International Journal of Qualitative Theory of Differential Equations and Applications
Vol. 5, No. 1 (January-June, 2019)

On the fundamental solution of a linear delay differential equation

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Dedicated to the 65th birthday of Professor István Győri

Received: 06th January 2018 Revised : 24th February 2018 Accepted: 16th March 2018

Abstract. We consider the integral $\Psi(\alpha) = \int_0^\infty |X(t,\alpha)|\,dt$ for the fundamental solution $X(\cdot,\alpha)$ of the linear delay differential equation $\dot{x}(t) = -\alpha x(t-1)$. It is shown that $(\pi/2-\alpha)\Psi(\alpha) \to 4\sqrt{4+\pi^2}/\pi^2$ as $\alpha \to \pi/2-$.

AMS Subject Classifications: 34K06, 34K20

Keywords: Delay differential equation; Fundamental solution; Asymptotic formula.

1. Introduction

The fundamental solution appears naturally in the variation of constants formula for linear inhomogeneous delay differential equations. Explicit estimations on the integral of its absolute value are important in perturbation results. This is a nontrivial task even for the simplest linear delay differential equations.

Let $\alpha > 0$, and consider the delay differential equation

$$\dot{x}(t) = -\alpha x(t-1) \tag{1.1}$$

with parameter $\alpha > 0$. Define $X : [-1,\infty) \to \mathbb{R}$ so that

$$X(t) = \begin{cases} 0 & \text{if } -1 \le t < 0 \\ 1 & \text{if } t = 0, \end{cases}$$

and for all $t > 0$ the equation

$$X(t) = 1 - \alpha \int_{-1}^{t-1} X(s)\,ds$$

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*Supported in part by the Hungarian Scientific Research Fund, Grant No. 75517.

holds. Clearly, X is well defined by the method of steps, $X|_{[0,\infty)}$ is continuous, $X|_{(1,\infty)}$ is differentiable, and X statisfies (1.1) for all $t \in (0,1) \cup (1,\infty)$. X is called the fundamental solution of (1.1). We write $X(\cdot,\alpha)$ if we want to emphasize the dependence on the parameter $\alpha > 0$. Define the integral

$$\Psi(\alpha) = \int_0^\infty |X(t,\alpha)|\, dt.$$

For $\alpha > 1/e$, the zeros of the characteristic function $\mathbb{C} \ni z \mapsto z + \alpha e^{-z} \in \mathbb{C}$ associated with (1.1) are simple, and appear in complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_{j=0}^{\infty}$ with

$$\operatorname{Re}\lambda_0 > \operatorname{Re}\lambda_1 > \ldots > \operatorname{Re}\lambda_{j-1} > \operatorname{Re}\lambda_j \to -\infty \quad (j \to \infty),$$
$$\operatorname{Im}\lambda_j \in (2j\pi, (2j+1)\pi) \quad \text{for all } j \in \{0,1,\ldots\}.$$

If $0 < \alpha \le 1/e$ then two real zeros λ_0, λ_0^* appear instead of $\lambda_0, \overline{\lambda_0}$ with $0 > \lambda_0 \ge \lambda_0^* > \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \ldots$.

It is well known that $\Psi(\alpha) < \infty$, the zero solution of (1.1) is asymptotically stable, $\alpha < \pi/2$ and $\operatorname{Re}\lambda_0 < 0$ are equivalent statements. Explicit estimations for $\Psi(\alpha)$, in particular upper bounds are important for stability results of perturbations of (1.1), see e.g. [2,3,4,5].

The following upper bound was obtained by Győri [2] for $0 < \alpha < \pi/2$:

$$\Psi(\alpha) \le \frac{1}{\alpha}\left[1 + \left(\frac{\operatorname{Im}\lambda_0(\alpha)}{\operatorname{Re}\lambda_0(\alpha)}\right)^2\right]. \tag{1.2}$$

This inequality is sharp if $0 < \alpha \le 1/e$.

