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An affirmative answer to "Gopalsamy and Liu's conjecture" on population models with multiple piecewise constant arguments

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Abstract. Consider the following differential equation with piecewise constant arguments:

$$\begin{cases} \frac{dN(t)}{dt} &= N(t)r(t)\left\{1 - aN(t) - \sum_{i=0}^{m} b_i N(n-i)\right\}, n \le t < n+1, n = 0, 1, 2, \dots, \\ N(0) &= N_0 > 0, \text{ and } N(-j) = N_{-j} \ge 0, j = 1, 2, \dots, m, \end{cases}$$

where $r(t)$ is a nonnegative continuous function on $[0, +\infty)$, $r(t) \not\equiv 0, a > 0, \sum_{i=0}^{m} b_i > 0$, and $b_i \ge 0, i = 0, 1, 2, \dots, m$. In this paper, we show that if $r(t) \equiv r$ (constant) and $(\sum_{i=1}^{m} b_i)/(\sum_{i=0}^{m} b_i)$ is sufficiently small, then Gopalsamy and Liu's criterion $r \le \frac{1+\alpha}{\alpha} \ln \frac{1+\alpha}{1-\alpha}$ of the global stability for $m = 0$ and $0 < \alpha = a/b_0 < 1$ still holds for any $m \ge 1$ and $0 < \alpha = a/(\sum_{i=0}^{m} b_i) < 1$. This generalizes the result in [G. Seifert, Certain systems with piecewise constant feedback controls with a time delay, Differential Integral Equations 6 (4) (1993) 937-947], that is, in the special case $a = 0$ and $m = 1$, there exists a constant $0 < \beta < 1$ such that for any $b_1/(b_0 + b_1) \le \beta$,

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the positive equilibrium $N^* = 1/(b_0 + b_1)$ of the above equation is global attractor if $r < 2$ to $0 < \alpha = a/(\sum_{i=0}^m b_i) < 1$ and $m \geq 1$.

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1. Introduction

Consider the following delay differential equation with piecewise constant delays for $m \geq 0$:

$$\begin{cases} \frac{dN(t)}{dt} = N(t)r(t)\left\{1 - aN(t) - \sum_{i=0}^m b_i N(n-i)\right\}, & n \leq t < n+1, \ n = 0, 1, 2, \ldots, \\ N(0) = N_0 > 0, \ \text{and} \ N(-j) = N_{-j} \geq 0, \ j = 1, 2, \ldots, m, \ \text{if} \ m \geq 1, \end{cases} \tag{1.1}$$

where $r(t)$ is a nonnegative continuous function on $[0, +\infty)$, $r(t) \not\equiv 0$, $\sum_{i=0}^m b_i > 0, b_i \geq 0, i = 0, 1, 2, \ldots, m$.

This population model has been studied in many literature (see for example, [1-14] and references therein). Concerning conditions for the positive equilibrium N^* of Eq.(1.1) with $a = 0$, to be globally asymptotically stable, Gopalsamy, Kulenovic and Ladas [1] have obtained $r < \frac{\ln 2}{m+1}$ for $r(t) \equiv r$ (constant), and So and Yu [10] improved this condition to $\int_0^\infty r(t)dt = +\infty$ and $\sup_{n\geq 0} \int_{n-m}^{n+1} r(t)dt \leq 3/2$.

For the case $m = 0$, $r(t) \equiv r$ (constant) and $0 < a < b_0$, Gopalsamy and Liu [2] offered a conjecture of the necessary and sufficient condition of the global asymptotic stability. Muroya and Kato [7] partially solved this conjecture. Recently, Li and Yuan [4] have solved completely this and Li, Muroya and Yuan [3] extended this to the variable case $r = r(t)$.

The following result is an affirmative answer to the Gopalsamy and Liu's conjecture for Eq.(1.1) with $m = 0$.

Theorem A. *(See [4, 7]). For $0 < \alpha = a/b_0 < 1$, the positive equilibrium $N^* = \frac{1}{a+b_0}$ of Eq.(1.1) with $m = 0$ and $r(t) \equiv r$ (constant), is globally asymptotic stable, if and only if,*

$$r \leq \hat{\bar{r}}(\alpha), \tag{1.2}$$

where

$$\hat{\bar{r}}(\alpha) = \frac{1+\alpha}{\alpha} \ln \frac{1+\alpha}{1-\alpha}, \quad \textit{for} \ \ 0 < \alpha < 1, \ \textit{and} \ \ \hat{\bar{r}}(0) = 2. \tag{1.3}$$

On the other hand, Seifert [9] has studied the local stability, global attractivity and the existence of the 2-periodic solution of the following logistic equation with piecewise constant delays

$$\frac{dN(t)}{dt} = N(t)r\left(1 - b_0 N(n) - b_1 N(n-1)\right), \quad n \leq t < n+1, \ n = 0, 1, 2, \ldots, \tag{1.4}$$

Eq.(1.4) is equivalent to Eq.(1.1) for $a = 0$, $r(t) \equiv r$ (constant) and $m = 1$. Seifert [9] has obtained the following theorem.

Theorem B. *(See [9, Theorem 3.4]). There exists a constant* $0 < \beta < 1$ *such that for any* $b_1/(b_0 + b_1) \leq \beta$*, the positive equilibrium* $N^* = 1/(b_0 + b_1)$ *of Eq.(1.4) is global attractor, if* $r < 2$.

We know that for Eq.(1.4) with $b_1 = 0$, the necessary and sufficient condition of the global stability is $r \leq \hat{\overline{r}}(0) = 2$ (See also, Matsunaga et al. [5]). Hence, Theorem B suggests us that the condition (1.2) for $m = 0$ and $\alpha = 0$ ($a = 0$) still guarantee the global stability of Eq.(1.4) even if there exists the effect of delay, b_1 (at least b_1 is sufficiently small). Motivated this result, Uesugi, Muroya and Ishiwata [11] showed that for any $(\sum_{i=1}^m b_i)/(\sum_{i=0}^m b_i) \leq \mathrm{e}/(\mathrm{e}+2)$, the positive equilibrium $N^* = 1/(\sum_{i=0}^m b_i)$ of Eq.(1.1) for $r(t) \equiv r$ (constant) and $a = 0$, is globally asymptotically stable, if $r \leq \hat{\overline{r}}(0) = 2$.

However, for the global asymptotic stability of Eq.(1.1) for the case $m \geq 1$ and $0 < \alpha = a/(\sum_{i=0}^m b_i) < 1$, only contractive conditions were established (see Muroya [6] and Nakata, Kuroda and Muroya [8]). How to extend the result in Uesugi, Muroya and Ishiwata [11] for $\alpha = 0$ to $0 < \alpha < 1$ is still an open problem. Motivated by Theorem B for $\alpha = 0$ and $m = 1$, we have a conjecture that for the case $m \geq 1$ and $0 < \alpha < 1$, there exists a sufficiently small constant $\overline{\beta}(\alpha) > 0$ such that for any $(\sum_{i=1}^m b_i)/(\sum_{i=0}^m b_i) \leq \overline{\beta}(\alpha)$, the positive equilibrium $N^* = 1/(a + \sum_{i=0}^m b_i)$ of Eq.(1.1) is globally asymptotically stable, if $r \leq \hat{\overline{r}}(\alpha)$ for $0 < \alpha < 1$.

In this paper, we establish the following affirmative answer to the above conjecture for the case $m \geq 1$ and $0 < \alpha < 1$.

