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Delay-Range-Dependent Absolute Stability Systems with Time-Varying Delay

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Abstract. In this paper we are concentrated on the problem of absolute stability for Lur'e systems with time-varying delay in a range. An appropriate Lyapunov-Krasovskii functional is proposed to investigate the delay-range-dependant stability problem.The time-varying delay is assumed to belong to an interval and no restriction on its derivative is needed. We introduce some relaxation matrices which allow the delay to be a fast time-varying function. Furthermore, numerical examples are given to prove effectiveness and benefits of the proposed criteria.

AMS Subject Classifications: 93D05, 93D09, 93D15, 34D23

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1. Introduction

The problem of stability of time-delay systems has received considerable attention in the last two decades, since they are often a principle source of instability and degradation in control performance in many control problems such as nuclear reactors, chemical engineering systems including communication network. Due to time-delay occurring in such practical systems, current efforts has been devoted on this topic. For the recent progress, the reader is referred to [9, Gu, Kharitonov and Chen],[10, Gu and Niculescu].

We shall note that studying of stability of time-delay systems have grown steadily. Indeed, since 1940 all the results were delay independent see for examples [3, Bliman],

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[8, Gan and Ge], [14, He and Wu], [18, Li], [20, Liao], [24, Popov and Halanay], [25, Somolines]. But, the problem is that when the time-delay is small, these results are often overly conservative, especially, they are not applicable to closed-loop systems which are open-loop unstable and are stabilized using delayed inputs. That's why, many efforts were sacrificed to provide delay-dependant stability criteria.

Since, the introduction of absolute stability by Lur'e (1957), the absolute stability problem of nonlinear control systems with a fixed matrix in the linear part of the system and one or multiple uncertain nonlinearities satisfying the sector constraints has been the subject of many researchers see [2, Aizerman and Gantmacher], [17, Khalil], [20, Liao], [21, Lur'e], [23, Popov], [28, Yakubovich and al.].

From the practical point of view and since that in general the delay is not known, it is worth considering it as time-varying ([4, Chen et al], [27, Yangling Wang],[29, Yan H et al]). For this object, one is interested in conditions that constrain the upper and lower bounds of the delay and the upper bound of the first derivative of the time-varying delay.

To the best of our knowledge, for the case where only the upper and lower bounds of the interval time-varying delay are precisely known and the lower bound of the delay is greater than zero, there is no result available for stability for such kinds of systems. It should also be mentioned that even for the case where the lower bound of the time-varying delay is zero and without considering the derivative of the time-varying delay, there are few works available in the existing literature [6, Fridman and Shaked], [12, Han and Jiang], [5, Fridman] by using Lyapunov-Krasovskii functional approach.

For this reason we are motivated to provide new stability criterion, in order to improve those in which some useful terms are ignored, when estimating the upper bound of the derivative of Lyapunov functional [7, Fridman and Shaked], [10, Gu and Niculescu].

Those resulting criteria are applicable to both fast and slow time-varying delay, in contrast with previous works in which the upper bound of the first derivative of the time-varying delay was either restricted to one or completely neglected, see [12, Han and Jiang], [26, Wu, He, She and Liu], [30, Zhang, Min, She and He]. It is important to mention that this became possible since the free matrices M_1 and M_2 of the proposition provide some extra freedom in their selection.

The stability criteria are formulated in the form of Linear Matrix Inequality (LMI). Moreover, we give examples to show the applicability of our main results.

Notation: Throughout this paper, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. I is the identity matrix matrix. The set $\mathcal{C}_{n,h_M} := \mathcal{C}([-h_M, 0], \mathbb{R}^n)$ is the space of continuous functions mapping the interval $[-h_M, 0]$ to $\mathbb{R}^n$. The notation $A > 0$ is that the matrix A is positive definite.