Győri and Hartung [3] proved several numerically computable upper bounds. Moreover, they obtained

$$\left(\frac{\pi}{2} - \alpha\right)\Psi(\alpha) \ge \frac{1}{\alpha} \quad \left(0 < \alpha < \frac{\pi}{2}\right). \tag{1.3}$$

In this note we investigate $\Psi(\alpha)$ near the critical value $\pi/2$. Our main result is that

$$\lim_{\alpha \to \pi/2-} \left(\frac{\pi}{2} - \alpha\right)\Psi(\alpha) = \frac{4\sqrt{4+\pi^2}}{\pi^2} \approx 2.371. \tag{1.4}$$

Remark that (1.3) implies $\liminf_{\alpha\to\pi/2-}(\pi/2-\alpha)\Psi(\alpha) \ge 2/\pi \approx 0.637$, while from (1.2) $\limsup_{\alpha\to\pi/2-}(\pi/2-\alpha)\Psi(\alpha) \le \infty$ follows.

The applied technique is suitable to obtain new explicit bounds for $\Psi(\alpha)$ for parameter values near $\pi/2$, see [6]. It can be used for equations with several delays as well.

Our estimation is based on the well known representation of the fundamental solution

$$X(t,\alpha) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma - iT}^{\gamma + iT} \frac{e^{zt}}{z + \alpha e^{-z}}\, dz + \sum_{z=\lambda_0,\overline{\lambda_0}} \operatorname{Res} \frac{e^{zt}}{z+\alpha e^{-z}} \tag{1.5}$$

which is valid for $t > 0$ and a $\gamma \in \mathbb{R}$ with $\operatorname{Re}\lambda_1 < \gamma < \operatorname{Re}\lambda_0$. Setting $\mu = \mu(\alpha) = \operatorname{Re}\lambda_0(\alpha)$ and $\nu = \nu(\alpha) = \operatorname{Im}\lambda_0(\alpha)$, we have

$$\sum_{z=\lambda_0,\overline{\lambda_0}} \operatorname{Res} \frac{e^{zt}}{z+\alpha e^{-z}} = \frac{e^{\lambda_0 t}}{1+\lambda_0} + \frac{e^{\overline{\lambda_0} t}}{1+\overline{\lambda_0}} = \frac{2e^{\mu t}}{(1+\mu)^2+\nu^2}\left[(1+\mu)\cos\nu t + \nu \sin\nu t\right].$$

Define the functions $p(\alpha), q(\alpha) : [-1,\infty) \to \mathbb{R}$ by

$$p(t,\alpha) = \frac{2e^{\mu t}}{(1+\mu)^2+\nu^2}\left[(1+\mu)\cos\nu t + \nu \sin\nu t\right]$$

and

$$q(t,\alpha) = X(t,\alpha) - p(t,\alpha).$$

We need information on $\mu(\alpha)$ and $\nu(\alpha)$.

Proposition 1.1. *If $3/2 \le \alpha < \pi/2$ then*

$$\operatorname{Re}\lambda_1(\alpha) < -1,\ -\frac{1}{2} < \mu(\alpha) < 0,\ 0 < \nu(\alpha) < \frac{\pi}{2},\ |\lambda_0| > 1.$$

Moreover,

$$\lim_{\alpha\to\pi/2-} \nu(\alpha) = \frac{\pi}{2},\ \lim_{\alpha\to\pi/2-} \frac{-\mu(\alpha)}{\frac{\pi}{2}-\alpha} = \frac{2\pi}{4+\pi^2}.$$

In the next step we estimate the integral of $\int_0^\infty |q(t,\alpha)|\,dt$. It is important that there is a uniform upper bound.

Proposition 1.2. *If $3/2 \le \alpha < \pi/2$ then*

$$\int_0^\infty |q(t,\alpha)|\,dt \le 3e^2.$$

The asymptotic behavior of $\int_0^\infty |p(t,\alpha)|\,dt$ can be obtained in an elementary way.

Proposition 1.3.

$$\lim_{\alpha\to\pi/2-} \left(\frac{\pi}{2}-\alpha\right)\int_0^\infty |p(t,\alpha)|\,dt = \frac{4\sqrt{4+\pi^2}}{\pi^2}.$$

The above propositions imply our result.

