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# On nonlinear functional parabolic equations with state-dependent delays of Volterra type

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**Abstract.** We consider second order quasilinear parabolic equations where also the main part contains functional dependence and state-dependent delay on the unknown function. Existence and some qualitative properties of the solutions are shown.

*AMS Subject Classifications:* 35R10, 35K59, 35K99

*Keywords:* Functional parabolic equation; State-dependent delay; Nonlinear parabolic equation.

## 1. Introduction

In the present paper we shall consider weak solutions of initial-boundary value problems for the equation

$$\begin{aligned} D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u,Du;u([\gamma_0(u)](t,x),x))] \\ +a_0^0(t,x,u,Du;u([\gamma_0(u)](t,x),x)) + a_0^1(t,x,u,Du;u([\gamma_1(u)](t,x),x)) \\ +a_0^2(t,x,u,Du;Du([\gamma_2(u)](t,x),x)) = f \end{aligned} \quad (1.1)$$

where the functions

$$a_i, a_i^j : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \to \mathbb{R}$$

satisfy modified conditions of [9] and $\gamma_j : L^2(Q_T) \to C(\overline{Q_T})$ are continuous (nonlinear) operators such that $[\gamma_j(u)](\cdot, x)$ is absolutely continuous for a.e. fixed $x$,

$$0 \le [\gamma_j(u)](t,x) \le t, \quad \frac{\partial}{\partial t}[\gamma_j(u)](t,x) \ge c_0$$

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with some constant $c_0 > 0$.

This work was motivated by works where nonlinear parabolic functional differential equations were considered which arise in certain applications. (See references in [8].) In [8] and [9] existence theorems and some qualitative properties were proved on solutions to initial value problems for the functional equations (connected with the above applications)

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u(t,x),Du(t,x);u)] + a_0(t,x,u(t,x),Du(t,x);u) = f. \quad (1.2)$$

In the present paper we consider (1.1) as a particular case of (1.2) and apply the results of [9] to the equation (1.1).

Differential equations and systems with state-dependent delay in one variable were considered thoroughly e.g. in [3] - [5] (see also the references there).

In Section 2 the existence of weak solutions will be proved and in Section 3 we shall formulate conditions which imply boundedness of solutions, further, stabilization of solutions will be shown as $t \to \infty$.

## 2. Existence of solutions

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform $C^1$ regularity property (see [1]), $Q_T = (0,T) \times \Omega$ and $p \geq 2$ be a real number. Let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace of the usual Sobolev space $W^{1,p}(\Omega)$ (of real valued functions) containing $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$). Denote by $L^p(0,T;V)$ the Banach space of the set of measurable functions $u : (0,T) \to V$ with the norm

$$\| u \|^p_{L^p(0,T;V)} = \int_0^T \| u(t) \|^p_V \, dt.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of $V$ (see, e.g., [11]).

First we formulate a slight modification of Theorem 1 in [9] which can be proved in the same way.

Assume that functions $\tilde{a}_i$ satisfy the following conditions.

$(A_1)$. The functions $\tilde{a}_i : Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $u \in L^p(0,T;V)$ $(i = 0,1,...,n)$.

$(A_2)$. There exist bounded (nonlinear) operators $g_1 : L^2(Q_T) \to \mathbb{R}^+$ and $k_1 : L^2(Q_T) \to L^q(\Omega)$ such that

$$|\tilde{a}_i(t,x,\zeta_0,\zeta;u)| \leq g_1(u)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(u)](x)$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$ and $u \in L^p(0,T;V)$.

$$(A_3). \quad \sum_{i=1}^{n} [\tilde{a}_i(t,x,\zeta_0,\zeta;u) - \tilde{a}_i(t,x,\zeta_0,\zeta^\star;u)](\zeta_i - \zeta_i^\star) \geq [g_2(u)](t)|\zeta - \zeta^\star|^p \quad (2.1)$$

where

$$[g_2(u)](t) \geq c^\star \left[1+ \| u \|_{L^p(0,t;V)}\right]^{-\sigma^\star}, \quad t \in [0,T] \tag{2.2}$$

$c^\star$ is some positive constant, $0 \leq \sigma^\star < p-1$.

$(A_4)$. $\sum_{i=0}^n \tilde{a}_i(t,x,\zeta_0,\zeta;u)\zeta_i \geq [g_2(u)](t)[|\zeta_0|^p + |\zeta|^p] - [k_2(u)](t,x)$
where $k_2(u) \in L^1(Q_T)$ satisfies with some positive constant $\sigma < p - \sigma^\star$

$$\| k_2(u) \|_{L^1(Q_t)} \leq \text{const} \left[1+ \| u \|_{L^p(0,t;V)}\right]^{\sigma}, \quad t \in [0,T].$$

$(A_5)$. If $(u_k) \to u$ weakly in $L^p(0,T;V)$, $(D_t u_k) \to D_t u$ weakly in $L^q(0,T;V^\star)$, $(\zeta_0^k) \to \zeta_0$ in $\mathbb{R}$ and $(\zeta^k) \to \zeta$ in $\mathbb{R}^n$ then for a.e. $(t,x) \in Q_T$

$$\lim_{k\to\infty} \tilde{a}_i(t,x,\zeta_0^k,\zeta^k;u_k) = \tilde{a}_i(t,x,\zeta_0,\zeta;u), \quad i = 0,1,...,n,$$

for a subsequence, in the case $i=0$ assuming that $(D_l u_k) \to D_l u$ in $L^2(Q_T)$ $(l = 1,...,n)$ holds, too.

**Remark 2.1.** Assumption $(A_5)$ is weaker than the corresponding assumption in [9], assumptions $(A_1)$ - $(A_4)$ are the same.