Theorem 1.1. *For* $0 < \alpha = a/\left(\sum_{i=0}^m b_i\right) < 1$*, there exists a constant* $0 < \overline{\beta}(\alpha) < 1$ *such that for any* $\left(\sum_{i=1}^m b_i\right) / \left(\sum_{i=0}^m b_i\right) \leq \overline{\beta}(\alpha)$*, the positive equilibrium* N^* *of Eq.(1.1) for* $r(t) \equiv r$ *(constant), is globally asymptotically stable, if* $r \leq \hat{\overline{r}}(\alpha)$.

Similarly, we obtain the following result for nonautonomous case Eq.(1.1).

Theorem 1.2. *For* $0 < \alpha = a/\left(\sum_{i=0}^m b_i\right) < 1$*, there exists a constant* $0 < \overline{\beta}(\alpha) < 1$ *such that for any* $\left(\sum_{i=1}^m b_i\right) / \left(\sum_{i=0}^m b_i\right) \leq \overline{\beta}(\alpha)$*, the positive equilibrium* N^* *of Eq.(1.1) is globally asymptotically stable, if* $\limsup_{n\to\infty} r_n > 0$ *and* $r_n \leq \hat{\overline{r}}(\alpha)$ *where* $r_n = \int_n^{n+1} r(t)dt$.

The organization of this paper is as follows. In Section 2, for preparations, we introduce some basic results. In Section 3, we offer two more results (Lemmas 3.2 and 3.4) from results in Li and Yuan [4] and Muroya and Kato [7] for Eq.(1.1) with $m = 0$. Applying these results for $m \geq 1$ in Section 4, we prove Theorems 1.1 and 1.2. Finally, in Section 5, numerical simulations are presented. These may be some supports for the existence of $\overline{\beta}(\alpha)$ in Theorem 1.1.

2. Preliminaries

In this section, we prepare some known results and important lemmas which are related to Section 3.

At first, by Gopalsamy and Liu [2], we see that for Eq.(1.1),

$$N(t) = N(n)\exp\left\{\int_n^t r(s)\left(1 - aN(s) - \sum_{i=0}^m b_i N(n-i)\right)ds\right\}, \quad n \le t < n+1,$$

$n = 0, 1, 2, \ldots$, and so $N(t) > 0$ for all $t > 0$. An easy computation yields that for $t \in [n, n+1)$,

$$\frac{d}{dt}\left[\frac{1}{N(t)}\exp\left(\int_n^t r(s)ds\left(1 - \sum_{i=0}^m b_i N(n-i)\right)\right)\right] = ar(t)\exp\left(\int_n^t r(s)ds\left(1 - \sum_{i=0}^m b_i N(n-i)\right)\right).$$

Put

$$r_n = \int_n^{n+1} r(t)dt, \quad t_n = 1 - \sum_{i=0}^m b_i N(n-i) \quad \text{and} \quad N^* = \frac{1}{a + \sum_{i=0}^m b_i}.$$

We introduce the following results in Muroya [6]. In particular, the discretized equations Eqs.(2.1) and (2.2) are important throughout this paper.

Lemma 2.1. *(See [6, Lemma 3.1]). If*

$$1 + aN(n)\frac{\exp\{r_n^t t_n\} - 1}{t_n} > 0, \quad \textit{for} \ \ t_n \ne 0,$$

and

$$1 + aN(n)r_n^t > 0, \quad \textit{for} \ \ t_n = 0,$$

then we have for $n \le t < n+1$,

$$N(t) = \begin{cases} \frac{N(n)\exp\{r_n^t t_n\}}{1 + aN(n)\frac{\exp\{r_n^t t_n\}-1}{t_n}}, & \textit{for } t_n \ne 0, \\ \frac{N(n)}{1 + aN(n)r_n^t}, & \textit{for } t_n = 0, \end{cases}$$

and

$$N(t) - N^* = \begin{cases} \frac{1 - b_0 N(n)\frac{\exp\{r_n^t t_n\}-1}{t_n}}{1 + aN(n)\frac{\exp\{r_n^t t_n\}-1}{t_n}}(N(n) - N^*) - \sum_{i=1}^m \frac{b_i N(n)\frac{\exp\{r_n^t t_n\}-1}{t_n}}{1 + aN(n)\frac{\exp\{r_n^t t_n\}-1}{t_n}}(N(n-i) - N^*), \\ \qquad\qquad \textit{for } t_n \ne 0, \\ \frac{1 - b_0 N(n) r_n^t}{1 + aN(n)r_n^t}(N(n) - N^*) - \sum_{i=1}^m \frac{b_i N(n) r_n^t}{1 + aN(n)r_n^t}(N(n-i) - N^*), \\ \qquad\qquad \textit{for } t_n = 0, \end{cases}$$

where $r_n^t = \int_n^t r(s)ds$.

In particular,

$$N(n+1) = \begin{cases} \frac{N(n)\exp\{r_n t_n\}}{1+aN(n)\frac{\exp\{r_n t_n\}-1}{t_n}}, & \text{for } t_n \neq 0, \\ \frac{N(n)}{1+aN(n)r_n}, & \text{for } t_n = 0, \end{cases} \tag{2.1}$$

and

$$\begin{aligned} &N(n+1) - N^* \\ &= \begin{cases} \frac{1-b_0 N(n)\frac{\exp\{r_n t_n\}-1}{t_n}}{1+aN(n)\frac{\exp\{r_n t_n\}-1}{t_n}}(N(n)-N^*) - \sum_{i=1}^m \frac{b_i N(n)\frac{\exp\{r_n t_n\}-1}{t_n}}{1+aN(n)\frac{\exp\{r_n t_n\}-1}{t_n}}(N(n-i)-N^*), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t_n \neq 0, \\ \frac{1-b_0 N(n) r_n}{1+aN(n)r_n}(N(n)-N^*) - \sum_{i=1}^m \frac{b_i N(n) r_n}{1+aN(n)r_n}(N(n-i)-N^*), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t_n = 0. \end{cases} \end{aligned} \tag{2.2}$$

For the case $a \geq \sum_{i=0}^m b_i > 0$, we easily get the following global stability result.

Theorem 2.1. *(See [6, Theorem 3.1]). If* $0 < r_n < +\infty$ *and*

$$a \geq \sum_{i=0}^m b_i > 0,$$

then solutions of Eq.(1.1) have the contractivity, that is,

$$|N(n+1) - N^*| \leq \max_{0 \leq i \leq m} |N(n-i) - N^*|.$$

Moreover, if

$$\limsup_{n\to\infty} r_n > 0,$$

then

$$\lim_{n\to\infty} N(n) = N^*,$$

and hence, the positive equilibrium $N^* = 1/(a+\sum_{i=0}^m b_i)$ *of Eq.(1.1) is globally asymptotically stable.*

Hereafter in this section, we are interested in the case $0 < a < \sum_{i=0}^m b_i$. Note that if $r_n = 0$, then $N(n+1) = N(n)$. Hence, for simplicity, we assume $r_n > 0$ and put

$$\begin{cases} f(t;r) &= \begin{cases} (1-t)\frac{e^{rt}-1}{t}, & t \neq 0, \\ r, & t = 0, \end{cases} \\ \tilde{f}(t;r) &= \begin{cases} \frac{e^{rt}-1}{t}, & t \neq 0, \\ r, & t = 0. \end{cases} \end{cases} \tag{2.3}$$

