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Successive approximation technique for investigation of solutions of some linear boundary value problems for functional-differential equations with special deviation of argument*

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Abstract. We suggest successive approximation techniques for studying two-point boundary value problems for linear differential equations with argument deviations. We refine of certain estimates related to the convergence analysis of successive approximations in the cases where argument deviations possess certain special properties.

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1. Introduction

In studies of solutions of various types of boundary value problems for ordinary and functional differential equations, it is often useful to possess appropriate techniques based upon some types of successive approximations constructed in an analytic form. To this class of methods belongs, in particular, the approach suggested in [1–3]. The

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method mentioned was at first oriented for studying only periodic solutions of nonlinear first order ordinary differential systems

$$x'(t) = f(t, x(t)), \qquad -\infty < t < \infty, \tag{1.1}$$

where $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is T-periodic in the first argument. Later, appropriate versions of the method indicated were developed for handling more general two-point nonlinear boundary value problems for systems of first or second order differential equations, integro-differential equations, equations with retarded argument, parametrized boundary value problems. We refer, e. g., to the books [4–6], the papers [7–11], and the survey [12] for the related references. According to the basic idea, the given boundary value problem is replaced by the initial value problem for a suitably modified system of functional differential or integro-functional equations containing some artificially introduced vector parameter $\xi = (\xi_i)_{i=1}^n$, whose value is to be determined later. In most cases ξ has the meaning of the initial value of the solution at a certain point. For example, in the case of the two-point boundary value problem

$$x'(t) = f(t, x(t)), \qquad t \in [0, T], \tag{1.2}$$

$$Ax(0) + Bx(T) = d, \tag{1.3}$$

where $0 < T < +\infty$, $d \in \mathbb{R}^n$, $\{A, B\} \subset \mathrm{GL}_n(\mathbb{R})$, $\det B \neq 0$, one considers [6] the integro-functional equation

$$\begin{aligned} x(t,\xi) = \xi + \int_0^t f(s, x(s,\xi))\, ds - \frac{t}{T}\int_0^T f(s, x(s,\xi))\, ds \\ + \frac{t}{T}\left[B^{-1}d - \left(B^{-1}A + I\right)\xi\right], \qquad t \in [0,T], \end{aligned} \tag{1.4}$$

containing an unknown parameter ξ. The (unique, under appropriate assumptions) solution $x^*(\cdot,\xi)$ of equation (1.4) is sought for analytically by using the modified Picard-type iteration sequence

$$\begin{aligned} x_m(t,\xi) = \xi + \int_0^t \left(f(s, x_{m-1}(s,\xi))\, ds - \frac{1}{T}\int_0^T f(r, x_{m-1}(r,\xi))\, dr \right) ds \\ + \frac{t}{T}\left[B^{-1}d - \left(B^{-1}A + I\right)\xi\right], \qquad t \in [0,T],\ m = 1, 2, \ldots, \end{aligned} \tag{1.5}$$

where $x_0(\cdot,\xi) \equiv \xi$. Each of the functions (1.5) satisfies the given boundary conditions (1.3) and is such that

$$x_m(0,\xi) = \xi \tag{1.6}$$

for arbitrary values of the vector parameter ξ. The presence of "non-Picard" perturbation terms in (1.5) brings up the necessity to solve the so-called "determining equations," which produce the numerical values of the parameter ξ corresponding to the solutions of the given boundary value problem (1.2), (1.3):

$$\int_0^T f(s, x^*(s,\xi))\, ds = B^{-1}d - \left(B^{-1}A + I\right)\xi. \tag{1.7}$$

Practical computations are performed on approximate versions of equation (1.7), where the exact solution $x^*(\cdot,\xi)$ of equation (1.4) is replaced by some of its approximations (1.5). Problems other than (1.2), (1.3) require appropriate modifications of the above scheme.

We also note that somewhat similar approach used in [13–16] for the two-point problem (1.3) for functional-differential equations is based on a different way to construct successive approximations. In this paper, we give a refinement of certain estimates related to the convergence analysis of successive approximations of form (1.5) associated with some two-point boundary value problems for a class of linear systems of functional-differential equations involving several argument deviations possessing certain special properties.