2. Absolute stability analysis

We consider the following time-varying-delay system

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h(t)) + B\omega(t)\\ y(t) &= C_0x(t) + C_1x(t-h(t))\\ \omega(t) &= -\varphi(t, y(t))\end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^p$ the measured output, and the nonlinear function $\varphi(.,.) : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ is assumed to be continuous and belongs to sector $[0, K]$, i.e $\varphi(.,.)$ satisfies

$$\varphi^\top(t, y)\,[\varphi(t, y) - Ky] \leq 0, \quad \forall (t, y) \in \mathbb{R}_+ \times \mathbb{R}^p, \tag{2.2}$$

where K is a positive definite matrix. The matrices $A_0,\ A_1,\ B,\ C_0$, and C_1 are real matrices with appropriate dimensions.
The time delay $h(t)$ is a time-varying continuous function that satisfies

$$0 \leq h_m \leq h(t) < h_M \quad \textit{and} \quad \dot{h}(t) < \mu, \tag{2.3}$$

where h_m, h_M and μ are known constant reals.

Note that h_m may not be equal to 0. The initial condition of (2.1) is given by

$$x(t) = \phi(t), \qquad t \in [-h_M, 0],\ \phi \in \mathcal{C}_{n,h_M}.$$

It is assumed that the right-hand side of (2.1) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition ϕ.

We first introduce the following definition.

Definition 2.1. *The system* (2.1) *is said to be absolutely stable in the sector* $[0, K]$ *if the system is globally uniformly asymptotically stable for any nonlinear function* $\varphi(t, y(t))$ *satisfying* (2.2).

We will extend the work of [1, Ben Abdallah, Ben Hamed and Chaabane], in which, only systems with constant delay are studied, to systems with time-varying delay in a range.

In addition to this, we propose to discuss the absolute stability and the stabilization of a large class of systems, that is, the class of Lur'e systems with time-varying delay.

Furthermore, we will improve the results of ([12, Han and Jiang], [26, Wu, He, She and Liu], [30, Zhang, Min, She and He], etc), in which, the delay is assumed to be time-varying in a range, by keeping in our account all informations about the delay as well as its first derivative.

The development of the work in this paper requires the following lemma which can be found in Reference [30, Zhang, Min, She and He].

Lemma 2.1. *Let $x(t) \in \mathbb{R}^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ and $X = X^\top > 0$, and a scalar function $h := h(t) \geq 0$:*

$$-\int_{t-h(t)}^{t} \dot{x}^\top(s)X\dot{x}(s)ds \quad \leq \quad \xi^\top(t)\Upsilon\xi(t) + h(t)\xi^\top(t)\Gamma^\top X^{-1}\Gamma\xi(t), \tag{2.4}$$

where

$$\Upsilon := \begin{bmatrix} M_1^\top + M_1 & -M_1^\top + M_2 \\ * & -M_2^\top - M_2 \end{bmatrix},\ \Gamma^\top := \begin{bmatrix} M_1^\top \\ M_2^\top \end{bmatrix},\ \xi(t) := \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}.$$

Under the sector condition (2.2), we will give a sufficient condition for absolute stability of system (2.1).

We have the following theorem.

Theorem 2.1. *For given scalars $0 \leq h_m < h_M$, the system (2.1) with nonlinear function satisfying (2.2) is absolutely stable if there exist a scalar $\varepsilon > 0$ and a positive definite matrices $P > 0,\ Q_1 > 0,\ Q_2 > 0,\ Q_3 > 0,\ R_1 > 0,\ R_2 > 0,\ R_3 > 0$, and real matrices $M_1,\ M_2,\ N_1,\ N_2,\ S_1,\ S_2 \in \mathbb{R}^{n\times n}$, such that the following LMI*

$$\Xi_1 = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & 0 & \Xi_{14} & \Xi_{15} & h_M A_0^T R_1 & h_M M_1^T & \Xi_{18} & 0 & h_M A_0^T R_3 & h_M S_1^T \\
* & \Xi_{22} & 0 & 0 & \Xi_{25} & h_M A_1^T R_1 & 0 & \Xi_{28} & 0 & h_M A_1^T R_3 & h_M S_2^T \\
* & * & \Xi_{33} & \Xi_{34} & 0 & 0 & 0 & 0 & \Xi_{39} & 0 & 0 \\
* & * & * & \Xi_{44} & 0 & 0 & h_M M_2^T & 0 & \Xi_{49} & 0 & 0 \\
* & * & * & * & \Xi_{55} & h_M B^T R_1 & 0 & \Xi_{58} & 0 & h_M B^T R_3 & 0 \\
* & * & * & * & * & -h_M R_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -h_M R_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -h_M R_3 & 0 \\
* & * & * & * & * & * & * & * & * & * & -h_M R_3
\end{bmatrix} < 0 \tag{2.5}$$

where

$\Xi_{11} = A_0^\top P + PA_0 + Q_1 + Q_2 + Q_3 + M_1^\top + M_1 + S_1^\top + S_1,$