Note that $f(t;r_1) \le f(t;r_2)$ and $\tilde{f}(t;r_1) \le \tilde{f}(t;r_2)$ for any $0 < r_1 \le r_2$ and $t < 1$ (see the proofs of Muroya [6, Lemma 2.3]). Moreover we set

$$\begin{cases} x(n) & = \left(\sum_{i=0}^{m} b_i\right) N(n), \\ x^* & = \left(\sum_{i=0}^{m} b_i\right) N^* = \frac{1}{1+\alpha}, \\ \alpha & = \frac{a}{\sum_{i=0}^{m} b_i} > 0, \\ a_i & = \frac{b_i}{\sum_{i=0}^{m} b_i} \ge 0, \text{ for } 0 \le i \le m, \end{cases}$$

then for $t_n = 1 - \sum_{i=0}^{m} b_i N(n-i) = 1 - \sum_{i=0}^{m} a_i x(n-i)$, Eqs.(2.1) and (2.2) become respectively

$$x(n+1) = \frac{x(n)\exp\{r_n t_n\}}{1 + \alpha x(n)\tilde{f}(t_n;r_n)}, \tag{2.4}$$

and

$$x(n+1) - x^* = \frac{1 - a_0 x(n)\tilde{f}(t_n;r_n)}{1 + \alpha x(n)\tilde{f}(t_n;r_n)}(x(n) - x^*) - \sum_{i=1}^{m} \frac{a_i x(n)\tilde{f}(t_n;r_n)}{1 + \alpha x(n)\tilde{f}(t_n;r_n)}(x(n-i) - x^*). \tag{2.5}$$

We introduce the following relation between $f(t;r)$ and $\hat{\bar{r}}(\alpha)$.

Lemma 2.2. *(See [7, Lemma 2.4]).* $\hat{\bar{r}}(\alpha)$ *is a strictly monotone increasing continuous function of* α *on the interval* $(-1,1)$*, and for* $0 < \alpha < 1$*, it holds that*

$$\begin{cases} f(t^*;\hat{\bar{r}}(\alpha)) = f(t^{**};\hat{\bar{r}}(\alpha)) = \frac{2}{1-\alpha}, & \\ f(t;\hat{\bar{r}}(\alpha)) > \frac{2}{1-\alpha}, & \textit{for } (t-t^*)(t-t^{**}) < 0, \\ f(t;\hat{\bar{r}}(\alpha)) < \frac{2}{1-\alpha}, & \textit{otherwise}, \end{cases} \tag{2.6}$$

and

$$\begin{cases} f'(t^*;\hat{\bar{r}}(\alpha)) & = \frac{(1+\alpha)^2\left(\frac{1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}\right)}{(1-\alpha)\alpha} > 0, \\ f''(t^*;\hat{\bar{r}}(\alpha)) & = \frac{(1+\alpha)^3\left(\frac{1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}-2\right)\left(\ln\frac{1+\alpha}{1-\alpha}-2\right)}{(1-\alpha)\alpha^2}, \\ f'(t^{**};\hat{\bar{r}}(\alpha)) & = \frac{(1+\alpha)^2}{2\alpha(1-\alpha)}\left(\frac{1+\alpha}{\alpha}\ln\frac{1+\alpha}{1-\alpha} - \frac{2}{1-\alpha}\right) < 0, \end{cases} \tag{2.7}$$

where

$$t^* = 1 - x^* = \frac{\alpha}{1+\alpha}, \qquad \textit{and} \;\; t^{**} = 2t^* = \frac{2\alpha}{1+\alpha}. \tag{2.8}$$

Further, for any $r \le \hat{\bar{r}}(\alpha)$ *and* $0 < \alpha < 1$*,*

$$1 + \alpha f(t;r) > 0 \quad \textit{for any} \;\; t < 1.$$

Note that $0 < \hat{\bar{r}}(\alpha) < +\infty$ and, $0 < t^* < t^{**} < 1$ for $0 < \alpha < 1$. Figure 1 illustrates the function $\hat{\bar{r}}(\alpha)$. Also, we draw the graphs of $f(t;\hat{\bar{r}}(\alpha))$ for $\alpha = 0, 0.2, 0.5$ and 0.8 in Section 5.

The following relation is also used in Section 3.

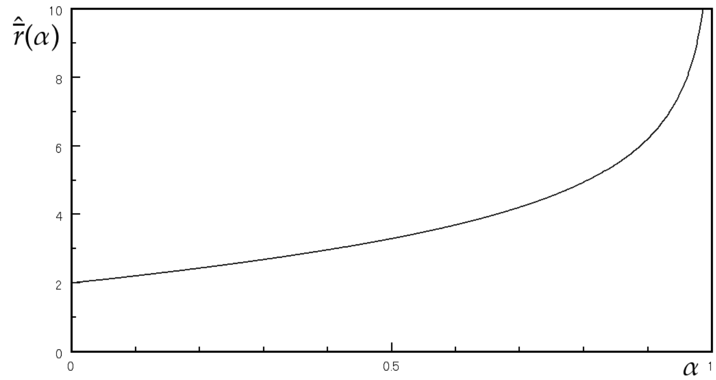

Figure 1: Graph of the function $\hat{\bar{r}}(\alpha)$

Lemma 2.3. *For $r > 0$, it holds that*

$$f''(t;r) < rf'(t;r) \quad \textit{for} \;\; t > 0. \tag{2.9}$$

Proof. From (2.3), we have that for $r > 0$ and $t > 0$

$$\begin{cases} tf(t;r) & = t - 1 + (1-t)\mathrm{e}^{rt}, \\ tf'(t;r) + f(t;r) & = 1 + \{-1 + (1-t)r\}\mathrm{e}^{rt}\ , \\ tf''(t;r) + 2f'(t;r) & = \{-2r + (1-t)r^2\}\mathrm{e}^{rt}. \end{cases}$$

It follows that

$$\begin{cases} t^2 f'(t;r) & = -\{t - 1 + (1-t)\mathrm{e}^{rt}\} + t[1 + \{-1 + (1-t)r\}\mathrm{e}^{rt}] \\ & = 1 - (1 - rt + rt^2)\mathrm{e}^{rt}, \\ t^3 f''(t;r) & = -2\{1 - (1 - rt + rt^2)\mathrm{e}^{rt}\} + t^2\{-2r + (1-t)r^2\}\mathrm{e}^{rt} \\ & = -2 + (2 - 2rt + r^2t^2 - r^2t^3)\mathrm{e}^{rt}, \end{cases}$$

from which, we obtain that

$$\begin{aligned} t^3 f''(t;r) &= -(2 + rt) + (2 - rt)\mathrm{e}^{rt} + rt^3 f'(t;r) \\ &< rt^3 f'(t;r) \text{ for } t > 0. \end{aligned}$$

In the above discussion, we use the following inequality

$$\begin{aligned}
-(2+y)+(2-y)\mathrm{e}^y &= -(2+y)+(2-y)\sum_{k=0}^{+\infty}\frac{y^k}{k!} \\
&= -(2+y)+(2-y)+(2-y)y+(2-y)\frac{y^2}{2!}+(2-y)\sum_{k=3}^{+\infty}\frac{y^k}{k!} \\
&= -\frac{y^3}{2!}+\frac{(2-y)y^3}{3!}+\frac{(2-y)y^4}{4!}+\frac{(2-y)y^5}{5!}+\dots \\
&= \left(\frac{2}{3!}-\frac{1}{2!}\right)y^3+\left(\frac{2}{4!}-\frac{1}{3!}\right)y^4+\left(\frac{2}{5!}-\frac{1}{4!}\right)y^5+\dots \\
&< 0 \text{ for } y>0.
\end{aligned}$$