Theorem 1.1.

$$\lim_{\alpha\to\pi/2-} \left(\frac{\pi}{2}-\alpha\right)\Psi(\alpha) = \frac{4\sqrt{4+\pi^2}}{\pi^2}.$$

2. Proofs

Proof of Proposition 1.1. We show only the last statement. All the others are found in the basic monographs [1,5] or can be easily obtained from them. [6] also contains a detailed proof.

For the zero $\lambda_0 = \mu + i\nu$ of the characteristic function $z + \alpha e^{-z}$ the equations

$$\mu + \alpha e^{-\mu} \cos\nu = 0, \quad \nu - \alpha e^{-\mu} \sin\nu = 0 \tag{2.1}$$

hold. It is well known that $\nu(\alpha) \in (0, \pi/2]$ for $1/e < \alpha \leq \pi/2$, and $\nu(\pi/2) = \pi/2$, see e.g. [1, Chapter XI]. Define the functions g, h from $(0, \pi/2]$ into $\mathbb{R}$ by

$$g(s) = -s\frac{\cos s}{\sin s}, \quad h(s) = \frac{s}{\sin s} e^{g(s)}.$$

Then the solutions of (6) with $\nu \in (0, \pi/2]$ satisfy

$$\mu = g(\nu) \quad \text{and} \quad h(\nu) = \alpha.$$

Both g and h are smooth functions. Elementary calculations give

$$g'(s) = \frac{s - \sin s \cos s}{\sin^2 s} > 0 \quad \left(0 < s \leq \frac{\pi}{2}\right),$$

$$g''(s) = 2\frac{\sin s - s\cos s}{\sin^3 s} > 0 \quad \left(0 < s \leq \frac{\pi}{2}\right).$$

Thus both g and g' are strictly increasing on $(0, \pi/2]$.

For the derivative of h we have

$$\begin{aligned} h'(s) &= e^{g(s)} \frac{s^2 + \sin^2 s - 2s \sin s \cos s}{\sin^3 s} \\ &> e^{g(s)} \frac{(s - \sin s)^2}{\sin^3 s} > 0 \quad \left(0 < s \leq \frac{\pi}{2}\right), \end{aligned}$$

that is, h is strictly increasing.

It is easy to check that

$$h(1.54) < \frac{3}{2}, \quad h\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

It follows that, for $\frac{3}{2} \leq \alpha \leq \frac{\pi}{2}$, the equation $h(\nu) = \alpha$ has a unique solution $\nu(\alpha)$ such that $1.54 < \nu(\alpha) \leq \frac{\pi}{2}$, $\nu(\pi/2) = \pi/2$, and $\nu(\alpha)$ is differentiable with

$$\nu'(\alpha) = \frac{1}{h'(\nu(\alpha))}.$$

This fact and the relation $\mu = g(\nu)$ combined yield for $\frac{3}{2} \leq \alpha < \frac{\pi}{2}$ that

$$\begin{aligned} \mu(\alpha) &= g(\nu(\alpha)) = g(\nu(\alpha)) - g(\nu(\pi/2)) \\ &= g'(\nu(\hat{\alpha}))\nu'(\hat{\alpha}) \left(\alpha - \frac{\pi}{2}\right) \\ &= \frac{g'(\nu(\hat{\alpha}))}{h'(\nu(\hat{\alpha}))} \left(\alpha - \frac{\pi}{2}\right) \end{aligned}$$

for some $\hat{\alpha} \in (3/2, \pi/2)$. If $\alpha \to \pi/2-$ then $\nu(\hat{\alpha}) \to \pi/2$. Therefore

$$\frac{-\mu(\alpha)}{\frac{\pi}{2} - \alpha} = \frac{g'(\nu(\hat{\alpha}))}{h'(\nu(\hat{\alpha}))} \to \frac{g'(\pi/2)}{h'(\pi/2)} = \frac{2\pi}{4 + \pi^2}.$$