$\Xi_{12} = PA_1 - S_1^\top + S_2,$

$\Xi_{14} = -M_1^\top + M_2,$

$\Xi_{15} = PB - \epsilon\, C_0^\top K,$

$\Xi_{22} = -(1-\mu)Q_3 - S_2^\top - S_2,$

$\Xi_{25} = -\epsilon\, C_1^\top K,$

$\Xi_{33} = -Q_1 + N_1^\top + N_1,$

$\Xi_{34} = -N_1^\top + N_2,$

$\Xi_{44} = -Q_2 - M_2^\top - M_2 - N_2^\top - N_2,$

$\Xi_{55} = -2\epsilon \, I,$
$\Xi_{18} = (h_M - h_m)A_0^T R_2,$
$\Xi_{28} = (h_M - h_m)A_1^T R_2,$
$\Xi_{58} = (h_M - h_m)B^T R_2,$
$\Xi_{88} = -(h_M - h_m)R_2,$
$\Xi_{39} = (h_M - h_m)N_1^T,$
$\Xi_{49} = (h_M - h_m)N_2^T,$
$\Xi_{99} = -(h_M - h_m)R_2,$

holds.

Proof. We consider the Lyapunov-Krasovskii functional candidate

$$\begin{aligned} V(t, x_t) &= x^\top(t)Px(t) + \int_{t-h_m}^{t} x^\top(s)Q_1x(s)ds + \int_{t-h_M}^{t} x^\top(s)Q_2x(s)ds \\ &+ \int_{t-h(t)}^{t} x^\top(s)Q_3x(s)ds + \int_{-h_M}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)R_1\dot{x}(s)dsd\theta \\ &+ \int_{-h_M}^{-h_m}\int_{t+\theta}^{t} \dot{x}^\top(s)R_2\dot{x}(s)dsd\theta \\ &+ \int_{-h(t)}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)R_3\dot{x}(s)dsd\theta \end{aligned}$$

where the matrices $P,\ Q_1,\ Q_2,\ Q_3,\ R_1,\ R_2,$ and R_3 are positive definite.

The derivative of V along the trajectories of system (2.1) is given by

$$\begin{aligned} \dot{V}(t, x_t) &= 2\dot{x}^\top(t)Px(t) + x^\top(t)Q_1x(t) - x^\top(t-h_m)Q_1x(t-h_m) \\ &+x^\top(t)Q_2x(t) - x^\top(t-h_M)Q_2x(t-h_M) \\ &+x^\top(t)Q_3x(t) - (1-\dot{h}(t))x^\top(t-h(t))Q_3x(t-h(t)) \\ &+h_M\dot{x}^\top(t)R_1\dot{x}(t) - \int_{t-h_M}^{t} \dot{x}^\top(s)R_1\dot{x}(s)ds \\ &+(h_M-h_m)\dot{x}^\top(t)R_2\dot{x}(t) - \int_{t-h_M}^{t} -h_m\dot{x}^\top(s)R_2\dot{x}(s)ds \\ &+h(t)\dot{x}^\top(t)R_3\dot{x}(t) - \int_{t-h(t)}^{t} \dot{x}^\top(s)R_3\dot{x}(s)ds \end{aligned} \tag{2.6}$$