Hence, Eq.(2.9) holds and the proof is complete. □

3. More results for $m = 0$

In this section, we offer two more results (Lemmas 3.2 and 3.4) from the known results in Li and Yuan [4] and Muroya and Kato [7] for the case $m = 0$. For simplicity, we consider the special case $r_n = \hat{\bar{r}}(\alpha), n = 0, 1, 2, \dots,$ for Eqs.(2.1) and (2.2). Eqs.(2.1) and (2.2) become

$$x(n+1) = \frac{x(n)\exp\left\{\hat{\bar{r}}(\alpha)t_n\right\}}{1+\alpha f(t_n;\hat{\bar{r}}(\alpha))}, \tag{3.1}$$

and

$$x(n+1)-x^* = \frac{1-f(t_n;\hat{\bar{r}}(\alpha))}{1+\alpha f(t_n;\hat{\bar{r}}(\alpha))}(x(n)-x^*), \tag{3.2}$$

where $t_n = 1-x(n)$, respectively. We put

$$\begin{cases} f(t) = f(t;\hat{\bar{r}}(\alpha)), \\ G(t) = F(t)(t-t^*)+t^* \text{ and } F(t) = \frac{1-f(t)}{1+\alpha f(t)}. \end{cases} \tag{3.3}$$

Then, Eq.(2.6) implies

$$\begin{cases} f(t^*) = f(t^{**}) = \frac{2}{1-\alpha}, & \\ f(t) > \frac{2}{1-\alpha}, & \text{for } 0 < t^* < t < t^{**} < 1, \\ f(t) < \frac{2}{1-\alpha}, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{cases} F(t^*) = F(t^{**}) = -1, \text{ and } F(t) < -1 & \text{for } 0 < t^* < t < t^{**}, \\ G(t^*) = t^*, G(t^{**}) = 0, \text{ and } G(t) < t^* & \text{for } 0 < t^* < t < t^{**}, \end{cases} \tag{3.4}$$

and
$$t^* = \frac{\alpha}{1+\alpha},$$
is a unique solution of $t = G(t)$ for $t < 1$. For the convenience of the reader, we draw the graphs of $f(t;\hat{\bar{r}}(\alpha))$ and $F(t)$ for $\alpha = 0, 0.2, 0.5$ and 0.8 in Section 5 (see Figures 5-12).

Now, for $t_n = 1 - x(n) < 1$, Eq.(3.2) is equivalent to
$$t_{n+1} = G(t_n), n = 0, 1, 2, \ldots.$$
By definitions, we have that $t_n - t^* = -(x(n) - x^*)$ and
$$\begin{cases} t_{n+1} - t^* = F(t_n)(t_n - t^*), \\ t_{n+2} - t^* = F(t_{n+1})(t_{n+1} - t^*) = F(G(t_n))F(t_n)(t_n - t^*). \end{cases} \tag{3.5}$$
Similar to Muroya and Kato [7, Lemmas 2.5-2.10], from Eq.(3.5), we obtain the following four lemmas.

Lemma 3.1. *(See [7, Lemmas 2.5]). a) Assume that for $t_n < 1$,*
$$|F(t_n)| < 1, \quad \textit{for} \ \ t_n \neq t^*.$$
Then,
$$|t_{n+1} - t^*| < |t_n - t^*|.$$
b) Suppose that for $t_n < 1$,
$$|F(G(t_n))F(t_n)| < 1, \quad \textit{for} \ \ t_n \neq t^*.$$
Then,
$$|t_{n+2} - t^*| < |t_n - t^*|.$$

Note that
$$f(t) > f(t^*) = \frac{2}{1-\alpha}, \quad \text{for} \ \ 0 < t^* < t < t^{**} < 1.$$
By Lemma 2.2, Eq.(2.6) and Lemma 2.3, there exists a unique $\bar{t}_1 = \bar{t}_1(\hat{\bar{r}}(\alpha))$ such that
$$f'(\bar{t}_1) = 0, \quad \text{for} \ \ 0 < t^* < \bar{t}_1 < t^{**} < 1.$$
Moreover, by Lemma 2.2, we can improve the result of Muroya and Kato [7, Lemma 2.10] as follows.

Lemma 3.2. *For $0 < \alpha < 1$, there exist two constants c_1 and c_2 such that*
$$c_1 < t^* < t^{**} < c_2 < 1, \tag{3.6}$$
and
$$\begin{cases} G'(t) < 0, \quad \textit{for } c_1 < t < c_2, \\ \textit{in particular, } G'(t) < -1, \quad \textit{for } t^* < t \leq \bar{t}_1. \end{cases} \tag{3.7}$$

Proof. By Eq.(2.6) in Lemma 2.2, we have $f'(\bar{t}_1) = 0$. Moreover, we see that $f''(t) < rf'(t) \leq 0$ for any $\bar{t}_1 \leq t \leq t^{**}$ by Eq.(2.9) in Lemma 2.3. By Eq.(2.6), we easily get

$$f(t) \geq f(t^{**}), \quad \text{and } \ 0 \geq f'(t) \geq f'(t^{**}), \quad \text{for } \ \bar{t}_1 \leq t \leq t^{**}.$$

Therefore, by Eq.(3.3) and (3.4), for $0 < \alpha < 1$ and $\bar{t}_1 \leq t \leq t^{**}$, it holds that

$$\begin{cases} 0 \leq F'(t) = -\frac{(1+\alpha)f'(t)}{(1+\alpha f(t))^2} \leq -\frac{(1+\alpha)f'(t^{**})}{(1+\alpha f(t^{**}))^2} = F'(t^{**}) = -\frac{(1-\alpha)^2}{1+\alpha} f'(t^{**}), \\ \text{and } F(t) \leq F(t^{**}) = -1. \end{cases}$$

On the other hand, it holds that $f'(t) \geq 0$ for $t^* < t \leq \bar{t}_1$, and hence,

$$F'(t) = -\frac{(1+\alpha)f'(t)}{(1+\alpha f(t))^2} \leq 0,$$

and

$$F(t) < F(t^*) = F(t^{**}) = -1.$$

Therefore, by Eqs.(3.3) and (2.7), we have that

$$\begin{aligned} G'(t) &= F'(t)(t-t^*) + F(t) \\ &\leq F'(t^{**})(t^{**}-t^*) + F(t^{**}) \\ &= G'(t^{**}) \\ &= \alpha \left(\frac{1-\alpha}{1+\alpha} \right)^2 (-f'(t^{**})) - 1 \\ &= -\frac{(1-\alpha)(1+\alpha)}{2\alpha} \ln \frac{1+\alpha}{1-\alpha} < 0 \quad \text{for } \ t^* < t \leq t^{**}, \end{aligned}$$

and in particular,

$$G'(t) \leq F'(t)(t-t^*) + F(t) < -1 \quad \text{for } \ t^* < t \leq \bar{t}_1.$$

Moreover, by $G'(1) > 0, G'(t^*) = -1$ and $G''(t^{**}) < 0$ and the above discussion, we can see that there exist two constants c_1 and c_2 such that (3.6) and (3.7) hold. □

Moreover, we have the following result (see Li and Yuan [4] and Muroya and Kato [7]).