**Definition 2.1.** *Assuming $(A_1)$–$(A_5)$, define operator $\tilde{A} : L^p(0,T;V) \to L^q(0,T;V^\star)$ by*

$$[\tilde{A}(u),v] = \int_{Q_T} \left\{ \sum_{i=1}^n \tilde{a}_i(t,x,u,Du;u) D_i v + \tilde{a}_0(t,x,u,Du;u)v \right\} dtdx \tag{2.3}$$

*where the brackets $[\cdot,\cdot]$ mean the dualities in spaces $L^q(0,T;V^\star)$, $L^p(0,T;V)$.*

Since the assumptions $(A_1)$ - $(A_4)$ are the same as in [9], we obtain that operator $A$ is bounded, demicontinuous and coercive. By using the same arguments as in [9], one gets by $(A_5)$ that $A$ is pseudomonotone with respect to $D(L) = \{u \in L^p(0,T;V) : D_t u \in L^q(0,T;V^\star), u(0) = 0\}$. According to the theory of monotone type operators (see, e.g. [2], [10]) we have

**Theorem 2.1.** *Assume $(A_1)$ - $(A_5)$. Then for any $f \in L^q(0,T;V^\star)$ and $u_0 \in L^2(\Omega)$ there exists $u \in L^p(0,T;V)$ such that $D_t u \in L^q(0,T;V^\star)$,*

$$D_t u + \tilde{A}(u) = f, \quad u(0) = u_0. \tag{2.4}$$

Now we formulate assumptions on functions $a_i, a_0^j$ in equation (1.1).

$(B_1)$. The functions $a_i, a_0^j : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $v \in L^2(Q_T)$ $(i = 1,...,n,\ j = 0,1,2)$.

$(B_2)$. There exist bounded (nonlinear) operators $g_1 : L^2(Q_T) \to \mathbb{R}^+$ and $k_1 : L^2(Q_T) \to L^q(\Omega)$ such that

$$|a_i(t,x,\zeta_0,\zeta;v)| \leq g_1(v)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(v)](x),$$

$$|a_0^j(t,x,\zeta_0,\zeta;v)| \le g_1(v)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(v)](x)$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$ and $v \in L^2(Q_T)$.

$$(B_3). \quad \sum_{i=1}^{n}[a_i(t,x,\zeta_0,\zeta;v) - a_i(t,x,\zeta_0,\zeta^\star;v)](\zeta_i - \zeta_i^\star) \ge [g_2(v)](t)|\zeta - \zeta^\star|^p \qquad (2.5)$$

where

$$[g_2(v)](t) \ge c^\star \left[1+ \| v \|_{L^2(Q_t)}\right]^{-\sigma^\star}, \quad t \in [0,T], \qquad (2.6)$$

$c^\star$ is some positive constant, $0 \le \sigma^\star < p-1$.

$(B_4)$. $\sum_{i=0}^{n} a_i(t,x,\zeta_0,\zeta;v)\zeta_i + a_0^0(t,x,\zeta_0,\zeta;v)\zeta_0 \ge [g_2(v)](t)[|\zeta_0|^p + |\zeta|^p]$,

$$|a_0^j(t,x,\zeta_0,\zeta;v)| \le [h_2^j(v)](t,x)[1 + |\zeta_0|^{\tilde\rho_j} + |\zeta|^{\tilde\rho_j}], \quad j=1,2$$

with $0 \le \tilde\rho_j < p-1$,

$$\int_{Q_t} |[h_2^j(v)](\tau,x)|^{q_1^j} d\tau dx \le \text{const}\left(1+ \| v \|_{L^2(Q_t)}\right)^\sigma \text{ where}$$

$$\sigma < p - \sigma^\star, \quad q_1^j = p_1^j/(p_1^j - 1), \quad p_1^j = p/(\tilde\rho_j + 1).$$

$(B_5)$. If $(v_k) \to v$ in $L^2(Q_T)$, $(\zeta_0^k) \to \zeta_0$ in $\mathbb{R}$ and $(\zeta^k) \to \zeta$ in $\mathbb{R}^n$ then for a.e. $(t,x) \in Q_T$, for a suitable subsequence

$$\lim_{k\to\infty} a_i(t,x,\zeta_0^k,\zeta^k;v_k) = a_i(t,x,\zeta_0,\zeta;v), \quad i=1,...,n,$$

$$\lim_{k\to\infty} a_0^j(t,x,\zeta_0^k,\zeta^k;v_k) = a_0^j(t,x,\zeta_0,\zeta;v), \quad j=0,1,2.$$

On operators $\gamma_j$ assume that

$(G)$ $\gamma_j : L^2(Q_T) \to C(\overline{Q_T})$, $(j=0,1,2)$ are continuous (nonlinear) operators such that $[\gamma_j(u)](\cdot,x)$ is absolutely continuous for a.e. fixed $x \in \Omega$,

$$\frac{\partial}{\partial t}[\gamma_j(u)](t,x) \ge c_0, \quad 0 \le [\gamma_j(u)](t,x) \le t$$

with some constant $c_0 > 0$ and $\frac{\partial}{\partial t}[\gamma_2(\cdot)] : L^2(Q_T) \to C(\overline{Q_T})$ is continuous operator.

**Example 2.1.** Condition $(G)$ is fulfilled e.g. by the operators of the form

$$[\gamma(u)](t,x) = t\beta\left\{\int_{Q_t} \Gamma(t,\tau,x,\xi)u^2(\tau,\xi)\, d\tau d\xi\right\}$$

where $\Gamma, \frac{\partial \Gamma}{\partial t}$ are continuous on $\overline{Q_T} \times \overline{Q_T}$, further, $\Gamma, \frac{\partial \Gamma}{\partial t} \ge 0$, $\beta \in C^1(\mathbb{R})$ satisfies $\delta_1 \le \beta \le 1$ with some constant $\delta_1 > 0$ and $\beta' \ge 0$.

**Definition 2.2.** *Assuming* $(B_1)$ - $(B_5)$, $(G)$, *define operator* $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ *by*

$$\begin{aligned}[A(u),v] &= \sum_{i=1}^{n}\int_{Q_T} a_i(t,x,u,Du;u([\gamma_0(u)](t,x),x))D_i v\,dtdx \\ &+\sum_{j=0}^{1}\int_{Q_T} a_0^j(t,x,u,Du;u([\gamma_j(u)](t,x),x))v\,dtdx \\ &+\int_{Q_T} a_0^2(t,x,u,Du;Du([\gamma_2(u)](t,x),x))v\,dtdx.\end{aligned}$$

**Theorem 2.2.** *Assume* $(B_1)$ - $(B_5)$, $(G)$. *Then for any* $f \in L^q(0,T;V^\star)$ *and* $u_0 \in L^2(\Omega)$ *there exists* $u \in L^p(0,T;V)$ *such that* $D_t u \in L^q(0,T;V^\star)$,

$$D_t u + A(u) = f, \quad u(0) = u_0. \tag{2.7}$$

*Proof.* Define functions $\tilde{a}_i$ by

$$\begin{aligned}\tilde{a}_i(t,x,\zeta_0,\zeta;u) &= a_i(t,x,\zeta_0,\zeta;u([\gamma_0(u)](t,x),x)), \qquad i=1,...,n \\ \tilde{a}_0(t,x,\zeta_0,\zeta;u) &= a_0^0(t,x,\zeta_0,\zeta;u([\gamma_0(u)](t,x),x)) \\ &+a_0^1(t,x,\zeta_0,\zeta;u([\gamma_1(u)](t,x),x)) \\ &+a_0^2(t,x,\zeta_0,\zeta;Du([\gamma_2(u)](t,x),x)).\end{aligned}$$

