}^{s}\sum_{j=1}^{s}\gamma_i\tau_{ij}k_{jx}^0(h)\Big)^l\\ &= I_N+hB(h),\end{aligned}$$

where $B(h):=\sum_{l=1}^{\infty}(-1)^l(h)^{l-1}\left(\sum_{i=1}^{s}\sum_{j=1}^{s}\gamma_i\tau_{ij}k_{jx}^0(h)\right)^l$, thus $\sigma(h)$ reads

$$\sigma(h)=v_1^T(A^{-1}(h))^Tp_1=v_1^T(I_N+hB(h))^Tp_1=hv_1^TB^T(h)p_1,$$

hence (iii) follows. It is left to show (iv). By (iii), we can choose some positive ρ_3' so that $|\sigma(h)|<\frac{|b|}{6|a|}$, for all $h\in(0,\rho_3')$. Then, take $\rho_3:=\min\left(\rho_2,\rho_3',\frac{|b|}{6|a||\omega|}\right)$, thereby it holds

$$|2(h\omega-\sigma(h))a|\le 2h|a||\omega|+2|a||\sigma(h)|<\frac{|b|}{3}+\frac{|b|}{3}=\frac{2|b|}{3},$$

therefore, (iv) follows. □

Lemma 5.3. *Let assumptions of Theorem 5.1 be fulfilled. Then, for all* $h\in(0,\rho_2)$ *the vectors* $\tilde v_0(h),\tilde v_1(h),\tilde p_0(h),\tilde p_1(h)$ *satisfy the set of equations* (2.9)*, where*

$$\begin{aligned}\tilde v_0(h) &:= v_0, & \tilde v_1(h) &:= \tfrac{1}{h}A^{-1}(h)v_1,\\ \tilde p_0(h) &:= hp_0, & \tilde p_1(h) &:= p_1-\sigma(h)p_0.\end{aligned}$$

Furthermore, this set of vectors is biorthogonal.

Proof. Assume $h\in(0,\rho_2)$. That $\tilde v_0(h),\tilde v_1(h)$ satisfy the first two equations of (2.9) follows immediately from Lemma 3.2, and (i) of Lemma 5.1. As for the remaining two equations, by Lemma 3.3, it holds that

$$\tilde p_0^T(h)(\psi_x^0(h)-I_N)=hp_0^T(\psi_x^0(h)-I_N)=0.$$

Finally, by Lemma 3.3, and (i) of Lemma 5.2, it is seen

$$\tilde p_1^T(h)(\psi_x^0(h)-I_N)=h\left(\frac{1}{h}p_1^T(\psi_x^0(h)-I_N)\right)=hp_0^T=\tilde p_0^T(h).$$

It is left to show biorthogonality. By (2.9), it holds

$$\tilde v_0^T(h)\tilde p_0(h)=\tilde v_1^T(h)(\psi_x^0(h)-I_N)^T\tilde p_0(h)=0.$$

On the other hand, by the biorthogonality of v_0, v_1, p_0, p_1, we have

$$\tilde{v}_0^T(h)\tilde{p}_1(h) = v_0^T p_1 - \sigma(h) v_0^T p_0 = 1.$$

Moreover, note that

$$\tilde{v}_1^T(h)\tilde{p}_0(h) = \tilde{v}_1^T(h)(\psi_x^0(h) - I_N)^T \tilde{p}_1(h) = \tilde{v}_0^T(h)\tilde{p}_1(h) = 1.$$

Lastly, it follows that

$$\tilde{p}_1^T(h)\tilde{v}_1(h) = (p_1 - \sigma(h)p_0)^T \left(\frac{1}{h}A^{-1}(h)v_1\right) = \frac{\sigma(h)}{h}(1 - \tilde{p}_0^T(h)\tilde{v}_1(h)) = 0. \qquad \square$$