Lemma 3.3. *For* $0 < \alpha < 1$*, it holds that*

$$F(G(t))F(t) < 1, \quad \textit{for } \ t^* < t \leq t^{**}.$$

By Lemmas 3.2, 3.3 and the continuity of the function $F(G(t))F(t)$, we have the following lemma which will be used to prove Theorem 1.1.

Lemma 3.4. *For* $0 < \alpha < 1$*, it holds that there exist two constants* c_1 *and* c_2 *such that (3.6) and (3.7) hold and*

$$F(G(t))F(t) < 1, \quad \textit{for } \ c_1 < t < c_2. \tag{3.8}$$

4. Proofs of Theorems 1.1 and 1.2

Now, consider the sequence $\{x(n)\}_{n=0}^{\infty}$ of Eq.(2.5) for $r = r_n = \hat{\bar{r}}(\alpha), n = 0, 1, 2, \ldots$. Put $\bar{t}_n = 1 - x(n), n = 0, 1, 2, \ldots$ and

$$\begin{cases} q(s_0, s_1; r) = \frac{1-a_0(1-s_0)\tilde{f}(a_0s_0+(1-a_0)s_1;r)}{1+\alpha(1-s_0)\tilde{f}(a_0s_0+(1-a_0)s_1;r)}, \\ \tilde{q}(s_0, s_1; r) = \frac{(1-a_0)(1-s_0)\tilde{f}(a_0s_0+(1-a_0)s_1;r)}{1+\alpha(1-s_0)\tilde{f}(a_0s_0+(1-a_0)s_1;r)}. \end{cases}$$

Hereafter, we restrict our attention to the case $m \geq 1$ and $\sum_{i=1}^{m} a_i > 0$, because the case $m = 0$ (and $\sum_{i=1}^{m} a_i = 0$) is already known by [4, 7]. Then, by Eq.(2.5), the sequence $\{\bar{t}_n\}_{n=0}^{\infty}$ satisfies the following equations:

$$\begin{cases} \bar{t}_{n+1} - t^* = q(\bar{t}_n, \tilde{t}_n; r)(\bar{t}_n - t^*) - \tilde{q}(t_n, \tilde{t}_n; r)(\tilde{t}_n - t^*), \\ \tilde{t}_n \equiv \left(\sum_{i=1}^{m} a_i t_{n-i}\right) / \left(\sum_{i=1}^{m} a_i\right), \quad n = 0, 1, 2, \ldots, \end{cases}$$

where $\sum_{i=1}^{m} a_i > 0$. It follows that $q(s_0, s_1; r) \in C^1(R^2)$ and $\tilde{q}(s_0, s_1; r) \in C^1(R^2)$ for any $(s_0, s_1) \in R^2$. Put

$$G_2(s_0, s_1; r) \equiv t^* + q(s_0, s_1; r)(s_0 - t^*) - \tilde{q}(s_0, s_1; r)(s_1 - t^*). \tag{4.1}$$

From Eq.(3.3), it holds that

$$G(s_0) - t^* = F(s_0)(s_0 - t^*). \tag{4.2}$$

Then, we have

$$\begin{aligned} G_2(s_0, s_1; r) \quad = \quad & G(s_0) + [(q(s_0, s_1; r) - \tilde{q}(s_0, s_1; r)) - F(s_0)]\,(s_0 - t^*) \\ & -\tilde{q}(s_0; s_1; r)(s_1 - s_0), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &(q(s_0, s_1; r) - \tilde{q}(s_0, s_1; r)) - F(s_0) \\ &\quad = -\frac{(1+\alpha)(1-s_0)(\tilde{f}(a_0s_0 + (1-a_0)s_1; r) - \tilde{f}(s_0; r))}{(1+\alpha(1-s_0)\tilde{f}(a_0s_0 + (1-a_0)s_1; r))(1+\alpha(1-s_0)\tilde{f}(s_0; r))} \\ &\quad\quad - (1-a_0)\frac{(1+\alpha)(1-s_0)}{(1+\alpha(1-s_0)\tilde{f}(s_0 + (1-a_0)(s_1 - s_0); r))(1+\alpha(1-s_0)\tilde{f}(s_0; r))} \\ &\quad\quad \times \frac{\tilde{f}(s_0 + (1-a_0)(s_1 - s_0); r) - \tilde{f}(s_0; r)}{1-a_0}. \end{aligned}$$

Therefore, we obtain that

$$G_2(s_0, s_1; r) = G(s_0) + (1-a_0)H(s_0, s_1; r), \tag{4.4}$$

where

$$\begin{aligned} H(s_0,s_1;r) = &-\frac{1-s_0}{1+\alpha(1-s_0)\tilde{f}(s_0+(1-a_0)(s_1-s_0);r)} \\ &\times\left[\frac{1+\alpha}{1+\alpha(1-s_0)\tilde{f}(s_0;r)}\frac{\tilde{f}(s_0+(1-a_0)(s_1-s_0);r)-\tilde{f}(s_0;r)}{1-a_0}(s_0-t^*)\right. \\ &\left.\quad+\tilde{f}(a_0s_0+(1-a_0)s_1;r)(s_1-s_0)\right], \end{aligned}$$

and there exists a constant $\theta \in (0,1)$ such that

$$\begin{aligned} &\frac{\partial}{\partial s_0}\left[\frac{\tilde{f}(s_0+(1-a_0)(s_1-s_0);r)-\tilde{f}(s_0;r)}{1-a_0}\right] \\ &\quad=\frac{\tilde{f}'(s_0+(1-a_0)(s_1-s_0);r)a_0-\tilde{f}'(s_0;r)}{1-a_0} \\ &\quad=\tilde{f}''(s_0+\theta(1-a_0)(s_1-s_0);r)(s_1-s_0)-\tilde{f}'(s_0+(1-a_0)(s_1-s_0);r), \end{aligned}$$

from which we see that $\frac{\partial}{\partial s_0}H(s_0,s_1;r)$ is bounded for a bounded domain $(s_0,s_1) \in D \subset R^2$. Let define $\epsilon_1(a_0;s_0,s_1,r)$, by

$$F(G(s_0)+(1-a_0)H(s_0,s_1;r)) = F(G(s_0))+\epsilon_1(a_0;s_0,s_1,r).$$

Then, for fixed constants $s_0, s_1 < 1$ and $r > 0$,

$$\epsilon_1(a_0;s_0,s_1,r) \to 0, \text{ as } 1-a_0 \to +0, \tag{4.5}$$

and for a fixed constant $s_2 < 1$,

$$\begin{aligned} &G_2(G_2(s_0,s_1;r),s_2;r)-t^* \\ &\quad= F(G_2(s_0,s_1;r))(G_2(s_0,s_1;r)-t^*)+(1-a_0)H(G_2(s_0,s_1;r),s_2;r) \\ &\quad= (F(G(s_0))+\epsilon_1(a_0;s_0,s_1,r))\,(F(s_0)(s_0-t^*)+(1-a_0)H(s_0,s_1;r)) \\ &\qquad+(1-a_0)H(G_2(s_0,s_1;r),s_2;r) \\ &\quad= F(G(s_0))F(s_0)(s_0-t^*)+\epsilon_2(a_0;s_0,s_1,s_2,r), \end{aligned}$$

where

$$\begin{aligned} \epsilon_2(a_0;s_0,s_1,s_2,r) \;=\;& F(G(s_0))(1-a_0)H(s_0,s_1;r) \\ &+\epsilon_1(a_0;s_0,s_1,r)(F(s_0)(s_0-t^*)+(1-a_0)H(s_0,s_1;r)) \\ &+(1-a_0)H(G_2(s_0,s_1;r),s_2;r) \\ \to\;& 0, \text{ as } 1-a_0 \to +0, \end{aligned}$$

that is,

$$G_2(G_2(s_0,s_1;r),s_2;r)-t^* \to F(G(s_0))F(s_0)(s_0-t^*) \text{ as } 1-a_0 \to +0. \tag{4.6}$$

As a result with Lemmas 3.2 and 3.4, if we chose a sufficiently small positive constant $1-a_0$, then we can obtain the following important result in this paper.