We shall show that these functions $\tilde{a}_i$ satisfy the assumptions $(A_1)$ - $(A_5)$. Clearly, $(A_1)$, $(A_3)$ are satisfied by $(B_1)$, $(B_3)$. Further, by using the notation $\psi_j(t,x) = [\gamma_j(u)](t,x)$ and $(G)$,

$$\begin{aligned}\| u([\gamma_j(u)](t,x),x) \|^2_{L^2(Q_{\tilde{t}})} &= \int_\Omega \left\{\int_0^{\tilde{t}} |u([\gamma_j(u)](t,x),x)|^2 dt\right\} dx \\ &\le \frac{1}{c_0}\int_\Omega \left\{\int_0^{\tilde{t}} |u(\psi_j(t,x),x)|^2 \frac{\partial \psi_j}{\partial t}(t,x)dt\right\} dx \\ &\le \frac{1}{c_0} \| u \|^2_{L^2(Q_{\tilde{t}})}, \qquad j=0,1 \quad 0<\tilde{t}\le T,\end{aligned} \tag{2.8}$$

and thus we obtain $(A_2)$ from $(B_2)$. Similarly,

$$\| Du([\gamma_2(u)](t,x),x) \|^2_{L^2(Q_{\tilde{t}})} \le \frac{1}{c_0} \| Du \|^2_{L^2(Q_{\tilde{t}})} .$$

Inequality (2.8) implies

$$[g_2(u([\gamma_0(u)](t,x),x))]\,(t) \ge \text{const}\left[1+ \| u \|_{L^2(Q_t)}\right]^{-\sigma^\star} \tag{2.9}$$

and by $(B_4)$

$$\int_{Q_t} \left[h_2^1(u([\gamma_1(u)](t,x),x))\right]^{q_1^1}(\tau,x)d\tau dx \leq \text{const}\left[1+\| u \|_{L^2(Q_t)}\right]^\sigma, \quad (2.10)$$
$$\int_{Q_t} \left[h_2^2(Du([\gamma_2(u)](t,x),x))\right]^{q_1^2}(\tau,x)d\tau dx \leq \text{const}\left[1+\| Du \|_{L^2(Q_t)}\right]^\sigma.$$

Hence, by using the notations

$$\begin{aligned} v^1(t,x) &= u([\gamma_1(u)](t,x),x), \\ v^2(t,x) &= Du([\gamma_2(u)](t,x),x), \\ |a_0^j(t,x,\zeta_0,\zeta;v^j)\zeta_0| &\leq [h_2^j(v^j)](t,x)\text{const}\left[1+|\zeta_0|^{\tilde\rho_j+1}+|\zeta|^{\tilde\rho_j+1}\right] \quad (2.11) \\ &\leq \varepsilon[g_2(v^j)](t)(|\zeta_0|^p+|\zeta|^p)+C(\varepsilon)\left\{[h_2^j(v^j)](t,x)^{q_1^j}+1\right\},\ j=1,2 \end{aligned}$$

where $q_1^j = p_1^j/(p_1^j-1)$, $p_1^j = p/(\tilde\rho_j+1)$. Choosing sufficiently small $\varepsilon > 0$, one obtains $(A_4)$ for functions $\tilde a_i$ from $(B_4)$ and (2.10) with

$$\begin{aligned} [k_2(u)](t,x) &= C(\varepsilon)\left\{[h_2^1(u[\gamma_1(u)](t,x),x)]^{q_1^1}(t,x)+1\right\} \\ &\quad +C(\varepsilon)[h_2^2(Du[\gamma_2(u)](t,x),x)]^{q_1^2}(t,x). \end{aligned}$$

Finally, we show that functions $\tilde a_i$ satisfy $(A_5)$. Assume that $(u_k)\to u$ weakly in $L^p(0,T;V)$, $(D_t u_k)\to D_t u$ weakly in $L^q(0,T;V^\star)$, $(\zeta_0^k)\to\zeta_0$ in $\mathbb{R}$, $(\zeta^k)\to\zeta$ in $\mathbb{R}^n$. Then $(u_k)\to u$ strongly in $L^2(Q_T)$, for a subsequence and for $j=0,1$

$$\begin{aligned} u_k([\gamma_j(u_k)](t,x),x)-u([\gamma_j(u)](t,x),x) &= \{u_k([\gamma_j(u_k)](t,x),x)-u([\gamma_j(u_k)](t,x),x)\} \\ &\quad +\{u([\gamma_j(u_k)](t,x),x)-u([\gamma_j(u)](t,x),x)\}. \end{aligned} \quad (2.12)$$

For the first term in the right hand side of (2.12) we have (by using the notation $\psi_j^k(t,x)=[\gamma_j(u_k)](t,x)$, $(G)$)

$$\begin{aligned} &\int_\Omega\left\{\int_0^T |u_k([\gamma_j(u_k)](t,x),x)-u([\gamma_j(u_k)](t,x),x)|^2 dt\right\}dx \\ &\leq \frac{1}{c_0}\int_\Omega\left\{\int_0^T |u_k(\psi_k(t,x),x)-u(\psi_k(t,x),x)|^2\frac{\partial\psi_k}{\partial t}dt\right\}dx \\ &\leq \frac{1}{c_0}\int_{Q_T}|u_k(\tau,x)-u(\tau,x)|^2 d\tau dx \to 0. \end{aligned}$$

Further, $u\in L^p(0,T;V)$, $D_t u\in L^q(0,T;V^\star)$ imply $u\in C([0,T];L^2(\Omega))$ (see, e.g. [11]). Thus $u:[0,T]\to L^2(\Omega)$ is uniformly continuous, hence for arbitrary $\varepsilon>0$ there exists $\delta>0$ such that

$$\int_\Omega |u([\gamma_j(u_k)](t,x),x)-u([\gamma_j(u)](t,x),x)|^2 dx < \varepsilon$$

$$\text{if } |[\gamma_j(u_k)](t,x) - [\gamma_j(u)](t,x)| < \delta \text{ for all } (t,x) \in Q_T.$$

Since the operator $\gamma_j : L^2(Q_T) \to C(\overline{Q_T})$ is continuous, there exists $k_0$ such that

$$|[\gamma_j(u_k)](t,x) - [\gamma_j(u)](t,x)| < \delta \text{ for } k \geq k_0.$$

Consequently, for $k \geq k_0$

$$\int_{Q_T} |u([\gamma_j(u_k)](t,x),x) - u([\gamma_j(u)](t,x),x)|^2 dxdt < \varepsilon T,$$

i.e the second term on the right hand side of (2.12) is converging to 0 in $L^2(Q_T)$.