With these preliminary results, we are ready to present the proof of Theorem 5.1.
Proof. [Proof of Theorem 5.1] What we have to show is that (2.1) has a nondegenerate $1:1$ resonance at the origin for all $h \in (0, \rho)$, ρ some positive constant. Indeed, take $\rho := \min(\rho_1, \rho_2, \rho_3)$. Then, Lemma 3.1 proves that the origin is an equilibrium of (2.1). Likewise, Lemma 3.2 shows that $\psi_x^0(h)$ has an eigenvalue equal to 1, with geometric multiplicity equal to 1. This means that the only Jordan block associated to this eigenvalue is of dimension ≥ 1. Nevertheless, Lemma 5.1 tells us that there exists one, and only one generalized eigenvector corresponding to this eigenvalue. It is left to show nondegeneracy. Firstly, we have to compute the normal form coefficients, denoted by $\tilde{a}(h), \tilde{b}(h)$. For the computations, Lemma 5.3 provides us with the required eigenvectors. So, Lemma 3.4 (with $\kappa := h$) shows that $\tilde{a}(h) = h^2 a \neq 0$ for all $h \in (0, \rho)$. Next, we compute $\tilde{b}(h)$ as follows.

$$\begin{aligned}\tilde{b}(h) &= \tilde{p}_1^T(h)B_\psi(\tilde{v}_0(h), \tilde{v}_0(h)) + \tilde{p}_0^T(h)B_\psi(\tilde{v}_0(h), \tilde{v}_1(h)) \\ &= (p_1^T - \sigma(h)p_0^T)\psi_{xx}^0(h)[v_0, v_0] + hp_0^T\psi_{xx}^0(h)[v_0, \tilde{v}_1(h)].\end{aligned}$$

By (3.4) and (3.5), we obtain

$$\begin{aligned}\tilde{b}(h) &= hp_1^T f_{xx}^0[v_0, v_0] + hp_0^T \sum_{i=1}^{s} \gamma_i W_{ixx}^0(h)[v_0, v_0] - h\sigma(h)p_0^T f_{xx}^0[v_0, v_0] \\ &\quad + h^2 p_0^T f_{xx}^0\left[v_0, \sum_{i=1}^{s} \gamma_i W_{ix}^0(h)\tilde{v}_1(h)\right].\end{aligned}$$

Finally, by taking into account (i) of Lemma 4.1 and (ii) of Lemma 5.2, we arrive at

$$\begin{aligned}\tilde{b}(h) &= 2h^2\omega a - 2h\sigma(h)a + hp_1^T B_f(v_0, v_0) + hp_0^T B_f(v_0, v_1) \\ &= 2h(h\omega - \sigma(h))a + hb. \qquad (5.2)\end{aligned}$$

Lastly, it is clear that $\tilde{b}(h) \neq 0$ for all $h \in (0, \rho)$. For if we assume $\tilde{b}(h_*) = 0$ for some $h_* \in (0, \rho)$, this would imply $\tilde{b}(h_*) = 2h_*(h_*\omega - \sigma(h_*))a + h_* b = 0$. Since $h_* \neq 0$, it follows $2(h_*\omega - \sigma(h_*))a + b = 0$, which clearly contradicts (iv) of Lemma 5.2. $\square$

In short, the discretized normal form of the Runge-Kutta map is

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} w_1 + w_2 \\ w_2 + (h^2 a) w_1^2 + (2h(h\omega - \sigma(h))a + hb) w_1 w_2 \end{pmatrix} + O(||w||^3),$$

and the critical coefficients and generalized eigenvectors read

$$\begin{array}{rclrcl} \tilde{v}_0(h) & = & v_0, & \tilde{v}_1(h) & = & \frac{1}{h} A^{-1}(h) v_1, \\ \tilde{p}_0(h) & = & h p_0, & \tilde{p}_1(h) & = & p_1 - \sigma(h) p_0, \\ \tilde{a}(h) & = & h^2 a, & \tilde{b}(h) & = & 2h(h\omega - \sigma(h))a + hb. \end{array}$$

Moreover, our analysis also shows that the two-dimensional center manifolds of systems (1.1) and (2.1) intersect at the origin in a nontransversal manner (see the generalized eigenvectors above).