□

Proof of Proposition 1.2. By Proposition 1.1, for all $3/2 \le \alpha < \pi/2$ we can choose $\gamma = -1$ in formula (1.5). Then, for $t > 0$

$$q(t, \alpha) = \frac{e^{-t}}{2\pi} Q(t, \alpha)$$

with

$$Q(t, \alpha) = \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{i\tau t}}{-1 + i\tau + \alpha e e^{-i\tau}} \, d\tau.$$

Integration by parts gives

$$\begin{aligned}\int_{-T}^{T} \frac{e^{i\tau t}}{-1 + i\tau + \alpha e e^{-i\tau}} \, d\tau = & \left[\frac{e^{i\tau t}}{it(-1 + i\tau + \alpha e e^{-i\tau})} \right]_{\tau=-T}^{\tau=T} \\ & - \int_{-T}^{T} \frac{e^{i\tau t}(1 - \alpha e e^{-i\tau})}{t(-1 + i\tau + \alpha e e^{-i\tau})^2} \, d\tau.\end{aligned}$$

Clearly

$$\frac{e^{i\tau t}}{it(-1 + i\tau + \alpha e e^{-i\tau})} \to 0 \quad \text{as } |\tau| \to \infty.$$

Hence, for $t \ge 2$

$$\begin{aligned}|Q(t, \alpha)| &\le \left| \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{i\tau t}(1 - \alpha e e^{-i\tau})}{t(-1 + i\tau + \alpha e e^{-i\tau})^2} \, d\tau \right| \\ &\le \frac{1 + \pi e/2}{2} 2 \int_0^{\infty} \frac{d\tau}{|-1 + i\tau + \alpha e e^{-i\tau}|^2}.\end{aligned}$$

If $\tau \ge 2\pi$ then

$$|-1 + i\tau + \alpha e e^{-i\tau}| \ge |-1 + i\tau| - \frac{\pi e}{2} = \sqrt{1 + \tau^2} - \frac{\pi e}{2} > \frac{1}{4}\sqrt{1 + \tau^2}.$$

For $\tau \in [0, 2\pi]$ we have

$$\begin{aligned}|-1 + i\tau + \alpha e e^{-i\tau}|^2 &= 1 + \tau^2 + \alpha^2 e^2 - 2\alpha e(\tau \sin\tau + \cos\tau) \\ &\ge 1 + \tau^2 + \alpha^2 e^2 - 2\alpha e \max_{0 \le \tau \le 2\pi} (\tau \sin\tau + \cos\tau) \\ &= 1 + \tau^2 + \alpha^2 e^2 - 2\alpha e \frac{\pi}{2} \\ &> 1 + \tau^2.\end{aligned}$$

Therefore, for $t \geq 2$

$$|Q(t,\alpha)| \leq \Big(1+\frac{\pi e}{2}\Big)\int_0^\infty \frac{16}{1+\tau^2}\,d\tau = \Big(1+\frac{\pi e}{2}\Big)8\pi,$$

and

$$|q(t,\alpha)| \leq (4+2\pi e)e^{-t}\,.$$

For $0 \leq t \leq 2$ we use the inequality $|q(t,\alpha)| \leq |X(t,\alpha)| + |p(t,\alpha)|$. Clearly $|X(t,\alpha)| \leq 1$, $0 \leq t \leq 2$, since $X(t,\alpha) = 1$, $0 \leq t \leq 1$, and $X(t,\alpha) = -\alpha(t-1)+1$, $1 \leq t \leq 2$.