Using (2.3) and applying the integral inequality (2.4) to the right-hand side of (2.6), we obtain

$$\begin{aligned}\dot V(t,x_t) &\leq 2\dot x^\top(t)Px(t)+x^\top(t)[Q_1+Q_2+Q_3]x(t)-x^\top(t-h_m)Q_1x(t-h_m)\\ &\quad -x^\top(t-h_M)Q_2x(t-h_M)-(1-\mu)x^\top(t-h(t))Q_3x(t-h(t))\\ &\quad +\dot x^\top(t)[h_MR_1+(h_M-h_m)R_2+h_MR_3]\dot x(t)\\ &\quad +\xi_1^\top(t)\Upsilon_1\xi_1(t)+h_M\xi_1^\top(t)\Gamma_1^\top R_1^{-1}\Gamma_1\xi_1(t)\\ &\quad +\xi_2^\top(t)\Upsilon_2\xi_2(t)+(h_M-h_m)\xi_2^\top(t)\Gamma_2^\top R_2^{-1}\Gamma_2\xi_2(t)\\ &\quad +\xi_3^\top(t)\Upsilon_3\xi_3(t)+h_M\xi_3^\top(t)\Gamma_3^\top R_3^{-1}\Gamma_3\xi_3(t)\end{aligned}$$

with

$$\begin{aligned}\xi_1(t) &= \begin{bmatrix} x(t)\\ x(t-h_M)\end{bmatrix};\ \Gamma_1^\top=\begin{bmatrix} M_1^\top\\ M_2^\top\end{bmatrix};\ \Upsilon_1=\begin{bmatrix} M_1^\top+M_1 & -M_1^\top+M_2\\ * & -M_2^\top-M_2\end{bmatrix}\\ \xi_2(t) &= \begin{bmatrix} x(t-h_m)\\ x(t-h_M)\end{bmatrix};\ \Gamma_2^\top=\begin{bmatrix} N_1^\top\\ N_2^\top\end{bmatrix};\ \Upsilon_2=\begin{bmatrix} N_1^\top+N_1 & -N_1^\top+N_2\\ * & -N_2^\top-N_2\end{bmatrix}\\ \xi_3(t) &= \begin{bmatrix} x(t)\\ x(t-h(t))\end{bmatrix};\ \Gamma_3^\top=\begin{bmatrix} S_1^\top\\ S_2^\top\end{bmatrix};\ \Upsilon_3=\begin{bmatrix} S_1^\top+S_1 & -S_1^\top+S_2\\ * & -S_2^\top-S_2\end{bmatrix}.\end{aligned}$$

Rearranging the terms of the right-hand side yields:

$$\dot V(t)\leq \eta^\top(t)\ \Pi\ \eta(t), \tag{2.7}$$

where

$$\Pi:=\begin{bmatrix}\Pi_{11} & \Pi_{12} & 0 & \Pi_{14} & \Pi_{15}\\ * & \Pi_{22} & 0 & 0 & \Pi_{25}\\ * & * & \Pi_{33} & \Pi_{34} & 0\\ * & * & * & \Pi_{44} & 0\\ * & * & * & * & \Pi_{55}\end{bmatrix},\qquad \eta(t):=\begin{bmatrix} x(t)\\ x(t-h(t))\\ x(t-h_m)\\ x(t-h_M)\\ \omega(t)\end{bmatrix}$$

with

$$\begin{aligned}\Pi_{11} &= A_0^\top P+PA_0+Q_1+Q_2+Q_3+h_MA_0^\top R_1A_0+(h_M-h_m)A_0^\top R_2A_0\\ &\quad +h_MA_0^\top R_3A_0+M_1^\top+M_1+h_MM_1^\top R_1^{-1}M_1+h_MS_1^\top R_3^{-1}S_1+S_1^\top+S_1,\\ \Pi_{12} &= PA_1+h_MA_0^\top R_1A_1+(h_M-h_m)A_0^\top R_2A_1+h_MA_0^\top R_3A_1-S_1^\top+S_2\\ &\quad +h_MS_1R_3^{-1}S_2,\\ \Pi_{14} &= -M_1^\top+M_2+h_MM_1^\top R_1^{-1}M_2,\\ \Pi_{15} &= PB+h_MA_0^\top R_1B+(h_M-h_m)A_0^\top R_2B+h_MA_0^\top R_3B,\\ \Pi_{22} &= -(1-\mu)Q_3-S_2^\top-S_2+h_MA_1^\top R_1A_1+(h_M-h_m)A_1^\top R_2A_1+h_MA_1^\top R_3A_1\\ &\quad +h_MS_2^\top R_3^{-1}S_2,\\ \Pi_{25} &= h_MA_1^\top R_1B+(h_M-h_m)A_1^\top R_2B+h_MA_1^\top R_3B,\\ \Pi_{33} &= -Q_1+N_1^\top+N_1+(h_M-h_m)N_1^\top R_2^{-1}N_1,\end{aligned}$$