6. Numerical example

In this section our aim is to numerically illustrate one of the main results of this article, namely, we will see that Bogdanov-Takens points persist at the same position and that they are turned into 1 : 1 resonances under Runge-Kutta methods (cf. Theorem 5.1). For this purpose, we consider the following continuous-time, dimensionless system:

$$\begin{array}{rcl} \dot{x} & = & -\left(\dfrac{\alpha+\beta}{R}\right) x + \dfrac{\alpha}{R} y - \dfrac{C}{R} x^3 + \dfrac{D}{R}(y-x)^3 - \dfrac{E}{R} x^5 + \dfrac{F}{R}(y-x)^5, \\ \dot{y} & = & \alpha x - (\alpha + G) y - z - D(y-x)^3 - H y^3 - F(y-x)^5 - I y^5, \\ \dot{z} & = & y, \end{array} \tag{6.1}$$

with state variables $(x, y, z) \in \mathbb{R}^3$, and parameters $\alpha, \beta, C, D, E, F, G, H, I,\ R \in \mathbb{R}$, $R > 0$. This system describes the dynamics of a modified van der Pol-Duffing oscillator. A thorough analysis of this oscillator concerning both local, as well as global phenomena can be found in [1, 2]. We assume α, β to be our bifurcation parameters, and we let $C = 1,\ D = -5,\ E = 1,\ F = 1,\ G = -1.5,\ H = 1,\ I = 1,\ R = 3$ fixed. Moreover, the numerical computations will be performed with the continuation software CONTENT ([6]), and numerical data will be exported to MATLAB for further numerical manipulations.

Let us firstly find a Bogdanov-Takens point for system (6.1). We choose $(x_{ini}, y_{ini}, z_{ini}) = (0, 0.5, 0)$, $(\alpha_{ini}, \beta_{ini}) = (9.5, -8)$ as initial data for the continuation of equilibria, and we then let β freely vary. The thus obtained curve is plotted in Figure 1. With this procedure we have found three neutral saddles, two Hopf points, one fold, and one branching point, labeled by NTS, H, LP and BP, respectively. The next step is to switch to a codimension one singularity that could lead us to the Bogdanov-Takens we are looking for, that is, to switch to an NTS, H or LP point. Thus, we switch to the NTS point that lies close to LP, which is located at $(x_{NTS}, y_{NTS}, z_{NTS}) \approx (-1.0541, 0, -5.459)$, $(\alpha_{NTS}, \beta_{NTS}) \approx (9.5, -7.5247)$. Along

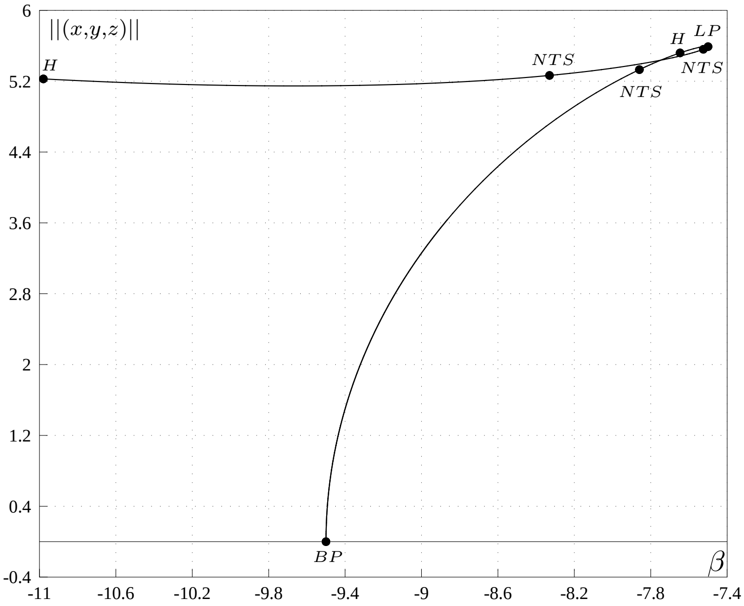

Figure 1: Continuation of equilibria of (6.1) for $\alpha = 9.5$ fixed.