Lemma 4.1. *For* $0 < \alpha < 1$, $r = \hat{\bar{r}}(\alpha)$, *two constants* c_1 *and* c_2 *such that (3.7) and (3.8) hold, there exists a constant* $\overline{\beta}(\alpha) > 0$ *such that for any* a_0 *such that* $1 - a_0 \leq \overline{\beta}(\alpha)$, *and* $s_1 \leq 1$,

$$|G_2(s_0, s_1; r) - t^*| < |s_0 - t^*| \quad \text{for } s_0 \leq c_1 \text{ or } s_0 \geq c_2, \tag{4.7}$$

and for a fixed constant $s_2 < 1$,

$$\begin{cases} |G_2(G_2(s_0, s_1; r), s_2; r) - t^*| < |s_0 - t^*|, \\ \frac{\partial}{\partial s_0} G_2(s_0, s_1; r) = G'(s_0) + (1 - a_0) \frac{\partial}{\partial s_0} H(s_0, s_1; r) < 0, \end{cases} \quad \text{for } c_1 < s_0 < c_2. \tag{4.8}$$

Proof. From (4.2) and (4.4), we have that

$$G_2(s_0, s_1; r) - t^* = F(s_0)(s_0 - t^*) + (1 - a_0) H(s_0, s_1; r).$$

We restrict $\overline{\beta}(\alpha) > 0$ as small as possible. From the boundedness of $H(s_0, s_1; r)$ for a bounded domain $(s_0, s_1) \in D \subset R^2$, there exists $\overline{H} < +\infty$ such that $|H(s_0, s_1; r)| \leq \overline{H}$. Since it holds that $|F(s_0)| < 1$ for $s_0 \leq c_1 < t^*$ or $s_0 \geq c_2 > t^{**}$ and $|H(s_0, s_1; r)| \leq \overline{H}$, we obtain (4.7).

Moreover, by Lemma 3.4 and the continuity of $F(G(s_0))F(s_0)$ on s_0, there exists $\overline{k}$ such that $F(G(s_0))F(s_0) \leq \overline{k} < 1$ for $c_1 < s_0 < c_2$. Hence, from (4.4) and (4.6), we obtain (4.8). Hence, the proof is complete. □

Then, we can prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By using the similar techniques in Li and Yuan [4] and Muroya and Kato [7] with Lemmas 3.4 and 4.1, we can prove Theorem 1.1. □

Proof of Theorem 1.2. By using the similar techniques in Li, Muroya and Yuan [3], we can derive Theorem 1.2 from Theorem 1.1. □

5. Numerical simulations

In this section, we consider the following logistic equation with two-piecewise constant arguments

$$\frac{dN(t)}{dt} = N(t) r \left(1 - aN(t) - b_0 N(n) - b_1 N(n-1)\right), \; n \leq t < n+1, \; n = 0, 1, 2, \ldots. \tag{5.1}$$

(5.1) has the positive equilibrium $N^* = 1/(a + b_0 + b_1)$ and $\alpha = a/(b_0 + b_1)$.

By Theorem 1.1, there exists a constant $0 < \overline{\beta}(\alpha) < 1$ such that for any $\frac{b_1}{b_0 + b_1} \leq \overline{\beta}(\alpha)$, the positive equilibrium N^* of (5.1) is globally asymptotically stable, if $r \leq \hat{\bar{r}}(\alpha)$. For the special case $\alpha = 0$, Uesugi et al, [11] determine $\overline{\beta}(0) = \mathrm{e}/(\mathrm{e} + 2)$. However, how to determine $\overline{\beta}(\alpha)$ for each $\alpha > 0$ is still remained as an open problem.

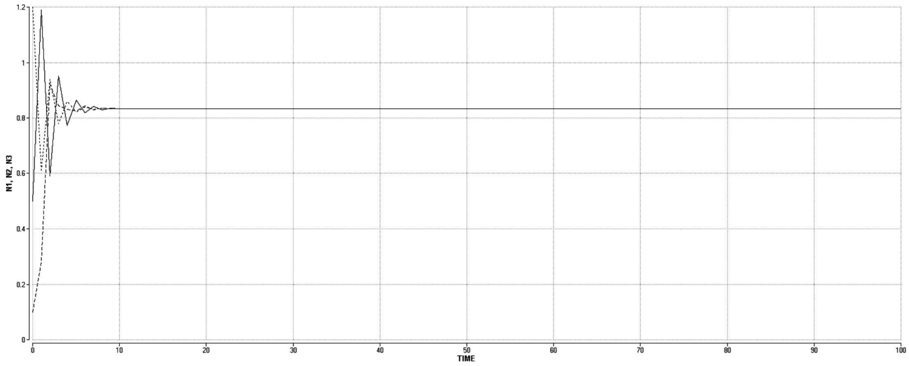

Figure 2: (5.1) with $a = 0.2, b_0 = 0.9, b_1 = 0.1$ and $r = \hat{\bar{r}}(\alpha), \alpha = 0.2$

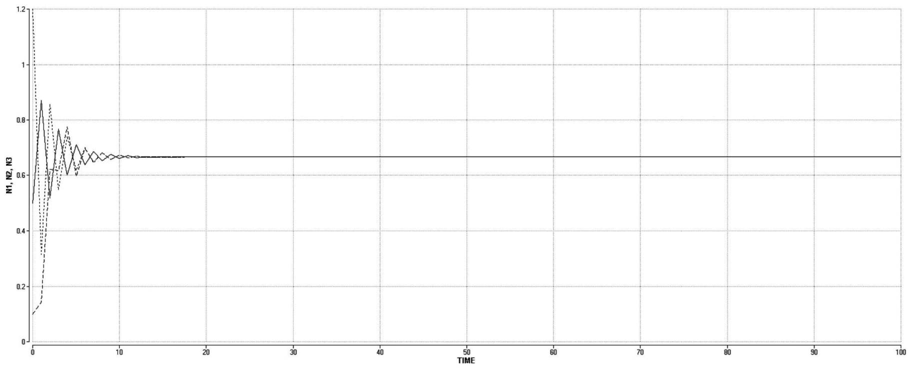

Figure 3: (5.1) with $a = 0.5, b_0 = 0.9, b_1 = 0.1$ and $r = \hat{\bar{r}}(\alpha), \alpha = 0.5$

