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On some properties of nonlinear functional parabolic equations*

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Abstract. We consider second order quasilinear parabolic equations where also the main part contains functional dependence on the unknown function. Existence and some qualitative properties of the solutions are shown.

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1. Introduction

This work was motivated by works where nonlinear parabolic functional differential equations were considered which arise in certain applications. (See references in [4].) In [4] existence theorems and some qualitative properties were proved on solutions to initial value problems for the functional equations (connected with the above applications)

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t, x, u(t, x), Du(t, x); u)] + a_0(t, x, u(t, x), Du(t, x); u) = f. \qquad (1.1)$$

The aim of the present paper is to formulate existence theorems if certain modified (in some sense more general) assumptions are fulfilled and to show further qualitative properties of solutions to such equations.

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2. Existence of solutions

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0,T)\times\Omega$ and $p \geq 2$ be a real number. Let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace of the usual Sobolev space $W^{1,p}(\Omega)$ (of real valued functions). Denote by $L^p(0,T;V)$ the Banach space of the set of measurable functions $u:(0,T)\to V$ with the norm

$$\| u \|^p_{L^p(0,T;V)} = \int_0^T \| u(t) \|^p_V \, dt.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p+1/q=1$ and $V^\star$ is the dual space of V (see, e.g., [6]).

On functions a_i assume

(A_1). The functions $a_i : Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V) \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $u \in L^p(0,T;V)$ $(i=0,1,...,n)$.

(A_2). There exist bounded (nonlinear) operators $g_1 : L^p(0,T;V) \to \mathbb{R}^+$ and $k_1 : L^p(0,T;V) \to L^q(\Omega)$ such that

$$|a_i(t,x,\zeta_0,\zeta;u)| \leq g_1(u)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(u)](x)$$

for a.e. $(t,x)\in Q_T$, each $(\zeta_0,\zeta)\in\mathbb{R}^{n+1}$ and $u \in L^p(0,T;V)$.

$$(A_3). \quad \sum_{i=1}^n [a_i(t,x,\zeta_0,\zeta;u) - a_i(t,x,\zeta_0,\zeta^\star;u)](\zeta_i - \zeta_i^\star) \geq [g_2(u)](t)|\zeta-\zeta^\star|^p \qquad (2.1)$$

where

$$[g_2(u)](t) \geq c^\star \left[1 + \| u \|_{L^p(0,t;V)}\right]^{-\sigma^\star}, \quad t\in[0,T] \qquad (2.2)$$

$c^\star$ is some positive constant, $0 \leq \sigma^\star < p-1$.

(A_4). $\sum_{i=0}^n a_i(t,x,\zeta_0,\zeta;u)\zeta_i \geq [g_2(u)](t)[|\zeta_0|^p + |\zeta|^p] - [k_2(u)](t,x)$

where $k_2(u) \in L^1(Q_T)$ satisfies with some positive constant $\sigma < p - \sigma^\star$, $t \in [0,T]$

$$\| k_2(u) \|_{L^1(Q_t)} \leq \text{const} \left[1 + \| u \|_{L^p(0,t;V)}\right]^\sigma \quad t\in[0,T].$$

(A_5). There exists $\delta > 0$ such that if $(u_k)\to u$ weakly in $L^p(0,T;V)$, strongly in $L^p(0,T;W^{1-\delta,p}(\Omega))$, $(\zeta_0^k)\to\zeta_0$ in $\mathbb{R}$ and $(\zeta^k)\to\zeta$ in $\mathbb{R}^n$ then for a.e. $(t,x)\in Q_T$

$$\lim_{k\to\infty} a_i(t,x,\zeta_0^k,\zeta^k;u_k) = a_i(t,x,\zeta_0,\zeta;u).$$

Remark 1. Assumption (A_5) is weaker than the corresponding assumption in [4] thus equation (1.1) may contain more general "nonlocal" terms in the present paper. (See the examples in Section 3.)

Definition Assuming (A_1) - (A_5), define operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ by

$$[A(u),v] = \int_{Q_T} \left\{ \sum_{i=1}^n a_i(t,x,u,Du;u)D_iv + a_0(t,x,u,Du;u)v \right\} dtdx \qquad (2.3)$$

where the brackets $[\cdot,\cdot]$ mean the dualities in spaces $L^q(0,T;V^\star)$, $L^p(0,T;V)$.

By using the theory of monotone type operators, one can prove the following modifications of Theorems 1.2, 2.1 in [4]. (See [5].)

Theorem 2.1. *Assume* (A_1) - (A_5)*. Then for any* $f \in L^q(0,T;V^\star)$ *and* $u_0 \in L^2(\Omega)$ *there exists* $u \in L^p(0,T;V)$ *such that* $D_t u \in L^q(0,T;V^\star)$*,*

$$D_t u + A(u) = f, \quad u(0) = u_0. \tag{2.4}$$

Now we formulate an existence theorem in $(0,\infty)$. Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $u:(0,\infty)\to V$ such that for each fixed finite $T>0$, $u|_{(0,T)} \in L^p(0,T;V)$ and let $Q_\infty = (0,\infty)\times\Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $u: Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T.