Thus we have for the functions

$$v_k^j(t,x) = u_k([\gamma_j(u_k)](t,x),x), \quad v^j(t,x) = u([\gamma_j(u)](t,x),x), \quad j = 0,1$$

that $(v_k^j) \to v^j$ strongly in $L^2(Q_T)$. Similarly, by using assumption (G) and the substitution in (2.8), for

$$v_k^2(t,x) = D_l u_k([\gamma_2(u_k)](t,x),x), \quad v^2(t,x) = D_l u([\gamma_2(u)](t,x),x) \quad (l = 1,...,n)$$

we have $(v_k^2) \to v^2$ strongly in $L^2(Q_T)$, assuming that $(D_l u_k) \to D_l u$ in $L^2(Q_T)$.

So by $(B_5)$ assumption $(A_5)$ is satisfied for $\tilde{a}_i$ $(i = 0,1,...,n)$. Therefore, by Theorem 2.1 we obtain the existence of solutions to (2.7). □

Now we formulate an existence theorem in $(0,\infty)$ which can be obtained from Theorem 2.2, by using a diagonal process and the Volterra property (see, e.g. [7]). Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $u : (0,\infty) \to V$ such that for each fixed finite $T > 0$, $u|_{(0,T)} \in L^p(0,T;V)$ and let $Q_\infty = (0,\infty) \times \Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $u : Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite $T$. On operators $\gamma_j$ assume

$(G_\infty)$ Operators $\gamma_j : L^2_{loc}(Q_\infty) \to C(\overline{Q_\infty})$ are of Volterra type, i.e. $\gamma_j(u)|_{Q_T}$ depends only on $u|_{Q_T}$, for any finite $T$ and $\gamma_j, \frac{\partial}{\partial t}[\gamma_2(\cdot)] : L^2(Q_T) \to C(\overline{Q_T})$ are continuous for every $T$. Further, $[\gamma_j(u)](\cdot,x)$ is absolutely continuous for a.e. fixed $x \in \Omega$,

$$\frac{\partial}{\partial t}[\gamma_j(u)](t,x) \geq c_0, \quad 0 \leq [\gamma_j(u)](t,x) \leq t$$

with some constant $c_0 > 0$.

**Theorem 2.3.** *Assume that*

$$a_i, a_0^j : Q_\infty \times \mathbb{R}^{n+1} \times L^2_{loc}(Q_\infty) \to \mathbb{R}, \quad \gamma_j : L^p_{loc}(Q_\infty) \to C(\overline{Q_\infty})$$

*satisfy assumptions* $(G_\infty)$ *and* $(B_1)$*–*$(B_5)$ *for any finite* $T$*, further,* $a_i(t,x,\zeta_0,\zeta;u)|_{Q_T}$*,* $a_0^j(t,x,\zeta_0,\zeta;u)|_{Q_T}$ *depend only on* $u|_{Q_T}$ *(Volterra property). Then for any* $f \in L^q_{loc}(0,\infty;V^\star)$*,* $u_0 \in L^2(\Omega)$ *there exists* $u \in L^p_{loc}(0,\infty;V)$ *which is a solution of (2.7) for any finite* $T$*.*

## 3. Boundedness and stabilization

**Theorem 3.1.** *Let the assumptions of Theorem 2.3 be satisfied such that $\gamma_0(u)$ is depending only on $t$ (not on $x$) and for all $u \in L^2_{loc}(Q_\infty)$, sufficiently large $t$ we have on operators $g_2$, $h_2^j$ (in $(B_4)$)*

$$[g_2(u)](t) \geq const \left[1 + \sup_{\tau\in[0,t]} \int_\Omega u^2(\tau, x)dx\right]^{-\sigma^\star/2}, \quad t \in (0, \infty), \tag{3.1}$$

$$\int_\Omega [h_2^1(u)](t, x)^{q_1^1} dx \leq const \left[1 + \left(\int_\Omega u^2(t, x)dx\right)^{\tilde\sigma/2}\right], \tag{3.2}$$

*where $0 < \tilde\sigma < p - \sigma^\star$, $\tilde\sigma \leq 2$. In the particular case when $\gamma_1(u)$ is depending only on $t$ and not on $x$, we assume (instead of (3.2))*

$$\int_\Omega [h_2^1(u)](t, x)^{q_1^1} dx \tag{3.3}$$

$$\leq const \left[1 + \sup_{\tau\in[0,t]} \left(\int_\Omega u^2(t, x)dx\right)^{\sigma/2} + \varphi(t) \left(\sup_{\tau\in[0,t]} \int_\Omega u^2(t, x)dx\right)^{(p-\sigma^\star)/2}\right]$$

*where $\lim_\infty \varphi = 0$. Further, for all $u \in L^p(0, \infty; V)$*

$$|h_2^2(u)| \leq const. \tag{3.4}$$

*Finally, $\| f(t) \|_{V^\star}$ is bounded for $t \in (0, \infty)$.*

*Then for a solution $u \in L^p_{loc}(0, \infty; V)$ of (2.7) in $(0, \infty)$, $\int_\Omega u^2(t, x)dx$ is bounded for $t \in (0, \infty)$.*