this curve we find a Bogdanov-Takens point located at $(x_{BT}, y_{BT}, z_{BT}) = (-1, 0, -4.26794919243109)$, $(\alpha_{BT}, \beta_{BT}) = (8.26794919243109, -6.26794919243109)$.

The next part of the experiment is to discretize system (6.1) by a Runge-Kutta method in order to see whether the Bogdanov-Takens point found is actually preserved by the method. For this purpose we choose the 3$^{\text{rd}}$ order method of Runge (cf. [3]) with an initial step-size $h_0 = 0.13$. By using the same procedure and initial data that were used for the continuous-time system, we found a 1 : 1 resonance located at $(x_{R1}, y_{R1}, z_{R1}) = (-1, 0, -4.26794919243116)$, $(\alpha_{R1}, \beta_{R1}) = (8.26794919243116, -6.26794919243116)$. Note that this point lies very close to the Bogdanov-Takens point obtained for the continuous-time system. The next step is to investigate how the 1 : 1 resonance of the one-step method is affected as we vary the step-size. To achieve this we define the distance function

$$\begin{aligned} Dist_{BT}(h) :=&||(x_{R1}(h), y_{R1}(h), z_{R1}(h), \alpha_{R1}(h), \beta_{R1}(h)) \\ &- (x_{BT}, y_{BT}, z_{BT}, \alpha_{BT}, \beta_{BT})||, \end{aligned}$$

for $h > 0$ small, and $|| \cdot ||$ representing the Euclidean norm. The result is shown in Figure 2. In this picture we let the step-size vary from $h = 0.05$ to $h = 0.3$, but we plotted the logarithm of the variables in order to detect any evidence of an $O(h^p)$-shift of the Bogdanov-Takens point. However, no such evidence appeared but the distance remained always below the tolerance used for the computations, which allows us to

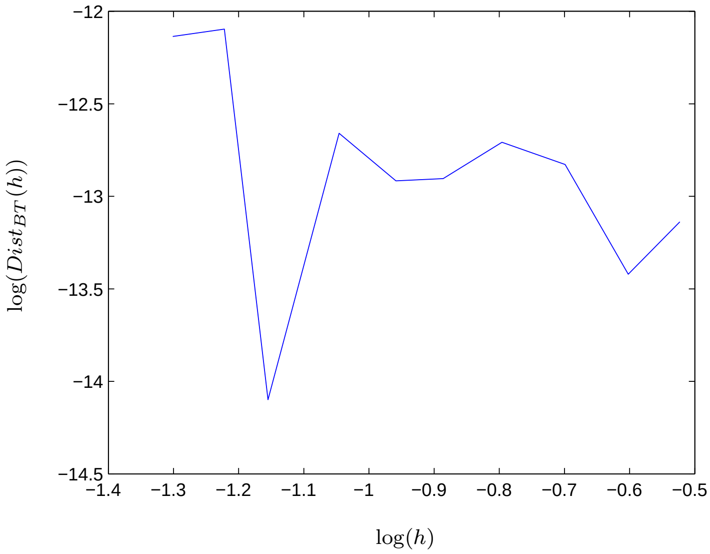

Figure 2: Distance between Bogdanov-Takens points and 1 : 1 resonances for different values of step-size.

confirm the prediction of Theorem 5.1, i.e., that the Bogdanov-Takens point persists at the same position under Runge-Kutta methods.

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