2. Problem setting

We consider the system of linear differential equations with argument deviations of the form

$$x'(t) = \sum_{j=0}^{k} P_j\,(t)\,x(\beta_j(t)) + f(t), \qquad t \in [0,T] \tag{2.1}$$

subjected to the inhomogeneous two-point boundary conditions

$$Ax\,(0) + Bx(T) = d. \tag{2.2}$$

Here, $T \in (0,+\infty)$, the elements of the matrix-valued functions $P_j : [0,T] \to \mathrm{GL}_n\,(\mathbb{R})$, $j = 0,1,\ldots,k$, are Lebesgue integrable, $f \in L_1\,([0,T]\,,\mathbb{R}^n)$, $\{A,B\} \subset \mathrm{GL}_n\,(\mathbb{R})$, and $\beta_j : [0,T] \to [0,T]$, $j = 1,\ldots,k$, are Lebesque measerable functions. Here, $\mathrm{GL}_n\,(\mathbb{R})$ is the algebra of square matrices of dimension n. The aim of this paper is to suggest a numerical-analytic scheme of type (1.5), (1.7) for the investigation of the solutions of the problem (2.1), (2.2) and improve the convergence conditions in the special cases where the argument deviations β_j, $j = 1,\ldots,k$, satisfy certain sign assumptions.

3. Notations and auxiliary statements

The following notations are used in the sequel.

1. $C\,([0,T]\,,\mathbb{R}^n)$ is the Banach space of the continuous functions $[0,T] \to \mathbb{R}^n$ with the standard uniform norm.
2. $L_1\,([0,T]\,,\mathbb{R}^n)$ is the usual Banach space of the vector functions $[0,T] \to \mathbb{R}^n$ with Lebesque integrable components.
3. $\mathrm{GL}_n\,(\mathbb{R})$ is the algebra of all the square matrices of dimension n with real elements.
4. $r\,(Q)$ is the maximal in module eigenvalue of the matrix $Q \in \mathrm{GL}_n\,(\mathbb{R})$.

5. I_n is the unit matrix of dimension n.

Let us define the sequence of functions $\{\alpha_m\}_{m=0}^{\infty} \subset C([0,T],\mathbb{R})$ by the recurrence relation

$$\alpha_m(t) = \left(1 - \frac{t}{T}\right)\int_0^t \alpha_{m-1}(s)\,ds + \frac{t}{T}\int_t^T \alpha_{m-1}(s)\,ds, \quad t \in [0,T], \tag{3.1}$$

where $m = 1, 2, \ldots$ and $\alpha_0(t) := 1$ for all $t \in [0,T]$. It is obvious that

$$\alpha_1(t) = 2t\left(1 - \frac{t}{T}\right), \qquad t \in [0,T], \tag{3.2}$$

and

$$\max_{t\in[0,T]} \alpha_1(t) = \frac{T}{2}. \tag{3.3}$$

Lemma 3.1. *For an arbitrary essentially bounded function $u : [0,T] \to \mathbb{R}$, the estimate*

$$\left|\int_0^t \left(u(\tau) - \frac{1}{T}\int_0^T u(s)\,ds\right)d\tau\right| \leq \frac{1}{2}\alpha_1(t)\Big(\operatorname*{ess\,sup}_{s\in[0,T]} u(s) - \operatorname*{ess\,inf}_{s\in[0,T]} u(s)\Big) \tag{3.4}$$

is true for all $t \in [0,T]$.

Proof. Inequality (3.4) is established similarly to [17], and the explicit proof is, therefore, omitted. □

Lemma 3.2. ([18, Lemma 1]) *Each of functions* (3.1) *is non-negative, takes zero values at the points* 0 *and T, and possesses the property*

$$\alpha_m\left(\frac{T}{2} - t\right) = \alpha_m\left(\frac{T}{2} + t\right), \qquad m \geq 1,\ t \in [0, T/2]. \tag{3.5}$$

Moreover,

$$\alpha_m'(t)\operatorname{sign}\left(t - \frac{T}{2}\right) \leq 0, \qquad t \in [0,T],\ m = 1, 2, \ldots, n. \tag{3.6}$$

Lemma 3.3. *For a measurable function $\beta : [0,T] \to [0,T]$ satisfying the condition*

$$\operatorname*{ess\,inf}_{t\in[0,T]} (\beta(t) - t)\operatorname{sign}\left(t - \frac{T}{2}\right) \geq 0, \tag{3.7}$$

the inequalities

$$\alpha_m(\beta(t)) \leq \alpha_m(t),\ m = 1, 2, \ldots,\ t \in [0,T], \tag{3.8}$$

are true, where α_m, $m = 1, 2, \ldots$, are the functions given by formula (3.1).