*Proof.* Let $u \in L^p_{loc}(0, \infty; V)$ be a solution of (2.7). Applying (2.7) to $u(t) \in V$, we obtain for

$$\tilde y(t) = \int_\Omega u^2(t, x)dx, \quad y_0(t) = \int_\Omega u^2([\gamma_0(u)](t), x)dx,$$

$$y_1(t) = \int_\Omega u^2([\gamma_1(u)](t, x), x)dx,$$

by using (2.11) with sufficiently small $\varepsilon > 0$,

$$\frac{1}{2}\tilde y'(t) + \frac{1}{2} g_2(u([\gamma_0)](t), x))(t) \| u(t) \|_V^p \tag{3.5}$$

$$-C \left\{\int_\Omega [h_2^1(u([\gamma_1(u)](t), x))]^{q_1^1} dx + \int_\Omega [h_2^2(Du([\gamma_2(u)](t), x))]^{q_1^2} dx + 1\right\}$$

$$\leq \| f(t) \|_{V^\star} \| u(t) \|_V$$

$$\leq \text{const} \| u(t) \|_V .$$

Young's inequality implies

$$\| u(t) \|_V \leq \frac{\varepsilon^p}{p} g_2(u([\gamma_0(u)](t),x))(t) \| u(t) \|_V^p + \frac{1}{q\varepsilon^q} \frac{1}{g_2(u([\gamma_0(u)](t),x))^{q/p}}. \tag{3.6}$$

Choosing sufficiently small $\varepsilon > 0$, by

$$\| u(t) \|_V^p \geq \text{const}\tilde{y}(t)^{p/2}$$

we obtain in the case (3.3), when $\gamma_1(u)$ is not depending on $x$,

$$\tilde{y}'(t) + c^\star \tilde{y}(t)^{p/2} \left[1 + \sup_{\tau\in[0,t]} y_0(\tau)\right]^{-\sigma^\star/2}$$
$$\leq \text{const} \left[1 + \sup_{\tau\in[0,t]} y_1(\tau)^{\sigma/2} + \varphi(t) \sup_{\tau\in[0,t]} y_1(\tau)^{(p-\sigma^\star)/2} + \sup_{\tau\in[0,t]} y_1(\tau)^{(q/p)(\sigma^\star/2)}\right].$$

Since $[\gamma_j(u)](\tau) \leq \tau$, $\sup_{\tau\in[0,t]} y_j(\tau) \leq \sup_{\tau\in[0,t]} \tilde{y}(\tau)$ $(j = 0, 1)$, thus we have

$$\tilde{y}'(t) + c^\star \tilde{y}(t)^{p/2} \left[1 + \sup_{\tau\in[0,t]} \tilde{y}(\tau)\right]^{-\sigma^\star/2}$$
$$\leq \text{const} \left[1 + \sup_{\tau\in[0,t]} \tilde{y}(\tau)^{\sigma/2} + \varphi(t) \sup_{\tau\in[0,t]} \tilde{y}(\tau)^{(p-\sigma^\star)/2} + \sup_{\tau\in[0,t]} \tilde{y}(\tau)^{(q/p)(\sigma^\star/2)}\right].$$

Since $\sigma < p - \sigma^\star$, $\lim_\infty \varphi = 0$ and $(q/p)\sigma^\star < p - \sigma^\star$, one obtains (as in [8]) that the above inequality implies the boundedness of $\tilde{y}(t)$.

In the case (3.2) (when $\gamma_1(u)$ may depend on $t, x$),

$$\int_\Omega |h_2^1(u([\gamma_1(u)](t,x),x))|^{q_1^1} dx \leq \text{const} \left[1 + \left(\int_\Omega |u([\gamma_1(u)](t,x),x)|^2 dx\right)^{\tilde{\sigma}/2}\right].$$

Hence, by using the notation $\psi(t,x) = [\gamma_1(u)](t,x)$

$$\int_{T_1}^{T_2} \left\{\int_\Omega |h_2^1(u([\gamma_1(u)](t,x),x))|^{q_1^1} dx\right\}$$
$$\leq \text{const}(T_2 - T_1) + \text{const} \int_{T_1}^{T_2} \left[\int_\Omega |u([\gamma_1(u)](t,x),x)|^2 dx\right]^{\tilde{\sigma}/2} dt$$
$$\leq \text{const}(T_2 - T_1) + \text{const} \left\{\int_{\psi(T_1,x)}^{\psi(T_2,x)} \left[\int_\Omega |u(\tau,x)|^2 dx\right] d\tau\right\}^{\tilde{\sigma}/2}.$$

Thus from (3.5), (3.6) one obtains

$$\begin{aligned}
&\tilde{y}(T_2) - \tilde{y}(T_1) + \tilde{c}^\star \int_{T_1}^{T_2} \tilde{y}(t)^{p/2} \left[1 + \sup_{\tau\in[0,t]} \tilde{y}(\tau)\right]^{-\sigma^\star/2} dt \\
\leq\quad & \mathrm{const}(T_2 - T_1) + \mathrm{const} \left\{ \int_\Omega \left[ \int_{\psi(T_1,x)}^{\psi(T_2,x)} |u(\tau,x)|^2 d\tau \right] dx \right\}^{\tilde{\sigma}/2} \\
\leq\quad & \mathrm{const}(T_2 - T_1) + \mathrm{const} \left[ \int_{d_1 T_1}^{d_2 T_2} \tilde{y}(\tau) d\tau \right]^{\tilde{\sigma}/2}
\end{aligned}$$

with some constants $d_1, d_2 > 0$. Since $\tilde{\sigma} < p - \sigma^\star$, the last inequality implies (as above) the boundedness of $\tilde{y}$. □

Now we formulate a theorem on stabilization of $u(t)$ as $t \to \infty$.

**Theorem 3.2.** *Assume that the conditions of Theorem 3.1 are fulfilled with Volterra operators*

$$g_1 : L^2_{loc}(Q_\infty) \to \mathbb{R}^+, \tag{3.7}$$

$$k_1 : L^2_{loc}(Q_\infty) \to L^q(\Omega) \tag{3.8}$$

*such that* $a_0^1(t, x, \zeta_0, \zeta; 0) = 0$ *and the following monotonicity condition is satisfied with some constant* $c_2 > 0$*:*

$$\begin{aligned}
&\sum_{i=1}^n [a_i(t, x, \zeta_0, \zeta; u) - a_i(t, x, \zeta_0^\star, \zeta^\star; u)](\zeta_i - \zeta_i^\star) \\
&\quad + \quad [a_0^0(t, x, \zeta_0, \zeta; u) - a_0^0(t, x, \zeta_0^\star, \zeta^\star; u)](\zeta_0 - \zeta_0^\star) \\
&\qquad \geq [g_2(u)](t)[|\zeta - \zeta^\star|^p + |\zeta_0 - \zeta_0^\star|^p] + c_2(\zeta_0 - \zeta_0^\star)^2.
\end{aligned} \tag{3.9}$$