By Proposition 1.1 and the Schwarz inequality

$$|p(t,\alpha)| \leq \frac{2}{(1+\mu)^2+\nu^2}\sqrt{(1+\mu)^2+\nu^2} = \frac{2}{\sqrt{(1+\mu)^2+\nu^2}} \leq 2.$$

Thus

$$|q(t,\alpha)| \leq 3 \quad (0 \leq t \leq 2).$$

It follows that

$$|q(t,\alpha)| \leq e^{-t}\max\{3e^2, 4+2\pi e\} = 3e^2e^{-t} \quad (t \geq 0).$$

Consequently

$$\int_0^\infty |q(t,\alpha)|\,dt \leq 3e^2.$$

□

Proof of Proposition 1.3. We have

$$p(t,\alpha) = \frac{2}{\sqrt{(1+\mu)^2+\nu^2}}e^{\mu t}\sin(\nu t+\omega)$$

with $\omega = \tan^{-1}\frac{1+\mu}{\nu} \in (0,\pi/2)$. Setting $t_j = (j\pi-\omega)/\nu$ for $j \in \{1,2,\ldots\}$, $\sin(\nu t+\omega) < 0$ for some $t \geq 0$ if and only if $t \in (t_1,t_2)\cup(t_3,t_4)\cup(t_5,t_6)\cup\ldots$. Therefore

$$\begin{aligned}
\int_{t_1}^\infty |p(t,\alpha)|\,dt &= \sum_{j=1}^\infty(-1)^j\int_{t_j}^{t_{j+1}} p(t,\alpha)\,dt \\
&= \frac{2}{\nu\sqrt{(1+\mu)^2+\nu^2}}\sum_{j=1}^\infty e^{\mu t_j}\int_0^\pi e^{\mu s/\nu}\sin s\,ds \\
&= \frac{2e^{-\mu\omega/\nu}}{\nu\sqrt{(1+\mu)^2+\nu^2}}\sum_{j=1}^\infty e^{j\mu\pi/\nu}\Big[\frac{1}{1+(\mu/\nu)^2}e^{\mu s/\nu}(\frac{\mu}{\nu}\sin s-\cos s)\Big]_{s=0}^{s=\pi} \\
&= \frac{2e^{-\mu\omega/\nu}(e^{\mu\pi/\nu}+1)}{\nu\sqrt{(1+\mu)^2+\nu^2}(1+\mu^2/\nu^2)}\sum_{j=1}^\infty e^{j\mu\pi/\nu} \\
&= \frac{2e^{-\mu\omega/\nu}(e^{\mu\pi/\nu}+1)}{\nu\sqrt{(1+\mu)^2+\nu^2}(1+\mu^2/\nu^2)}\frac{e^{\mu\pi/\nu}}{1-e^{\mu\pi/\nu}}.
\end{aligned}$$

Combining the above equality multiplied by $\pi/2-\alpha$, Proposition 1.1 and $\lim_{x\to 0}(e^x - 1)/x = 1$, it follows that

$$\left(\frac{\pi}{2}-\alpha\right)\int_{t_1}^{\infty} |p(t,\alpha)|\,dt \to \frac{4\sqrt{4+\pi^2}}{\pi^2}$$

as $\alpha \to \pi/2-$. It is obvious that

$$\left(\frac{\pi}{2}-\alpha\right)\int_{0}^{t_1} |p(t,\alpha)|\,dt \to 0$$

as $\alpha \to \pi/2-$. This completes the proof. □

Proof of Theorem 1.1. From $X(t,\alpha) = p(t,\alpha) + q(t,\alpha)$ one obtains

$$\int_0^{\infty} |p(t,\alpha)|\,dt - \int_0^{\infty} |q(t,\alpha)|\,dt \le \int_0^{\infty} |X(t,\alpha)|\,dt \le \int_0^{\infty} |p(t,\alpha)|\,dt + \int_0^{\infty} |q(t,\alpha)|\,dt.$$

From Proposition 1.2

$$\left(\frac{\pi}{2}-\alpha\right)\int_{0}^{\infty} |q(t,\alpha)|\,dt \to 0 \quad (\alpha \to \frac{\pi}{2}-)$$

follows. Therefore

$$\lim_{\alpha\to\pi/2-}\left(\frac{\pi}{2}-\alpha\right)\int_{0}^{\infty} |X(t,\alpha)|\,dt = \lim_{\alpha\to\pi/2-}\left(\frac{\pi}{2}-\alpha\right)\int_{0}^{\infty} |p(t,\alpha)|\,dt,$$

and by Proposition 1.3 the proof is complete. □

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