Proof. It follows from Lemma 3.2 that (3.6) holds for any $t \in [0, T]$ and $m \geq 1$, i. e., each α_m, $m = 1, 2, \ldots$, is non-decreasing on the interval $[0, T/2]$ and non-increasing on $[T/2, T]$. Combining (3.6) and (3.7), we obtain the required estimate (3.8). □

For example, condition (3.7) is satisfied with $T = 1$ for the continuous piecewise linear function defined on the interval $[0, 1]$ by the formula

$$\beta(t) := \begin{cases} 0.625\, t & \text{for } t \in [0, 0.4], \\ 2.5\, t - 0.75 & \text{for } t \in [0.4, 0.6], \\ 0.625\, t + 0.375 & \text{for } t \in [0.6, 1]. \end{cases}$$

Note that, in general, if $\beta : [0, T] \to [0, T]$ is a continuous function satisfying condition (3.7), then, moreover, it has the properties $\beta(0) = 0$, $\beta(\frac{T}{2}) = \frac{T}{2}$, and $\beta(T) = T$.

4. Iterative scheme and convergence analysis for the case of general type deviation functions

To study the solution of the boundary value problem (2.1), (2.2) let us introduce the sequence of functions $x_m : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $m \geq 0$, by putting

$$x_{m+1}(t, \xi) := \xi + \tilde{f}(t) + \sum_{j=0}^{k} \int_0^t P_j(s)\, x_m(\beta_j(s))\, ds - \frac{t}{T} \int_0^T \sum_{j=0}^{k} P_j(s)\, x_m(\beta_j(s))\, ds + \frac{t}{T} \left(B^{-1} d - \left(B^{-1} A + I_n \right) \xi \right), \quad (4.1)$$

where $m = 0, 1, 2, \ldots$,

$$\tilde{f}(t) := \int_0^t f(s) ds - \frac{t}{T} \int_0^T f(s) ds, \qquad t \in [0, T], \quad (4.2)$$

and $x_0(t, \xi) := \xi$ for any $t \in [0, T]$ and $\xi \in \mathbb{R}^n$. Let us establish the convergence of sequence (4.1) for arbitrary deviation functions $\beta_j : [0, T] \to [0, T]$.

Lemma 4.1. *For all $m \geq 1$ and $\xi \in \mathbb{R}^n$, the function $x_m(\cdot, \xi)$ is absolutely continuous and possesses the properties*

$$x_m(0, \xi) = \xi, \quad (4.3)$$

$$x_m(T, \xi) = B^{-1}(d - A\xi). \quad (4.4)$$

Proof. This statement is an immediate consequence of formulae (4.1). □

It follows from Proposition 4.1 that starting from $m = 1$, all the functions of sequence (4.1) satisfy the boundary condition (2.2).

Definition 4.1. *Let $G \in \mathrm{GL}_n(\mathbb{R})$ and $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. We say that the successive approximations scheme* (4.1) *is* applicable *to the boundary value problem* (3.1)*,* (3.2) *with an estimate of type (G, g) if:*

1. $r(G) < 1$;
2. $\sup_{t\in[0,T]} g(t,\xi) < +\infty$ *for all* $\xi \in \mathbb{R}^n$;
3. *For any $\xi \in \mathbb{R}^n$ there exists a continuous function $x^*(\cdot,\xi) : [0,T] \to \mathbb{R}^n$ such that the pointwise and componentwise estimates*

$$|x^*(t,\xi) - x_m(t,\xi)| \le G^m (I_n - G)^{-1} g(t,\xi), \; t \in [0,T], \; m = 1,2,\ldots, \tag{4.5}$$

are true.

Proposition 4.1. *Let for certain $G \in \mathrm{GL}_n(\mathbb{R})$ and $g : [0,T] \to [0,+\infty)$ the successive approximation method* (4.1) *be applicable to boundary value problem* (2.1)*,* (2.2) *with the estimate of the type (G,g). Then:*