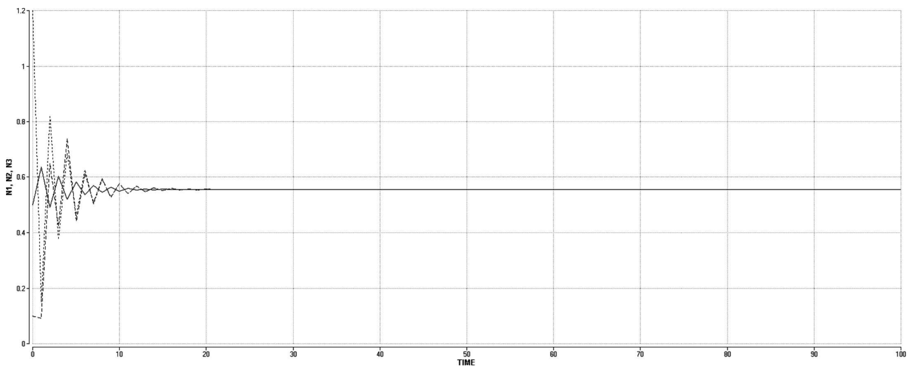

Figure 4: (5.1) with $a = 0.8, b_0 = 0.9, b_1 = 0.1$ and $r = \hat{\bar{r}}(\alpha), \alpha = 0.8$

For $a = 0.2, b_0 = 0.9, b_1 = 0.1$ and $r = \hat{\overline{r}}(\alpha), \alpha = 0.2$, we investigate for initial conditions $(N_0, N_{-1}) = (0.1m, 0.1n)$ for all $m = 1, 2, 3, \ldots, 30$ and $n = 0, 1, 2, \ldots, 30$, that each solutions converges to the positive equilibrium. From these numerical simulations, we may guess that every solution converges to the positive equilibrium $N^* = \frac{1}{1.2} \backsimeq 0.833333\ldots,$. Thus, we conjecture that $\overline{\beta}(0.2) \geq 0.1$.

For the case $a = 0.5, b_0 = 0.9, b_1 = 0.1$ and $r = \hat{\overline{r}}(\alpha), \alpha = 0.5$, we also investigate each solutions for same initial conditions. From the observation of the computation, every solution converges to the positive equilibrium $N^* = \frac{1}{1.5} \backsimeq 0.666667\ldots,$. Hence, we also conjecture that $\overline{\beta}(0.5) \geq 0.1$.

Moreover, for $a = 0.8, b_0 = 0.9, b_1 = 0.1$, and $r = \hat{\overline{r}}(\alpha), \alpha = 0.8$, we also investigate each solutions for the same initial conditions. We see that every solution converges to the positive equilibrium $N^* = \frac{1}{1.8} \backsimeq 0.555556\ldots,$. Hence, we conjecture that $\beta(0.8) \geq 0.1$. Figures 2-4 illustrate the orbit of the solutions with three initial conditions $(N_0, N_{-1}) = (0.1, 0.3), (0.5, 0.3)$ and $(1.2, 0.3)$ for each cases $\alpha = 0.2, 0.5$ and 0.8, respectively. These numerical simulations support the existence of $\overline{\beta}(\alpha) > 0$ in Theorem 1.1.

Finally, for the convenience of the reader, we draw the graphs of the function $f(t; \hat{\overline{r}}(\alpha))$ and $F(t)$ for $\alpha = 0, 0.2, 0.5$, and 0.8, in Figures 5-12, respectively.

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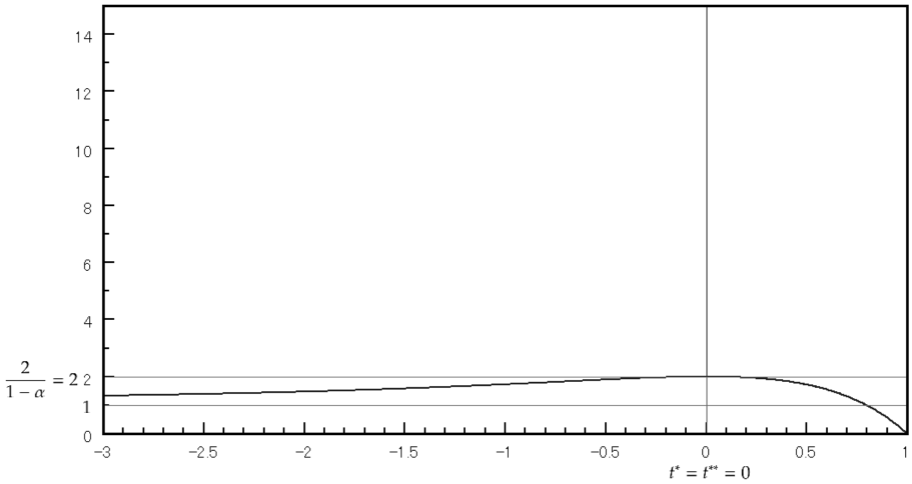