$$\begin{aligned}
\Pi_{34} &= -N_1^\top + N_2 + (h_M - h_m) N_1^\top R_2^{-1} N_1, \\
\Pi_{44} &= -Q_2 - M_2^\top - M_2 + h_M M_2^\top R_1^{-1} M_2 + (h_M - h_m) N_2^\top R_2^{-1} N_2 - N_2^\top - N_2, \\
\Pi_{55} &= h_M B^\top R_1 B + (h_M - h_m) B^\top R_2 B + h_M B^\top R_3 B.
\end{aligned}$$

A sufficient condition for absolute stability of the system (2.1) is that there exist real matrices $P > 0,\ Q_1 > 0, Q_2 > 0, Q_3 > 0, R_1 > 0, R_2 > 0$ and $R_3 > 0$, such that

$$\dot{V}(t) \leq \eta^\top(t) \Pi \eta(t) < 0, \tag{2.8}$$

for all $\eta(t) \neq 0$.

In order to show that $\Sigma < 0$, we shall use Shur complement and the $\mathcal{S}$-procedure. Using (2.2) implies

$$\omega^\top(t)\omega(t) + \omega^\top(t)\left[KC_0 x(t) + KC_1 x(t - h(t))\right] \leq 0.$$

Using the $\mathcal{S}$-procedure, we can find $\epsilon > 0$ such that

$$\eta^\top(t)\ \Pi\ \eta(t) - 2\epsilon\ \omega^\top(t)\omega(t) - 2\epsilon\ \omega^\top(t)\left[KC_0 x(t) + KC_1 x(t-h)\right] < 0, \tag{2.9}$$

for all $\eta(t) \neq 0$.

Rewrite (2.9) as

$$\eta^\top(t)\ \Sigma\ \eta(t) < 0, \tag{2.10}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} \\ * & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} \\ * & * & * & \Sigma_{44} & \Sigma_{45} \\ * & * & * & * & \Sigma_{55} \end{bmatrix},$$

with

$$\begin{aligned}
\Sigma_{ij} &= \Pi_{ij}, \quad (i, j = 1, 2, 3, 4), \\
\Sigma_{15} &= \Pi_{15} - \epsilon\, C_0^\top K, \\
\Sigma_{25} &= \Pi_{25} - \epsilon\, C_1^\top K, \\
\Sigma_{35} &= \Pi_{35}, \\
\Sigma_{45} &= \Pi_{45}, \\
\Sigma_{55} &= \Pi_{55} - 2\epsilon\, I.
\end{aligned}$$

Finally, from Shur complement, the LMI ($\Sigma < 0$) is equivalent to the LMI (2.5). This completes the proof. □

Next, we give examples showing a slight amelioration in the allowable upper bound of $h(t)$.

3. Numerical examples

Example 3.1.

Consider the time delay system (2.1) with the nonlinear function satisfying (2.2) and

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 160 & 0 \\ 0 & 1.25 \end{bmatrix}, \; C_1 = \begin{bmatrix} 10 & 0 \\ 0 & 7.5 \end{bmatrix}, \; K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.1}$$

For given μ, the computed upper bounds h_2, which guarantee the stability of system (2.1) for given lower bounds h_1, are listed in Table 1. When $h_1 = 0$, it is clear that our result is improvement over those in [7, Fridman and Shaked] and [13, Han and Jiang]. This comparison prove the merits of Theorem 2.1.

h_1	h_2 (Fridman and Shaked)	h_2 (Han and Jiang)	h_2 (new criterion)
0	0.7692	0.8654	0.8963022
0.05	-	0.8763	0.8847
0.10	-	0.8873	0.88866
0.15	-	0.8984	0.98248
0.20	-	0.9097	0.91412
0.30	-	0.9330	0.95324
0.40	-	0.9575	0.9724

Table 1: Allowable upper bound of h_2 with given h_1

Example 3.2.

