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Asymptotic Stability for Three-Dimensional Nonlinear Systems including a Hamilton System

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Dedicated to Professor István Győri on the occasion of his 65th birthday

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Abstract. In this paper the following three-dimensional nonlinear system is considered:

$$x' = \frac{\partial}{\partial y}H(x,y), \quad y' = -\frac{\partial}{\partial x}H(x,y) + z, \quad z' = -\frac{\partial}{\partial y}H(x,y) - z.$$

This system contains a subsystem described by a Hamiltonian function. Under the assumption that all orbits of the Hamiltonian system near to the origin are isolated closed curves surrounding the origin, sufficient conditions are given for the zero solution to tend to the origin as $t \to \infty$.

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1. Introduction

Let $B_\rho = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho^2\}$ for any $\rho > 0$ and let $H(x,y)$ be a continuous function on B_ρ having continuous first partial derivatives. Suppose there exist constants α_1, α_2, β_1, β_2, γ and μ with $0 < \alpha_1 \le \alpha_2$, $0 < \beta_1 \le \beta_2$, $\gamma > 0$ and $0 < \mu \le 1$ such that

$$\alpha_1(x^2+y^2) \le H(x,y) \le \alpha_2(x^2+y^2)\,, \tag{C_1}$$

$$\beta_1(x^2+y^2) \le x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y) \le \beta_2(x^2+y^2)\,, \tag{C_2}$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial x}H(x,y) = \lim_{(x,y)\to(0,0)} \frac{\partial}{\partial y}H(x,y) = 0, \tag{C_3}$$

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$$0 \le |x| < \mu|y| \quad \text{implies} \quad \gamma|y| \le \left|\frac{\partial}{\partial y}H(x,y)\right|. \tag{C_4}$$

Then, in a neighborhood of the origin $(0,0)$, all solutions of the system

$$x' = \frac{\partial}{\partial y}H(x,y), \qquad y' = -\frac{\partial}{\partial x}H(x,y) \tag{1.1}$$

are periodic, namely, all orbits near to the origin are isolated closed curves surrounding the origin. Hence, the zero solution of (1.1) is stable, but not attractive (for the definition, see Section 2).

One of the most simple examples of (1.1) is the pendulum system without friction,

$$x' = y, \qquad y' = -\sin x. \tag{1.2}$$

In this case, we may consider $H(x,y) = 1 - \cos x + y^2/2$. Hence, for $\rho > 0$ sufficiently small, conditions (C_1)–(C_4) are satisfied with $\alpha_1 = 1/4$, $\alpha_2 = 1/2$, $\beta_1 = 1/2$, $\beta_2 = 1$, $\gamma = 1$ and $\mu = 1$. To take another example of (1.1) , we consider the Lotka–Volterra system

$$X' = aX - bXY, \qquad Y' = -cY + dXY$$

on $\mathbb{R}^2_+$, $\mathbb{R}_+ = (0,\infty)$, where a, b, c and d are positive constants; X and Y are the densities of the prey and predator, respectively. Let

$$x = -\log(bY/a) \qquad \text{and} \qquad y = -\log(dX/c).$$

Then, we can transform the Lotka–Volterra system into the system

$$x' = c(1 - e^{-y}), \qquad y' = a(e^{-x} - 1), \tag{1.3}$$

which has the form of (1.1) with

$$H(x,y) = a(e^{-x} + x - 1) + c(e^{-y} + y - 1).$$

It is clear that for $\rho > 0$ sufficiently small, conditions (C_1)–(C_4) are satisfied with $\alpha_1 = \min\{a,c\}/4$, $\alpha_2 = \max\{a,c\}$, $\beta_1 = \min\{a,c\}/2$, $\beta_2 = 2\max\{a,c\}$, $\gamma = c/2$ and $\mu = 1$. It is easy to find other nonlinear phenomena described by system (1.1) in pure and applied science.

All solutions $(x(t), y(t))$ of (1.1) do not converge to the origin. Then, can $x(t)$ and $y(t)$ converge to zero by adding the third variable to system (1.1)? To deal with this problem, we consider the three-dimensional time-varying nonlinear system

$$x' = \frac{\partial}{\partial y}H(x,y), \quad y' = -\frac{\partial}{\partial x}H(x,y) + z, \quad z' = -\frac{\partial}{\partial y}H(x,y) - z. \tag{1.4}$$

If subsystem (1.1) is linear, then the well-known Routh-Hurwitz criterion may be useful for our problem. However, if system (1.4) contains a nonlinear subsystem, such as system (1.2) or (1.3), then the Routh-Hurwitz criterion is of no use to system (1.4) directly.