*Further, for arbitrary fixed* $u \in L^p_{loc}(0, \infty; V) \cap L^\infty(0, \infty; L^2(\Omega))$*, with* $D_t u \in L^q_{loc}(0, \infty; V^\star)$*,* $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$*, a.a.* $x \in \Omega$

$$\begin{aligned}
|a_i(t, x, \zeta_0, \zeta; u) - a_{i,\infty}(x, \zeta_0, \zeta)| &\leq \Phi(t)[1 + |\zeta_0|^{p-1} + |\zeta|^{p-1}], \quad i = 1, ..., n, \\
|a_0^0(t, x, \zeta_0, \zeta; u) - a_{0,\infty}^0(x, \zeta_0, \zeta)| &\leq \Phi(t)[1 + |\zeta_0|^{p-1} + |\zeta|^{p-1}], \\
|a_0^2(t, x, \zeta_0, \zeta; u)| &\leq \Psi(t)[1 + |\zeta_0|]
\end{aligned} \tag{3.10}$$

*with some Carathéodory functions* $a_{i,\infty}$*,* $a_{0,\infty}^0$ *where* $\int_0^\infty \Phi(t)^q dt < \infty$*,* $\int_0^\infty \Psi(t)^2 dt < \infty$*. On* $a_0^1$ *we assume for every* $w \in V$ *the inequality*

$$|a_0^1(t, x, \zeta_0, \zeta; w) - a_{0,\infty}^1(x; w)| \leq \Psi(t)[1 + |\zeta_0|]$$

*where* $a_{0,\infty}^1 : \Omega \times V \to \mathbb{R}$ *is such that* $a_{0,\infty}^1(\cdot; w)$ *is measurable; there exist constants* $a > 0, 0 < c_3 < c_2$ *such that for all* $u, u^\star, v \in L^p_{loc}(0, \infty; V) \cap L^\infty(0, \infty; L^2(\Omega))$*,*

$T_1 > T_2 \geq 0$,

$$\int_{T_1}^{T_2} \int_\Omega |a_0^1(t,x,v,Dv;u) - a_0^1(t,x,v,Dv;u^\star)|^2 dtdx \tag{3.11}$$
$$\leq \quad c_0 c_3^2 \int_{\max\{0,T_1-a\}}^{T_2} \int_\Omega [u-u^\star]^2 dtdx, \qquad [\gamma_1(u)](t,x) \geq t-a.$$

*Finally, there exists* $f_\infty \in V^\star$ *such that*

$$\| f(t) - f_\infty \|_{V^\star} \leq \Phi(t). \tag{3.12}$$

*Then for a solution of (2.4) in* $(0,\infty)$ *we have*

$$\int_0^\infty \| u(t) - u_\infty \|_V^p \, dt \quad < \quad \infty, \qquad \int_0^\infty \| u(t) - u_\infty \|_{L^2(\Omega)}^2 \, dt < \infty, \tag{3.13}$$
$$\lim_{t\to\infty} \| u(t) - u_\infty \|_{L^2(\Omega)} \quad = \quad 0, \tag{3.14}$$
$$\int_T^\infty \| u(t) - u_\infty \|_{L^2(\Omega)}^2 \, dt \quad \leq \quad const \Bigg\{ e^{-\tilde{\gamma} T} + \int_0^T \left[ e^{-\tilde{\gamma}(T-t)} \int_t^\infty (\Phi(\tau)^q + \Psi(\tau)^2) d\tau \right] dt \Bigg\} \tag{3.15}$$

*with some constant* $\tilde{\gamma} > 0$ *where* $u_\infty \in V$ *is the unique solution to*

$$A_\infty(u_\infty) = f_\infty \tag{3.16}$$

*and the operator* $A_\infty : V \to V^\star$ *is defined for* $z, v \in V$ *by*

$$\langle A_\infty(z), v \rangle = \sum_{i=1}^n \int_\Omega a_{i,\infty}(x,z,Dz) D_i v dx + \int_\Omega a_{0,\infty}^0(x,z,Dz) v dx + \int_\Omega a_{0,\infty}^1(x;z) v dx.$$

*Proof.* Since the functions $a_{i,\infty}$ and $a_{0,\infty}^0$ satisfy the Carathéodory condition and $a_{0,\infty}^1(\cdot;z)$ is measurable, we obtain from $(B_2)$, (3.10) that $A_\infty : V \to V^\star$ is bounded and demicontinuous. From (3.9), (3.10), (3.11) it is not difficult to derive that $A_\infty$ is strictly monotone and by $(B_4)$ $A_\infty$ is coercive. Thus there exists a unique solution of (3.16).

If $u$ is a solution of (2.4) in $(0,\infty)$ then by (3.16) one obtains

$$\langle D_t[u(t) - u_\infty], u(t) - u_\infty \rangle + \langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty \rangle = \langle f(t) - f_\infty, u(t) - u_\infty \rangle. \tag{3.17}$$

By using the notation

$$\begin{aligned}
\langle [A_u(u_\infty)](t), z\rangle &= \int_\Omega \sum_{i=1}^n a_i(t,x,u_\infty(x), Du_\infty(x); u([\gamma_0(u)](t,x),x))D_i z\, dx \\
&+ \int_\Omega a_0^0(t,x,u_\infty(x), Du_\infty(x); u([\gamma_0(u)](t,x),x))z\, dx \\
&+ \int_\Omega a_0^1(t,x,u,Du; u_\infty)z\, dx \\
&+ \int_\Omega a_0^2(t,x,u,Du; Du([\gamma_2(u)](t,x),x))z\, dx,
\end{aligned}$$

(3.9) and Young's inequality, we obtain for the second term in (3.17)

$$\begin{aligned}
&\langle [A(u)](t) - A_\infty(u_\infty), u(t)-u_\infty\rangle \qquad (3.18)\\
&= \langle [A(u)](t) - [A_u(u_\infty)](t), u(t)-u_\infty\rangle + \langle [A_u(u_\infty)](t) - A_\infty(u_\infty), u(t)-u_\infty\rangle \\
&\geq [g_2(u([\gamma_0(u)](t,x),x))](t) \parallel u(t)-u_\infty \parallel_V^p + c_2 \int_\Omega |u(t)-u_\infty|^2 dx \\
&- \Big| \int_\Omega [a_0^1(t,x,u,Du; u([\gamma_1(u)](t,x),x)) - a_0^1(t,x,u,Du;u_\infty)][u(t)-u_\infty] dx \Big| \\
&- |\langle [A_u(u_\infty)](t) - A_\infty(u_\infty), u(t)-u_\infty\rangle|.
\end{aligned}$$