Consider the following system with

$$\begin{aligned} A_0 &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \\ B &= C_0 = C_1 = K = 0. \end{aligned}$$

Table 2 summarizes the results obtained in the literature and compares them to the upper bounds h_2 obtained by using Theorem 2.1 given in this paper. h_2 is the maximal allowable delay proved by each method. It is obtained by a linear search. Clearly, our method produces much less conservative results, thus demonstrating its validity.This example demonstrates the benefits of the proposed criterion for linear systems with time varying-delay.

Table 2: Allowable upper bound of h_2 with given h_1 for unknown μ

h_1	Methods	h_2
0	Li and De Sousa(1997)	0.8571
	Niculescu and al(1995)	0.99
	Our method	1.5309
0.01	Li and De Sousa(1997)	-
	Niculescu and al(1995)	-
	Our method	1.5323
1	Jiang and Han(2005)	1.64
	He,Wang,Lin and Wu(2006)	1.74
	Our method	1.88

4. Stabilization of nonlinear delay system

This section presents the delay-dependent stabilization condition obtained by using the absolute stability proposed in section 2.

Consider the following nonlinear control time-varying delay system

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h(t)) + B\omega(t) + Gu(t) \\ y(t) &= C_0x(t) + C_1x(t-h(t)) \\ \omega(t) &= -\psi(t, y(t)) \end{aligned} \tag{4.1}$$

where $A_0,\ A_1,\ B,\ G,\ C_0,\ C_1$ are real matrices with appropriate dimensions, and the nonlinearity $\psi(t, y)$ belongs to the sector $[0, K]$, $K > 0$.

The initial condition of (4.1) is given by

$$x(t) = \phi(t), \qquad t \in [-h_M, 0],\ \phi \in \mathcal{C}_{n,h_M}.$$

The time delay $h(t)$ is a time-varying continuous function that satisfies

$$0 \leq h_m \leq h(t) < h_M \quad \text{and} \quad \dot{h}(t) < \mu,$$

where h_m, h_M and μ are known constant reals. It is assumed that the right-hand side of (4.1) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition ϕ.

The closed-loop system with the state control feedback

$$u(t) = Nx(t), \tag{4.2}$$

is given by

$$\dot{x}(t) = (A_0 + GN)\, x(t) + A_1 x(t - h(t)) + B\omega(t). \tag{4.3}$$

The following theorem gives a sufficient condition for stabilization of the system by means a state feedback when the nonlinearity $\psi(t, y)$ belongs to the sector $[0, K]$.

Theorem 4.1. *For given scalars $0 \leq h_m < h_M,\ \lambda_i,\ \alpha_i,\ \beta_i \in \mathbb{R},\ i = 1, 2,$ if there exist a scalar $\epsilon > 0$, positive definite matrices $\overline{P} > 0,\ \overline{Q_1} > 0,\ \overline{Q_2} > 0,\ \overline{Q_3} > 0,\ \overline{R_1} > 0\ \overline{R2} > 0,\ \overline{R_3} > 0$, and a matrix $Y \in \mathbb{R}^{r \times n}$ such that the LMI*

$$\Xi_2 = \begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & 0 & \Xi_{110} & \Xi_{111} \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & \Xi_{26} & 0 & \Xi_{28} & 0 & \Xi_{210} & \Xi_{211} \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & 0 & 0 & \Xi_{39} & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & 0 & \Xi_{49} & 0 & 0 \\ * & * & * & * & -2\varepsilon.I & h_M B^T & 0 & \Xi_{58} & 0 & h_M B^T & 0 \\ * & * & * & * & * & -h_M\overline{R_1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -h_M\overline{R_1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -h_M\overline{R_3} & 0 \\ * & * & * & * & * & * & * & * & * & * & -h_M\overline{R_3} \end{bmatrix} < 0 \tag{4.4}$$