1. *For any fixed $\xi \in \mathbb{R}^n$, the function $x^* : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ appearing in* (4.5) *coincides with the uniform limit of sequence* (4.1)*:*

$$\lim_{m\to\infty} \max_{t\in[0,T]} |x^*(t,\xi) - x_m(t,\xi)| = 0.$$

2. *The function $x^*(\cdot,\xi)$ for arbitrary $\xi \in \mathbb{R}^n$ satisfies the initial condition $x^*(0,\xi) = \xi$ and the boundary condition* (3.2)*;*
3. *For any $\xi \in \mathbb{R}^n$, the function $x^*(\cdot,\xi)$ is an absolutely continuous solution of the integro-functional equation*

$$x(t) = \xi + \tilde{f}(t) + \frac{t}{T}\left[B^{-1}d - \left(B^{-1}A + I_n\right)\xi\right] + \sum_{j=0}^{k} \int_0^t P_j(s)\, x(\beta_j(s))\, ds - \frac{t}{T}\sum_{j=0}^{k}\int_0^T P_j(s)\, x(\beta_j(s))\, ds, \qquad t \in [0,T]. \tag{4.6}$$

Proof. This assertion is easily obtained by taking Definition 4.1 and Lemma 4.1 into account. □

It is clear that the function $x^*(\cdot,\xi)$ in Proposition 4.1 is also a solution of the initial value problem

$$x(0) = \xi \tag{4.7}$$

for the forced functional differential equation

$$x'(t) = \sum_{j=0}^{k} P_j(t)\, x(\beta_j(t)) + f(t) + \mu_\xi(x), \qquad t \in [0,T], \tag{4.8}$$

where the linear mapping $\mu_\xi : C([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is defined by the formula

$$\mu_\xi(x) := \frac{1}{T}\left[B^{-1}d - \left(B^{-1}A + I_n\right)\xi\right] - \frac{1}{T}\int_0^T \left(\sum_{j=0}^{k} P_j\left(s\right) x\left(\beta_j\left(s\right)\right) + f\left(s\right)\right) ds \quad (4.9)$$

for any $x \in C([a,b],\mathbb{R}^n)$.

Let us establish the relation between the limit function $x^*(\cdot,\xi)$ of the sequence (4.1) and the solutions of the given two-point boundary value problem (2.1), (2.2). For this purpose, fix some $\xi \in \mathbb{R}^n$ and consider the initial value problem (4.7) for the forced equation

$$x'(t) = \sum_{j=0}^{k} P_j\left(t\right) x\left(\beta_j(t)\right) + f(t) + \mu, \qquad t \in [0,T], \quad (4.10)$$

where $\mu \in \mathbb{R}^n$ is a vector parameter.

Proposition 4.2. *Let us fix an arbitrary $\xi \in \mathbb{R}^n$ and assume that the successive approximation method* (4.1) *for boundary value problem* (2.1), (2.2) *is applicable with an estimate of type* (G,g) *for certain* $G \in \mathrm{GL}_n(\mathbb{R})$ *and* $g : [0,T] \to [0,+\infty)$. *Then a solution* $x(\cdot)$ *of the initial value problem* (4.10), (4.7) *satisfies the boundary condition* (2.2) *if, and only if*

$$\mu = \mu_\xi(x^*(\cdot,\xi)) \quad (4.11)$$

where $x^*(\cdot,\xi) : [0,T] \to \mathbb{R}^n$ *is the uniform limit of sequence* (4.1)*:*

$$x^*(\cdot,\xi) = \lim_{m\to\infty} x_m(\cdot,\xi). \quad (4.12)$$

Proof. By virtue of Proposition 4.1, for an arbitrary $\xi \in \mathbb{R}^n$, the function $x^*(\cdot,\xi)$ defined by formula (4.12) satisfies the integro-functional equation (4.6). Differentiating (4.6), we find that $x = x^*(\cdot,\xi)$ is a solution of the initial value problem (4.10), (4.7) with the value of the parameter μ given by equality (4.11). □

Proposition 4.2 implies the following statement.

Proposition 4.3. *Let the successive approximation method* (4.1) *be applicable to problem* (2.1), (2.2) *with an estimate of type* (G,g) *for certain* $G \in \mathrm{GL}_n(\mathbb{R})$ *and* $g : [0,T] \to [0,+\infty)$. *Then the limit function* $x^*(\cdot,\xi)$ *of the recurrence sequence* (4.1) *is a solution of the boundary value problem* (2.1), (2.2) *if, and only if the value of the vector parameter* $\xi \in \mathbb{R}^n$ *in* (4.1) *satisfies the system of determining equations*

$$B^{-1}d - \left(B^{-1}A + I_n\right)\xi - \int_0^T \left(\sum_{j=0}^{k} P_j\left(s\right) x^*\left(\beta_j\left(s\right),\xi\right) + f\left(s\right)\right) ds = 0. \quad (4.13)$$