2. Main Result

Consider a system of differential equations of the form

$$x' = \frac{\partial}{\partial y}H(x,y), \quad y' = -\frac{\partial}{\partial x}H(x,y) + z, \quad z' = -\frac{\partial}{\partial y}H(x,y) - z. \tag{E}$$

Let $\mathbf{x}(t) = (x(t), y(t), z(t))$ and $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$, and let $\|\cdot\|$ be the Euclidean norm. We denote the solution of (E) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is clear that system (E) has the zero solution $\mathbf{x}(t) \equiv \mathbf{0}$.

The zero solution is said to be *stable*, if for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *attractive*, if for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The zero solution of (E) is said to be *asymptotically stable* if it is stable and attractive. The stability and the attractivity are completely different concepts in nonlinear systems, such as (E) (refer to the books [1, 2, 3, 4, 5]).

The following theorem is our main result.

Theorem. *Suppose that conditions (C_1)–(C_4) are satisfied. Then the zero solution of (E) is asymptotically stable.*

Proof. (i): Define

$$M_1 = \min\left\{\alpha_1, \frac{1}{2}\right\} \quad \text{and} \quad M_2 = \max\left\{\alpha_2, \frac{1}{2}\right\}.$$

To prove the stability of the zero solution of (E), for a given $\varepsilon \in (0, \rho)$, we select

$$\delta(\varepsilon) = \sqrt{\frac{M_1}{M_2}}\,\varepsilon.$$

Recall that ρ is the constant given in (C_1). Needless to say, $\delta < \varepsilon$. Let $t_0 \geq 0$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ be given. We will show that $\|\mathbf{x}_0\| = \sqrt{x_0^2 + y_0^2 + z_0^2} < \delta$ and $t \geq t_0$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$. For convenience of notation, we write $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and $(x(t), y(t), z(t)) = \mathbf{x}(t)$.

Suppose that there exists $t_1 > t_0$ with $\|\mathbf{x}(t_1)\| = \varepsilon$ and

$$\|\mathbf{x}(t)\| < \varepsilon < \rho \quad \text{for} \;\; t_0 \leq t < t_1.$$

Note that $(x(t), y(t)) \in B_\rho$ for $t_0 \leq t \leq t_1$. Let

$$v(t) = H(x(t), y(t)) + \frac{1}{2}z^2(t)$$

for $t \geq t_0$. Then, from (C_1) it turns out that

$$v(t) \geq \alpha_1\big(x^2(t) + y^2(t)\big) + \frac{1}{2}z^2(t) \geq M_1\mathbf{x}^2(t) \tag{2.1}$$

for $t_0 \le t \le t_1$. Since

$$v'(t) = -z^2(t) \le 0 \quad \text{for} \ \ t \ge t_0,$$

it follows that

$$v(t) \le v(t_0) = H(x_0, y_0) + \frac{1}{2}z_0^2 \le \alpha_2(x_0^2 + y_0^2) + \frac{1}{2}z_0^2 < M_2\delta^2 = M_1\varepsilon^2$$

for $t \ge t_0$. Hence, together with (2.1), we obtain

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for} \ \ t_0 \le t \le t_1.$$

This contradicts the assumption that $\|\mathbf{x}(t_1)\| = \varepsilon$. Thus, we see that

$$\|\mathbf{x}(t)\| < \varepsilon < \rho \quad \text{for} \ \ t \ge t_0, \tag{2.2}$$

and therefore, the zero solution of (E) is stable. This completes the proof of part (i).

Hereafter, we will show that the zero solution of (E) is asymptotically stable. To this end, it is enough to show that it is attractive, namely, $\mathbf{x}(t)$ converges to $\mathbf{0}$ as t increases. Since $v'(t)$ is nonpositive for $t \ge t_0$, the function $v(t)$ has a limiting value $v_0 \ge 0$. If $v_0 = 0$, then by (2.1), the solution $\mathbf{x}(t)$ tends to $\mathbf{0}$ as $t \to \infty$. This completes the proof. Hence, the remainder is the case in which $v_0 > 0$. We will demonstrate that this case does not occur.