Figure 5: $f(t; \hat{\bar{r}}(\alpha))$, $\alpha = 0$

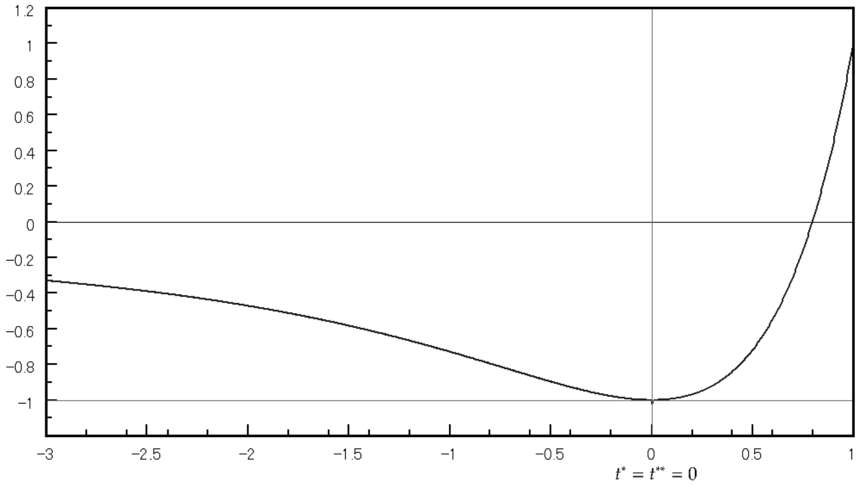

Figure 6: $F(t)$, $\alpha = 0$

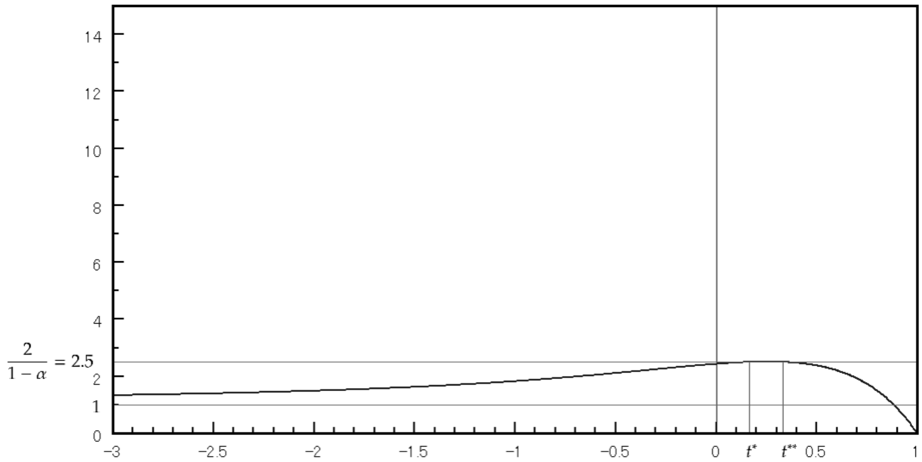

Figure 7: $f(t; \hat{\bar{r}}(\alpha))$, $\alpha = 0.2$

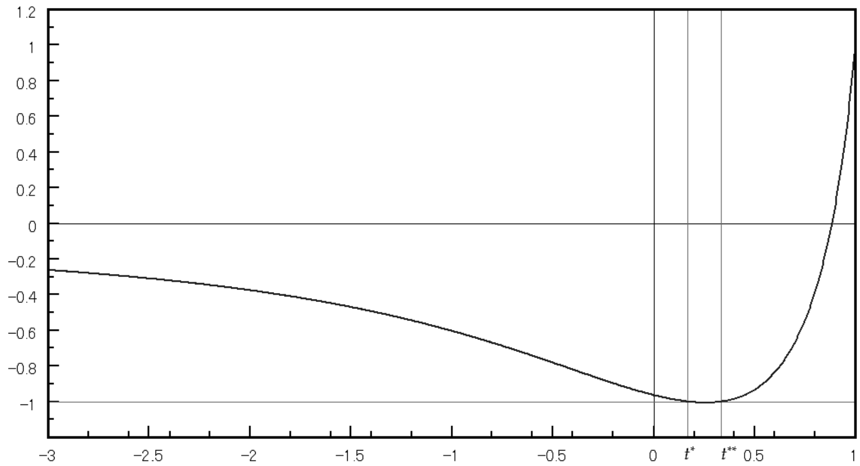

Figure 8: $F(t)$, $\alpha = 0.2$

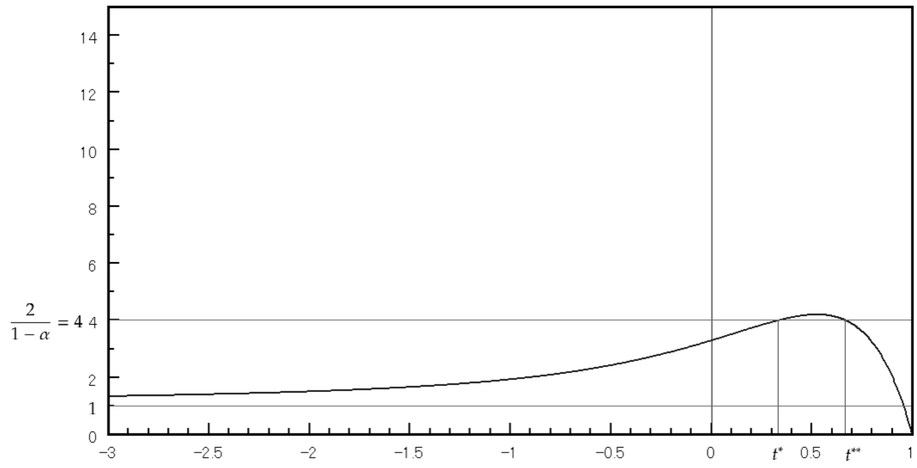

Figure 9: $f(t;\hat{\bar{r}}(\alpha))$, $\alpha = 0.5$

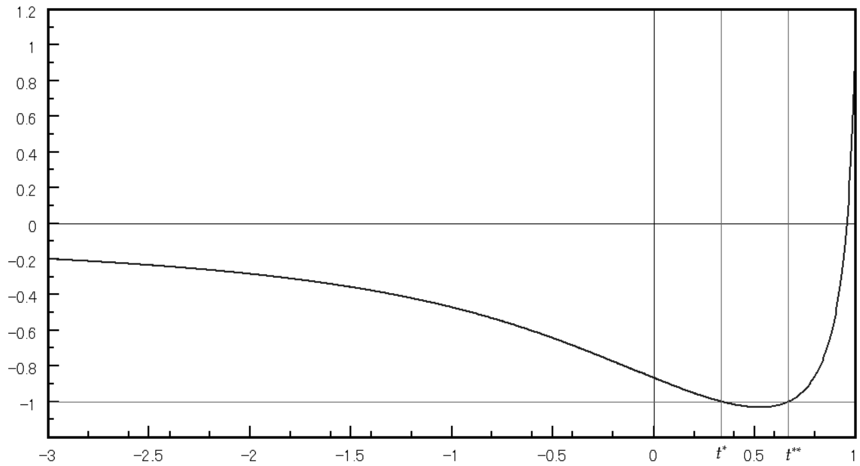

Figure 10: $F(t)$, $\alpha = 0.5$

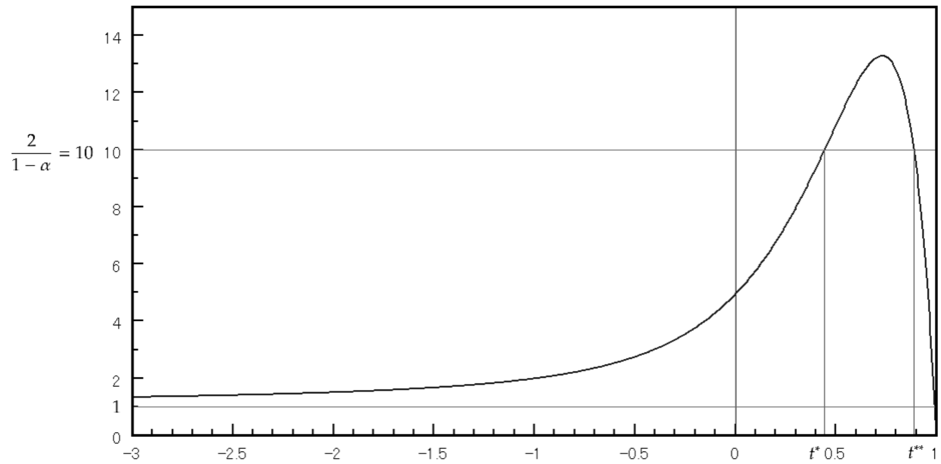

Figure 11: $f(t;\hat{\bar{r}}(\alpha))$, $\alpha = 0.8$

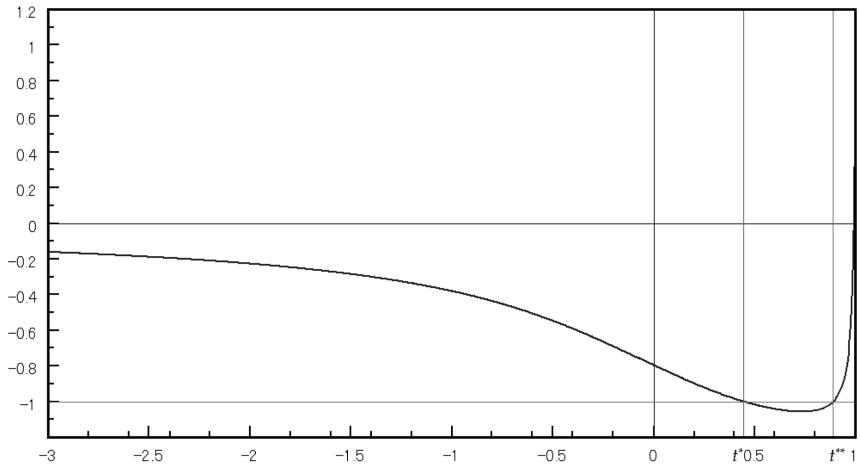

Figure 12: $F(t)$, $\alpha = 0.8$

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