For arbitrary $\varepsilon > 0$

$$\begin{aligned}
&|\langle [A_u(u_\infty)](t) - A_\infty(u_\infty), u(t)-u_\infty\rangle| \\
&\leq \frac{\varepsilon^p}{p} \parallel u(t)-u_\infty \parallel_V^p + \frac{\varepsilon^2}{2} \parallel u(t)-u_\infty \parallel_{L^2(\Omega)}^2 \\
&+ C(\varepsilon) \sum_{i=1}^n \int_\Omega |a_i(t,x,u_\infty, Du_\infty; u([\gamma_0(u)](t,x),x)) - a_{i,\infty}(x,u_\infty,Du_\infty|^q dx \\
&+ C(\varepsilon) \int_\Omega |a_0^0(t,x,u_\infty,Du_\infty; u([\gamma_0(u)](t,x),x)) - a_{0,\infty}^0(x,u_\infty,Du_\infty|^q dx \\
&+ C(\varepsilon) \int_\Omega |a_0^1(t,x,u,Du;u_\infty) - a_{0,\infty}^1(x;u_\infty)|^2 dx \\
&+ C(\varepsilon) \int_\Omega |a_0^2(t,x,u,Du; Du([\gamma_2(u)](t,x),x))|^2 dx
\end{aligned}$$

and

$$|\langle f(t) - f_\infty, u(t)-u_\infty\rangle| \leq \varepsilon^p/p \parallel u(t)-u_\infty \parallel_V^p + C(\varepsilon) \parallel f(t)-f_\infty \parallel_{V^\star}^q .$$

Thus, since $\int_\Omega u^2(t)dx$ is bounded and

$$\langle D_t[u(t)-u_\infty], u(t)-u_\infty\rangle = \frac{1}{2}y'(t) \text{ where } y(t) = \int_\Omega |u(t)-u_\infty|^2 dx,$$

integrating (3.17) over $(T_1, T_2)$, we obtain with sufficiently small $\varepsilon$

$$y(T_2) - y(T_1) + c^\star \int_{T_1}^{T_2} \| u(t) - u_\infty \|_V^p \, dt + \tilde{c}_2 \int_{T_1}^{T_2} y(t)dt \tag{3.19}$$
$$- \left[ \int_{T_1}^{T_2} \int_\Omega |a_0^1(t,x,u,Du;u([\gamma_1(u)](t,x),x)) - a_0^1(t,x,u,Du;u_\infty)|^2 dt dx \right]^{1/2}$$
$$\times \left[ \int_{T_1}^{T_2} y dt \right]^{1/2} \le c \left( \int_{T_1}^{T_2} \Phi^q dt + \int_{T_1}^{T_2} \Psi^2 dt \right)$$

with some constants $c^\star, c, \tilde{c}_2 > 0$ where $c_3 < \tilde{c}_2 < c_2$. By (3.11) for sufficiently large $T_1$

$$\int_{T_1}^{T_2} \int_\Omega |a_0^1(t,x,u,Du;u([\gamma_1(u)](t,x),x)) - a_0^1(t,x,u,Du;u_\infty)|^2 dt dx$$
$$\le \quad c_0 c_3^2 \int_{\max\{0,T_1-a\}}^{T_2} \left[ \int_\Omega |u([\gamma_1(u)](t,x),x) - u_\infty(x)|^2 dx \right] dt$$
$$\le \quad c_3^2 \int_{\max\{0,T_1-2a\}}^{T_2} \left[ \int_\Omega |u(\tau,x) - u_\infty(x)|^2 dx \right] d\tau.$$

Since $y$ is bounded, $c_3 < \tilde{c}_2$ and $\int_0^\infty \Phi^q dt < \infty$, $\int_0^\infty \Psi^2 dt < \infty$, we obtain from (3.19) with $T_1 = 0$ (3.13).

Thus by (3.11), (3.19)

$$\int_0^\infty \int_\Omega |a_0^1(t,x,u,Du;u([\gamma_1(u)](t,x),x)) - a_0^1(t,x,u,Du;u_\infty)|^2 dt dx < \infty,$$

and, consequently, $\lim_\infty y = 0$.

Because, first observe that by (3.13)

$$\liminf_{t\to\infty} y(t) = 0$$

Hence there exist

$$T_1 < T_2 < ... < T_k < ... \to +\infty \text{ such that } \lim_{k\to\infty} y(T_k) = 0.$$

Applying (3.19) to $T_1 = T_k$ and $T_2 = T$ with $T > T_k$, we obtain

$$0 \le y(T) \le y(T_k) + a_k \quad \text{ where } \lim_{k\to\infty} a_k = 0$$

and so $\lim_\infty y = 0$.

Finally, from (3.11), (3.14), (3.19) we obtain as $T_2 \to \infty$

$$-y(T_1) + c^\star \int_{T_1}^\infty \| u(t) - u_\infty \|_V^p \, dt + \tilde{c}_2 \int_{T_1}^\infty y dt$$
$$-c_3 \left[ \int_{T_1-2a}^\infty y dt \right]^{1/2} \left[ \int_{T_1}^\infty y dt \right]^{1/2} \le \text{const} \left[ \int_{T_1}^\infty \Phi(t)^q dt + \int_{T_1}^\infty \Psi(t)^2 dt \right].$$

Hence, by using the notation $Y(T) = \int_T^\infty y(t)dt$,

$$\begin{aligned} Y'(T_1) + (\tilde{c}_2 - c_3/2)Y(T_1) - (c_3/2)Y(T_1 - 2a) & \\ \leq \; Y'(T_1) + \tilde{c}_2 Y(T_1) - c_3 Y(T_1 - 2a)^{1/2} Y(T_1)^{1/2} & \\ \leq \; \text{const}\left[\int_{T_1}^\infty \Phi^q dt + \int_{T_1}^\infty \Psi^2 dt\right]. & \end{aligned} \tag{3.20}$$