For the sake of simplicity, let

$$u(t) = \frac{1}{2}z^2(t).$$

Then, we have $v(t) = H(x(t), y(t)) + u(t)$ and $v'(t) = -2u(t)$. From (2.2), we see that $u(t)$ is bounded. Hence, $u(t)$ has the inferior limit and the superior limit.

(ii): We will show that the inferior limit of $u(t)$ is zero. Suppose that

$$\liminf_{t\to\infty} u(t) > 0.$$

Then there exist an $\varepsilon_1 > 0$ and a $T_1 \ge t_0$ such that $u(t) > \varepsilon_1$ for $t \ge T_1$. Hence,

$$\int_{t_0}^{\infty} v'(s)ds = -2\int_{t_0}^{\infty} u(s)ds \le -2\varepsilon_1 \int_{t_0}^{\infty} ds = -\infty.$$

On the other hand, since $v(t) \ge 0$ for $t \ge t_0$,

$$\int_{t_0}^{\infty} v'(s)ds \ge -v(t_0).$$

This is a contradiction. Thus, we see that $\liminf_{t\to\infty} u(t) = 0$. This completes the proof of part (ii).

(iii): We next show that the superior limit of $u(t)$ is zero. The proof is by contradiction. Suppose that $\nu \overset{\text{def}}{=} \limsup_{t\to\infty} u(t) > 0$. Since $\liminf_{t\to\infty} u(t) = 0$, we can choose two sequences $\{t_n\}$ and $\{s_n\}$ with $t_0 < t_n < s_n < t_{n+1}$ such that $u(t_n) = \nu/2$, $u(s_n) = 3\nu/4$ and

$$\frac{\nu}{2} < u(t) < \frac{3\nu}{4} \quad \text{for } \ t_n < t < s_n.$$

Let $I = \bigcup_{n=1}^{\infty} [t_n, s_n]$. Then, we have

$$-v(t_0) \leq \int_{t_0}^{\infty} v'(s)ds = -2\int_{t_0}^{\infty} u(s)ds < -2\int_I u(s)ds < -\nu \sum_{n=1}^{\infty}(s_n - t_n),$$

so that

$$\sum_{n=1}^{\infty}(s_n - t_n) < \frac{v(t_0)}{\nu}.$$

Hence, it turns out that

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{2.3}$$

Since $\partial H(x,y)/\partial y$ is continuous, it follows from (C_3) and (2.2) that there exists an $l > 0$ such that

$$\left|\frac{\partial}{\partial y}H(x(t), y(t))\right| \leq l \quad \text{for } \ t \geq t_0.$$

Hence, we obtain

$$u'(t) = v'(t) - \frac{\partial}{\partial y}H(x(t), y(t))z(t) \leq v'(t) + \left|\frac{\partial}{\partial y}H(x(t), y(t))\right| |z(t)| \leq v'(t) + l\,\varepsilon$$

for $t \geq t_0$. Integrating this inequality from t_n to s_n, we get

$$\frac{\nu}{4} = u(s_n) - u(t_n) \leq \int_{t_n}^{s_n} v'(s)ds + l\,\varepsilon(s_n - t_n) = v(s_n) - v(t_n) + l\,\varepsilon(s_n - t_n)$$

for each $n \in \mathbb{N}$. This contradicts (2.3), thereby completing the proof of part (iii).

Since $v(t)$ tends to a positive value v_0 as $t \to \infty$, we can choose a $T_2 \geq t_0$ such that

$$0 < \frac{v_0}{2} < v(t) < \frac{3v_0}{2} \quad \text{for } \ t \geq T_2. \tag{2.4}$$

If $\beta_1 \geq \gamma/(\sqrt{2}\beta_2)$, then we have

$$\widetilde{\beta}_1\big(x^2 + y^2\big) \leq x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y) \leq \beta_2\big(x^2 + y^2\big)\,,$$

where $\widetilde{\beta}_1 = \gamma/(\sqrt{2}\beta_2)$. Hence, we may assume without loss of generality that $\beta_1\beta_2 < \gamma/\sqrt{2}$. Let $\varepsilon_2 > 0$ be so small that $\varepsilon_2 < \nu/2$,

$$\sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}} < \frac{\beta_1}{2}, \tag{2.5}$$

$$\tan\left(\frac{\pi}{2} - \frac{\pi}{2\beta_1}\sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}}\right) > \frac{\varepsilon}{\mu}\sqrt{\frac{2\alpha_2}{v_0 - 2\varepsilon_2}}, \tag{2.6}$$

$$\sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}} + \beta_1\beta_2 < \frac{\gamma}{\sqrt{2}}, \tag{2.7}$$

where α_2, (β_1, β_2) and (γ, μ) are numbers given in (C_1), (C_2) and (C_4), respectively.