Proof. It suffices to apply the Proposition 4.2 and notice that equation (4.8) coincides with (2.1).if and only if relation (4.13) holds. □

Remark 4.1. *In practice, it is convenient to fix some natural m and, instead of (4.13), consider the "approximate" determining equation*

$$B^{-1}d - \left(B^{-1}A + I_n\right)\xi - \int_0^T \left(\sum_{j=0}^{k} P_j(s)\, x_m(\beta_j(s), \xi) + f(s)\right) ds = 0. \tag{4.14}$$

If equation (4.14) has an isolated solution $\xi = \xi_m$ in a certain open domain $D \subset \mathbb{R}^n$, then under some additional assumptions one can show that the corresponding exact determining equation (4.13) is also solvable and, by virtue of Proposition 4.3, the boundary value problem (2.1)*,* (2.2) *has a solution (see, e. g., [6, Theorem 3.1] or [19, Theorem 7.1]). In this case, the function*

$$[0, T] \ni t \longmapsto X_m(t) := x_m(t, \xi_m)$$

can be regarded as the mth approximation to a solution of the boundary value problem (2.1)*,* (2.2)*.*

Theorem 4.1. *Let $P_j : [0, T] \to \mathrm{GL}_n(\mathbb{R})$, $j = 0, 1, 2, \ldots, k$, be matrix-valued functions with essentially bounded elements, the argument deviation functions $\beta_j : [0, T] \to [0, T]$, $j = 0, 1, 2, \ldots, k$, be measurable, and the matrix B in the boundary conditions* (2.2) *be non-singular. Moreover, assume that the inequality*

$$r\left(\tilde{P}_0 + \tilde{P}_1 + \cdots + \tilde{P}_k\right) < \frac{2}{T}, \tag{4.15}$$

is satisfied, where

$$\tilde{P}_j := \operatorname*{ess\,sup}_{t \in [0,T]} |P_j(t)|, \qquad j = 0, 1, 2, \ldots, k. \tag{4.16}$$

Then the successive approximation method (4.1) *is applicable to the boundary value problem* (2.1)*,* (2.2) *with an estimate of the type $\left(\frac{1}{2}T\sum_{j=0}^{k}\tilde{P}_j, \gamma\right)$, where*

$$\gamma(\xi) := \frac{1}{2}T\delta(\xi) + \left|B^{-1}d - \left(B^{-1}A + I_n\right)\xi\right| \tag{4.17}$$

and

$$\delta(\xi) := \frac{1}{2}\left(\operatorname*{ess\,sup}_{t \in [0,T]} (Q(t)\xi + f(t)) - \operatorname*{ess\,inf}_{t \in [0,T]} (Q(t)\xi + f(t))\right)$$

for all $\xi \in \mathbb{R}^n$, and $Q := \sum_{j=0}^{k} P_j$.

Here and below, the symbols ess sup, ess inf, $\leq$, $\geq$, $|\cdot|$, and similar relations for vector-valued and matrix-valued functions are understood componentwise. For the sake of convenience, we put

$$(\mathcal{J}y)(t) := \int_0^t \left(y\,(s) - \frac{1}{T}\int_0^T y(\tau)d\tau \right) ds, \qquad t \in [0,T],$$

for all $y \in L_1([0,T],\mathbb{R})$. Similarly, for a matrix-valued function $Y = [y_1,\ldots,y_n]$ with the columns $y_i \in L_1([0,T],\mathbb{R})$, $i = 1,2,\ldots,n$, we use the notation $\mathcal{J}Y := [\mathcal{J}y_1,\ldots,\mathcal{J}y_n]$.

Proof. Let us show that, under the conditions assumed, sequence (4.1) is a Cauchy sequence in the Banach space $C\left([0,T],\mathbb{R}^n\right)$ equipped with the usual uniform norm. Due to Lemma 3.1, it follows from (4.1) that for $m = 0$ and an arbitrary fixed $\xi \in \mathbb{R}^n$

$$\begin{aligned} |x_1\,(t,\xi) - \xi| &= \left|(\mathcal{J}Q)(t)\,\xi + (\mathcal{J}f)(t) + tT^{-1}\left[B^{-1}d - \left(B^{-1}A + I_n\right)\xi\right]\right| \\ &\leq \alpha_1(t)\delta\,(\xi) + \delta_1\,(\xi)\,, \end{aligned} \tag{4.18}$$

where the function α_1 is given by equality (3.2) and

$$\delta_1\,(\xi) := \left|B^{-1}d - \left(B^{-1}A + I_n\right)\xi\right|, \qquad \xi \in \mathbb{R}^n. \tag{4.19}$$