Since the real part of the roots of the characteristic equation

$$\lambda + (\tilde{c}_2 - c_3/2) - (c_3/2)\mathrm{e}^{-2\lambda a} = 0$$

is negative, we obtain for the solution the inequality (3.15). □

**Example 3.1.** Consider examples of the following type:

$$\begin{aligned} a_i(t,x,\zeta_0,\zeta;u) &= b(t,x,[H(u)](t,x))\zeta_i|\zeta|^{p-2}, \quad i = 1,...,n, \\ a_0^0(t,x,\zeta_0,\zeta;u) &= b_0(t,x,[H_0(u)](t,x))\zeta_0|\zeta_0|^{p-2} + c_2\zeta_0, \quad c_2 \geq 0 \end{aligned}$$

where $b, b_0$ are bounded Carathéodory functions satisfying with some positive constant $c_3$

$$b(t,x,\theta) \geq \frac{c_3}{1+|\theta|^{\sigma^\star}}, \qquad b_0(t,x,\theta) \geq \frac{c_3}{1+|\theta|^{\sigma^\star}};$$

$$a_0^j(t,x,\zeta_0,\zeta;u) = b_0^j(t,x,[F_j(u)](t,x))\alpha_0^j(t,x,\zeta_0,\zeta), \qquad j = 1,2 \tag{3.21}$$

(or $a_0^j$ is a sum of such products), where functions $\alpha_0^j, b_0^j$ satisfy

$$|\alpha_0^j(t,x,\zeta_0,\zeta)| \leq \text{const}[1 + |\zeta_0|^{\tilde{\rho}_j} + |\zeta|^{\tilde{\rho}_j}], \qquad |b_0^j(t,x,\theta)|^{q_1^j} \leq \text{const}(1+|\theta|^2).$$

Finally,

$$H, H_0 : L^2(Q_T) \to C(\overline{Q_T}), \quad F_j : L^2(Q_T) \to L^2(Q_T)$$

are continuous operators of Volterra type, satisfying

$$\| H(u) \|_{C(\overline{Q_t})} \leq \text{const} \| u \|_{L^2(Q_t)}, \quad \| H_0(u) \|_{C(\overline{Q_t})} \leq \text{const} \| u \|_{L^2(Q_t)},$$

$$\int_{Q_t} |F_j(u)|^2 \leq \text{const}\left(\int_{Q_t} |u|^2\right)^{\sigma/2}, \quad t > 0.$$

It is not difficult to show that the conditions of the existence Theorem 2.2 are fulfilled. If the above conditions hold for all $T > 0$ and $t > 0$ then the conditions of Theorem 2.3 are satisfied.

Further, assumptions of Theorem 3.1 are fulfilled if the following additional assumptions are satisfied. Assumption (3.1) is satisfied if

$$\begin{aligned} \| H(u) \|_{C(\overline{Q_t})} &\leq \text{const} \sup_{\tau\in[0,t]} \left[\int_\Omega u^2(\tau,x)dx\right]^{1/2}, \\ \| H_0(u) \|_{C(\overline{Q_t})} &\leq \text{const} \sup_{\tau\in[0,t]} \left[\int_\Omega u^2(\tau,x)dx\right]^{1/2} \qquad t > 0, \end{aligned}$$

(3.2) is satisfied if

$$\int_\Omega [F_1(u)]^2(t,x)dx \le \text{const}\left(\int_\Omega u^2(t,x)dx\right)^{\tilde\sigma/2} \quad \text{for all } t>0,$$

(3.3) is satisfied if for all $t > 0$

$$\begin{aligned}\int_\Omega |F_1(u)|^2(t,x)dx \quad &\le \quad \text{const}\left\{1+\left[\sup_{\tau\in[0,t]}\int_\Omega u^2(t,x)dx\right]^{\sigma/2}\right\} \\ &+\varphi(t)\left[\sup_{\tau\in[0,t]}\int_\Omega u^2(t,x)dx\right]^{(p-\sigma^\star)/2}.\end{aligned}$$

Inequality (3.4) is satisfied if

$$|b_0^2(t,x,\theta)| \le \text{const}.$$

Finally, the assumptions of Theorem 3.2 are fulfilled if the following additional conditions are satisfied for our example. $c_2 > 0$, there exist measurable functions $b_\infty, b_{0,\infty}$ such that for all fixed $u \in L^p_{loc}(0,\infty;V)\cap L^\infty(0,\infty;L^2(\Omega))$, with $D_t u \in L^q_{loc}(0,\infty;V^\star)$

$$|b(t,x,[H(u)](t,x)) - b_\infty(x)| \le \Phi(t), \quad |b_0(t,x,[H(u)](t,x)) - b_{0,\infty}(x)| \le \Phi(t),$$

$$|b_0^2(t,x,\theta)| \le \Phi(t).$$

Functions $b, b_0$ may have the form

$$b(t,x,\theta) = \frac{b_\infty(x)}{1+\Phi(t)|\theta|^{\sigma^\star}}, \quad b_0(t,x,\theta) = \frac{b_{0,\infty}(x)}{1+\Phi(t)|\theta|^{\sigma^\star}}$$

where $b_\infty, b_{0,\infty}$ are measurable functions having values between two positive constants. Further,

$$a_0^1(t,x,\zeta_0,\zeta;u) = b_{0,\infty}^1(x,F_1(u)) + \beta(t,\zeta_0,\zeta), \tag{3.22}$$

where

$$|\beta(t,\zeta_0,\zeta)| \le \Psi(t)(1+|\zeta_0|),$$

the Carathéodory function $b_{0,\infty}^1$ satisfies the Lipschitz condition

$$|b_{0,\infty}^1(x,\theta) - b_{0,\infty}^1(x,\theta^\star)| \le \tilde c_3|\theta-\theta^\star|$$

and the operator $F_1$ satisfies

$$\int_{T_1}^{T_2}\int_\Omega |F_1(u)-F_1(u^\star)|^2 dtdx \le c_0\hat c_3^2 \int_{\max\{0,T_1-a\}}^{T_2}\int_\Omega |u-u^\star|^2 dtdx, \quad c_2 > \tilde c_3\hat c_3.$$

(In this case $a_0^1$ is a sum of two products of the form (3.21).)

A simple example satisfying the conditions of Theorem 3.1 is

$$D_t u - \triangle_p u + |u|^{p-2}u + c_2 u + b_0^1(t, x, u([\gamma_1(u)](t), x)) + b_0^2(t, x, Du([\gamma_2(u)](t), x)) = f$$

where $\triangle_p$ is the $p$-Laplacian, defined by $\triangle_p u = \sum_{j=1}^n D_j(|Du|^{p-2} D_j u)$.

If the fourth term is given by (3.22) and $|b_0^2(t, x, \theta)| \leq \Psi(t)$ then Theorem 3.2 holds.

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