(iv): From parts (ii) and (iii) above, we see that $\lim_{t\to\infty} u(t) = 0$. Hence, we can choose a $T_3 \geq T_2$ such that $0 \leq u(t) < \varepsilon_2$ for $t \geq T_3$, and therefore,

$$|z(t)| \leq \sqrt{2u(t)} \leq \sqrt{2\varepsilon_2} \quad \text{for } \ t \geq T_3. \tag{2.8}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then we can rewrite system (E) as the form

$$\begin{aligned} r' &= \frac{\partial}{\partial y}H(x,y)\cos\theta - \frac{\partial}{\partial x}H(x,y)\sin\theta + z\sin\theta, \\ \theta' &= \frac{z}{r}\cos\theta - \frac{1}{r^2}\left\{x\frac{\partial}{\partial x}H(x,y) + y\frac{\partial}{\partial y}H(x,y)\right\}, \\ z' &= -\frac{\partial}{\partial y}H(x,y) - z. \end{aligned} \tag{$\widetilde{E}$}$$

Let $(r(t), \theta(t), z(t))$ be the solution of $(\widetilde{E})$ corresponding to $\mathbf{x}(t)$. Using (2.2), (2.4), (2.8) and (C_1), we obtain

$$v_0 - 2\varepsilon_2 < 2(v(t) - u(t)) = 2H(x(t), y(t)) \leq 2\alpha_2\big(x^2(t) + y^2(t)\big) < 2\alpha_2\varepsilon^2$$

for $t \geq T_3$, so that

$$\sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r(t) < \varepsilon \quad \text{for } \ t \geq T_3. \tag{2.9}$$

Taking into account of (2.8) and (2.9), we see that the solution $(r(t), \theta(t), z(t))$ of $(\widetilde{E})$ stays in the thin disc

$$D = \left\{(r,\theta,z)\colon \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r < \varepsilon,\ -\pi < \theta \leq \pi \ \text{ and } \ |z| \leq \sqrt{2\varepsilon_2}\right\}$$

for $t \geq T_3$. It follows from (C_2), (2.8) and (2.9) that

$$-\sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}} - \beta_2 < -\frac{|z(t)|}{r(t)} - \beta_2 \leq \theta'(t) \leq \frac{|z(t)|}{r(t)} - \beta_1 < \sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}} - \beta_1$$

for $t \geq T_3$. Let

$$\omega_- = \beta_1 - \sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}} \qquad \text{and} \qquad \omega_+ = \beta_2 + \sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}}.$$

Then, we have

$$-\omega_+ < \theta'(t) < -\omega_- < 0 \quad \text{for } \ t \geq T_3. \tag{2.10}$$

From (2.5), we can estimate that

$$\frac{\beta_1}{2} < \omega_- < \beta_1 \le \beta_2 < \omega_+ < \beta_2 + \frac{\beta_1}{2} \le \frac{3}{2}\beta_2. \tag{2.11}$$

Define a region Ω by

$$\Omega = \left\{(r,\theta)\colon \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r < \varepsilon \ \text{ and } \ \frac{\pi\omega_-}{2\beta_1} \le \theta \le \pi\left(1 - \frac{\omega_-}{2\beta_1}\right)\right\}.$$

The region Ω is non-empty because $\omega_- < \beta_1$. Consider the movement of $(r(t), \theta(t))$. Then, from (2.9) and (2.10), we see that $(r(t), \theta(t))$ stays in the annulus

$$A = \left\{(r,\theta)\colon \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} < r < \varepsilon \ \text{ and } \ -\pi < \theta \le \pi\right\} \supset \Omega$$

for $t \ge T_3$ and it moves clockwise. Hence, we can find two divergent sequences $\{a_n\}$ and $\{b_n\}$ with $T_3 < a_n < b_n$ such that $\theta(a_n) - \theta(b_n) = \pi(1 - \omega_-/\beta_1)$ and

$$(r(t), \theta(t)) \in \Omega \quad \text{for } \ a_n \le t \le b_n. \tag{2.12}$$