According to formulae (4.1), for all $t \in [0,T]\,, \xi \in \mathbb{R}^n$, and $m = 1,2,,\ldots,$ we can write

$$\begin{aligned} r_m(t,\xi) := x_m\,(t,\xi) - x_{m-1}\,(t,\xi) &= \left(\mathcal{J}\sum_{j=0}^{k} P_j(\cdot)\,r_{m-1}(\beta_j(\cdot),\xi)\right)(t) \\ &= \left(1 - \frac{t}{T}\right)\sum_{j=0}^{k}\int_0^t P_j\,(s)\,r_{m-1}(\beta_j(s),\xi)ds \\ &\quad - \frac{t}{T}\sum_{j=0}^{k}\int_t^T P_j\,(s)\,r_{m-1}(\beta_j(s),\xi)ds. \end{aligned} \tag{4.20}$$

Equalities (4.20) imply that

$$\begin{aligned} |r_{m+1}\,(t,\xi)| \leq \sum_{j=0}^{k}\tilde{P}_j\Bigg(\left(1 - \frac{t}{T}\right)\int_0^t |r_m\,(\beta_j\,(s)\,,\xi)|\,ds \\ + \frac{t}{T}\int_t^T |r_m\,(\beta_j\,(s)\,,\xi)|\,ds\Bigg) \end{aligned} \tag{4.21}$$

for all $t \in [0,T]\,, \xi \in \mathbb{R}^n$, and $m = 1,2,\ldots.$ From (4.18) we obtain

$$|r_1\,(t,\xi)| = |x_1\,(t,\xi) - \xi| \leq \alpha_1(t)\delta\,(\xi) + \delta_1\,(\xi) \leq \frac{T}{2}\delta\,(\xi) + \delta_1\,(\xi) = \gamma(\xi). \tag{4.22}$$

Let us now estimate $r_2(t,\xi)$ using (4.20) and (4.22):

$$|r_2(t,\xi)| \le \sum_{j=0}^{k} \tilde{P}_j \left(\left(1 - \frac{t}{T}\right) \int_0^t |r_1(\beta_j(s),\xi)|\, ds \right. \tag{4.23}$$

$$\left. + \frac{t}{T} \int_t^T |r_1(\beta_j(s),\xi)|\, ds \right) \tag{4.24}$$

$$\le \sum_{j=0}^{k} \tilde{P}_j \left(\left(1 - \frac{t}{T}\right) \int_0^t \gamma(\xi) ds + \frac{t}{T} \int_t^T \gamma(\xi) ds \right) \tag{4.25}$$

$$\le \sum_{j=0}^{k} \tilde{P}_j \gamma(\xi) \alpha_1(t) \tag{4.26}$$

$$\le \frac{T}{2} \sum_{j=0}^{k} \tilde{P}_j \gamma(\xi). \tag{4.27}$$

By the method of mathematical induction we then obtain that the estimate

$$|r_m(t,\xi)| \le \left(\frac{T}{2}\right)^{m-1} \left(\tilde{P}_0 + \tilde{P}_1 + \cdots + \tilde{P}_k\right)^{m-1} \gamma(\xi) = G^{m-1}\gamma(\xi), \tag{4.28}$$

is true for all $t \in [0,T]$, $\xi \in \mathbb{R}^n$, and $m = 1, 2, \ldots$, where

$$G := \frac{T}{2} \sum_{j=0}^{k} \tilde{P}_j.$$

Estimate (4.28), according to (4.20), yields

$$|x_{m+j}(t,\xi) - x_m(t,\xi)| \le \sum_{i=1}^{j} |r_{m+i}(t,\xi)| \le G^m \sum_{i=0}^{j-1} G^i \gamma(\xi),$$

whence we obtain

$$|x_{m+j}(t,\xi) - x_m(t,\xi)| \le G^m \sum_{i=0}^{\infty} G^i \gamma(\xi) = G^m (I - G)^{-1} \gamma(\xi), \tag{4.29}$$