where

$$\begin{aligned}
\Xi_{11} &= \overline{P}(A_0 + (\lambda_1 + \alpha_1)I)^\top + (A_0 + (\lambda_1 + \alpha_1)I)\overline{P} + GY + Y^\top G^\top + \overline{Q_1} + \overline{Q_2} + \overline{Q_3}, \\
\Xi_{12} &= A_1\overline{P} + (\alpha_2 - \alpha_1)\overline{P}, \\
\Xi_{14} &= (\lambda_2 - \lambda_1)\overline{P}, \\
\Xi_{15} &= B - \epsilon\overline{P}C_0^\top K, \\
\Xi_{16} &= h_M\overline{P}A_0^\top + h_M Y^\top G^\top, \\
\Xi_{17} &= \lambda_1 h_M \overline{R_1}, \\
\Xi_{18} &= (h_M - h_m)\overline{P}A_0^\top + (h_M - h_m)Y^\top G^\top, \\
\Xi_{110} &= h_M\overline{P}A_0^\top + h_M Y^\top G^\top, \\
\Xi_{111} &= \alpha_1 h_M \overline{R_3}, \\
\Xi_{22} &= -(1 - \mu)\overline{Q_3} - 2\alpha_2\overline{P}, \\
\Xi_{25} &= -\epsilon\overline{P}C_1^\top K,
\end{aligned}$$

$$
\begin{aligned}
\Xi_{26} &= h_M \overline{P} A_1^\top, \\
\Xi_{28} &= (h_M - h_m)\overline{P} A_1^\top, \\
\Xi_{210} &= h_M \overline{P} A_1^\top, \\
\Xi_{211} &= \alpha_2 h_M \overline{R_3}, \\
\Xi_{33} &= -\overline{Q_1} + 2\beta_1 \overline{P}, \\
\Xi_{34} &= (\beta_2 - \beta_1)\overline{P}, \\
\Xi_{39} &= \beta_1 (h_M - h_m)\overline{R_2}, \\
\Xi_{44} &= -\overline{Q_2} - 2(\lambda_2 + \beta_1)\overline{P}, \\
\Xi_{47} &= \lambda_2 h_M \overline{R_1}, \\
\Xi_{49} &= \beta_2 (h_M - h_m)\overline{R_2}, \\
\Xi_{58} &= (h_M - h_m) B^T, \\
\Xi_{88} &= -(h_M - h_m)\overline{R_2}, \\
\Xi_{99} &= -(h_M - h_m)\overline{R_2}
\end{aligned}
$$

hold. Then the origin of the controlled system (4.1) *is stabilized by the linear state feedback* (4.2) *where*

$$N = Y\overline{P}^{-1}.$$

Proof. Let $0 \le h_m < h_M$, λ_1, λ_2, α_1, α_2, β_1 and β_2 are fixed reals. Using Theorem 2.1, the closed-loop system is stable if there exist positive definite matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, and M_1, M_2, N_1, N_2, S_1, $S_2 \in \mathbb{R}^{n\times n}$ such that the LMI (2.5) with replacing A_0 by $A_0 + GN$ holds, then the origin of system (4.1) is globally uniformly asymptotically stable. This is equivalent to the feasibility of the following LMI

$$T^\top \, \Xi_2 \, T = \Xi_T < 0,$$

where

$$T = diag\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, I, R_1^{-1}, R_1^{-1}, R_2^{-1}, R_2^{-1}, R_3^{-1}, R_3^{-1}\}.$$

Denoting

$$\overline{P} = P^{-1}, \overline{Q_1} = P^{-1} Q_1 P^{-1}, \overline{Q_2} = P^{-1} Q_2 P^{-1}, \overline{Q_3} = P^{-1} Q_3 P^{-1},$$

$$\overline{R_1} = R_1^{-1}, \overline{R_2} = R_2^{-1}, \overline{R_3} = R_3^{-1}, NP^{-1} = Y,$$

and picking $M_i = \lambda_i P$, $N_i = \beta_i P$, $S_i = \alpha_i P$, $i = 1, 2$, we obtain the desired LMI (4.4). □

5. Conclusion

The problem of absolute stability of a class of time-varying delay systems with sector-bounded nonlinearity have been considered. New delay-dependant stability and stabilization criteria with sector condition have been proposed. Some new results are given and illustrated by numerical examples, treated with Matlab, in order to show effectiveness of the main results. Those criteria have been formulated in the form of linear matrix inequalities (LMI).

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