By (2.10) and (2.11), we have

$$\theta(a_n) - \theta(b_n) < \omega_+(b_n - a_n) < \frac{3}{2}\beta_2(b_n - a_n),$$

so that

$$b_n - a_n > \frac{2(\theta(a_n) - \theta(b_n))}{3\beta_2} = \frac{2\pi(\beta_1 - \omega_-)}{3\beta_1\beta_2} > \frac{2(\beta_1 - \omega_-)}{\beta_1\beta_2} \tag{2.13}$$

for each $n \in \mathbb{N}$. It follows from (2.5) that

$$\frac{\pi\omega_-}{2\beta_1} = \frac{\pi}{2}\left(1 - \frac{1}{\beta_1}\sqrt{\frac{4\alpha_2\varepsilon_2}{v_0 - 2\varepsilon_2}}\right) > \frac{\pi}{2}\left(1 - \frac{1}{\beta_1}\frac{\beta_1}{2}\right) = \frac{\pi}{4}.$$

Hence, together with (2.12), we obtain

$$|y(t)| > \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} \sin\frac{\pi\omega_-}{2\beta_1} > \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}}$$

for $a_n \le t \le b_n$. Since we can rewrite (2.7) as

$$\beta_1 - \omega_- + \beta_1\beta_2 < \frac{\gamma}{\sqrt{2}},$$

we have

$$\frac{1}{\sqrt{2}}\sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} > \frac{1 + \beta_1\beta_2/(\beta_1 - \omega_-)}{\gamma}\sqrt{2\varepsilon_2}.$$

We therefore conclude that

$$|y(t)| > \frac{1 + \beta_1\beta_2/(\beta_1 - \omega_-)}{\gamma}\sqrt{2\varepsilon_2} \quad \text{for } \ a_n \le t \le b_n. \tag{2.14}$$

From (2.12), we see that

$$0 \le |x(t)| < \varepsilon \cos \frac{\pi\omega_-}{2\beta_1} \quad \text{and} \quad |y(t)| > \sqrt{\frac{v_0 - 2\varepsilon_2}{2\alpha_2}} \sin \frac{\pi\omega_-}{2\beta_1} > 0$$

for $a_n \le t \le b_n$. Hence, by (2.6) we have

$$|x(t)| < \varepsilon \cos \frac{\pi\omega_-}{2\beta_1} < \varepsilon \sqrt{\frac{2\alpha_2}{v_0 - 2\varepsilon_2}} \frac{|y(t)|}{\tan\left(\pi\omega_-/(2\beta_1)\right)} < \mu |y(t)|$$

for $a_n \le t \le b_n$, and therefore, by (C_4) we get

$$\gamma |y(t)| \le \left| \frac{\partial}{\partial y} H(x(t), y(t)) \right| \quad \text{for } a_n \le t \le b_n. \tag{2.15}$$

From the third equation of (E) with (2.8), (2.14) and (2.15), we obtain

$$|z'(t)| \ge \gamma |y(t)| - |z(t)| > \gamma \frac{1 + \beta_1\beta_2/(\beta_1 - \omega_-)}{\gamma} \sqrt{2\varepsilon_2} - \sqrt{2\varepsilon_2} = \frac{\beta_1\beta_2}{\beta_1 - \omega_-} \sqrt{2\varepsilon_2} \tag{2.16}$$

for $a_n \le t \le b_n$. Since $z'(t)$ is continuous for $t \ge t_0$, we see that

$$\left| \int_{a_n}^{b_n} z'(s) ds \right| = \int_{a_n}^{b_n} |z'(s)| ds.$$

Hence, by (2.8), (2.13) and (2.16), we have

$$2\sqrt{2\varepsilon_2} \ge |z(a_n)| + |z(b_n)| \ge \int_{a_n}^{b_n} |z'(s)| ds > \frac{\beta_1\beta_2}{\beta_1 - \omega_-} \sqrt{2\varepsilon_2}(b_n - a_n) > 2\sqrt{2\varepsilon_2},$$

which is a contradiction. Thus, the case of $v_0 > 0$ does not happen. We therefore conclude that the zero solution of (E) is asymptotically stable.

The proof of the theorem is now complete. □

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