where, by virtue of assumption (4.15), $\lim_{m\to\infty} G^m = 0$. It follows from estimate (4.29) that $\{x_m(\cdot,\xi)\}_{m=0}^{\infty}$ is a Cauchy sequence in $C([0,T],\mathbb{R}^n)$, and therefore

$$\lim_{m\to\infty} x_m(t,\xi) = x^*(t,\xi)$$

uniformly in $t \in [0,T]$ for any fixed $\xi \in \mathbb{R}^n$. Passing to the limit as $j \to \infty$ in (4.29), we obtain the estimate

$$|x^*(t,\xi) - x_m(t,\xi)| \le G^m (I - G)^{-1} \gamma(\xi)$$

for all $t \in [0,T]$, $\xi \in \mathbb{R}^n$, and $m = 1, 2, \ldots$. According to Definition 4.1, this leads us to the assertion of Theorem 4.1. □

5. Convergence of successive approximations for the special deviation functions

If the cases where deviation functions $\beta_j : [0,T] \to [0,T]$, $j = 0,1,2,\ldots,k$, satisfy the condition

$$\operatorname*{ess\,inf}_{t\in[0,T]} (\beta_j(t) - t) \operatorname{sign}\left(t - \frac{T}{2}\right) \geq 0 \tag{5.1}$$

of type (3.7), the assumption (4.15) of Theorem 4.1 can be weakened.

Theorem 5.1. *If the deviation functions* $\beta_j : [0,T] \to [0,T]$, $j = 0,1,2,\ldots,k$, *possesses property* (5.1), *and, moreover, the inequality*

$$r\left(\tilde{P}_0 + \tilde{P}_1 + \cdots + \tilde{P}_k\right) < \frac{10}{3T} \tag{5.2}$$

is satisfied, then the successive approximation method (4.1) *is applicable to the boundary value problem* (2.1), (2.2) *with an estimate of the type* $\left(\frac{3T}{10}\sum_{j=0}^{k}\tilde{P}_j, g\right)$, *where*

$$g(t,\xi) := \frac{10}{9}\alpha_1(t)\gamma(\xi), \qquad t \in [0,T], \xi \in \mathbb{R}^n,$$

and the function γ *is defined by formula* (4.17).

Proof. It follows from Lemma 3.3 that condition (5.1) ensures the validity of estimates

$$\alpha_1(\beta_j(t)) \leq \alpha_1(t), \qquad t \in [0,T], j = 0,1,2,\ldots,k, \tag{5.3}$$

where α_1 is the function (3.2). Let us estimate the value of $r_2(t,\xi)$. Using (4.21), (4.22), and (5.3), we have

$$|r_2(t,\xi)| \leq \sum_{j=0}^{k}\tilde{P}_j\left(\left(1 - \frac{t}{T}\right)\int_0^t \left(\frac{T}{2}\delta(\xi) + \delta_1(\xi)\right) ds \right. \tag{5.4}$$

$$\left. + \frac{t}{T}\int_t^T \left(\frac{T}{2}\delta(\xi) + \delta_1(\xi)\right) ds\right)$$

$$\leq \sum_{j=0}^{k}\tilde{P}_j\ \gamma(\xi)\alpha_1(t) \tag{5.5}$$

for all $t \in [0,T]$ and $\xi \in \mathbb{R}^n$. By induction, we arrive at the pointwise and coordinatewise estimates

$$|r_{m+1}(t,\xi)| \leq \left(\sum_{j=0}^{k}\tilde{P}_j\right)^m \gamma(\xi)\,\alpha_m(t), \qquad t \in [0,T],\ \xi \in \mathbb{R}^n, \tag{5.6}$$

where α_m, $m \geq 0$, are the functions of sequence (3.1). By virtue of [6, Lemma 2.4], the estimates

$$\alpha_{m+1}(t) \leq \frac{3T}{10}\alpha_m(t) \leq \frac{10}{9}\left(\frac{3T}{10}\right)^m \alpha_1(t) \tag{5.7}$$

are true for all $t \in [0,T]$ and $m \geq 1$. Taking (5.7) into account and using (5.6), we arrive at the inequalities

$$|r_{m+1}(t,\xi)| \leq \frac{10}{9}\left(\frac{3T}{10}\sum_{j=0}^{k}\tilde{P}_j\right)^m \gamma(\xi)\,\alpha_1(t)$$

valid for all $t \in [0,T]$, $\xi \in \mathbb{R}^n$, and $m = 1, 2, \ldots$ The rest of the argument is similar to that of the proof of Theorem 4.1. □

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