Theorem 2.2. *Assume that the functions*

$$a_i : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{loc}(0,\infty;V) \to \mathbb{R}$$

satisfy the assumptions (A_1) - (A_5) *for any finite* T *and that* $a_i(t,x,\zeta_0,\zeta;u)|_{Q_T}$ *depend only on* $u|_{(0,T)}$ *(Volterra property). Then for any* $f \in L^q_{loc}(0,\infty;V^\star)$*,* $u_0 \in L^2(\Omega)$ *there exists* $u \in L^p_{loc}(0,\infty;V)$ *which is a solution of (2.4) for any finite* T*.*

3. Boundedness and stabilization

Theorem 3.1. *Let the assumptions of Theorem 2.2 be satisfied such that for all* $u \in L^p_{loc}(0,\infty;V)$*, sufficiently large* t

$$[g_2(u)](t) \geq const\, [1 + \sup_{\tau\in[0,t]} y(\tau)]^{-\sigma^\star/2} \tag{3.1}$$

$$\int_\Omega [k_2(u)](t,x)dx \leq const \left[\sup_{[0,t]} y^{\sigma/2} + \varphi(t) \sup_{[0,t]} y^{(p-\sigma^\star)/2} + 1\right] \tag{3.2}$$

with some positive constants where

$$y(t) = \int_\Omega u(t,x)^2 dx, \quad 0 < \sigma^\star < p-1, \quad \lim_\infty \varphi = 0, \quad 1 \leq \sigma < p - \sigma^\star.$$

Further, $\| f(t) \|_{V^\star}$ *is bounded for* $t \in (0,\infty)$*.*

Then for a solution $u \in L^p_{loc}(0,\infty;V)$ *of (2.4) in* $(0,\infty)$*,* y *is bounded in* $(0,\infty)$*.*

The proof of Theorem 3.1 is the same as that of Theorem 2.3 in [4].

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are fulfilled with Volterra operators*

$$g_1 : L^p_{loc}(0,\infty;V) \cap L^\infty(0,\infty;L^2(\Omega)) \to \mathbb{R}^+, \tag{3.3}$$

$$k_1 : L^p_{loc}(0,\infty;V) \cap L^\infty(0,\infty;L^2(\Omega)) \to L^q(\Omega) \tag{3.4}$$

such that the following monotonicity condition is satisfied:

$$\sum_{i=1}^n [a_i(t,x,\zeta_0,\zeta;u) - a_i(t,x,\zeta_0^\star,\zeta^\star;u)](\zeta_i - \zeta_i^\star) + \tag{3.5}$$

$$[a_0(t,x,\zeta_0,\zeta;u) - a_0(t,x,\zeta_0^\star,\zeta^\star;u)](\zeta_0 - \zeta_0^\star) \geq [g_2(u)](t)[|\zeta - \zeta^\star|^p + |\zeta_0 - \zeta_0^\star|^p].$$

We assume that for arbitrary fixed $u \in L^p_{loc}(0,\infty;V) \cap L^\infty(0,\infty;L^2(\Omega))$, $(\zeta_0,\zeta) \in \mathbb{R}^{n+1}$, *a.a.* $x \in \Omega$

$$\lim_{t\to\infty} a_i(t,x,\zeta_0,\zeta;u) = a_{i,\infty}(x,\zeta_0,\zeta), \quad i = 0,1,...,n, \tag{3.6}$$

exist and are finite where $a_{i,\infty}$ *satisfy the Carathéodory conditions (for fixed* u*). Finally, there exists* $f_\infty \in V^\star$ *such that*

$$\lim_{t\to\infty} \parallel f(t) - f_\infty \parallel_{V^\star} = 0. \tag{3.7}$$

Then for a solution $u \in L^p_{loc}(0,\infty;V)$ *of (2.4) we have*

$$\lim_{t\to\infty} \parallel u(t) - u_\infty \parallel_{L^2(\Omega)} = 0, \quad \lim_{T\to\infty} \int_{T-a}^{T+a} \parallel u(t) - u_\infty \parallel_V^p dt = 0 \tag{3.8}$$

for arbitrary fixed $a > 0$, *where* $u_\infty \in V$ *is the unique solution to*

$$A_\infty(u_\infty) = f_\infty \tag{3.9}$$

and the operator $A_\infty : V \to V^\star$ *is defined (for* $z, v \in V$*) by*

$$\langle A_\infty(z), v\rangle = \sum_{i=1}^n \int_\Omega a_{i,\infty}(x,z,Dz)D_i v dx + \int_\Omega a_{0,\infty}(x,z,Dz)v dx. \tag{3.10}$$

Proof. It is not difficult to show that $A_\infty : V \to V^\star$ is bounded, hemicontinuous, strictly monotone and coercive which implies the existence of a unique solution of (3.9) (see, e.g., [6]).

If u is a solution of (2.4) in $(0,\infty)$ then by (3.9) one obtains

$$\langle D_t[u(t) - u_\infty], u(t) - u_\infty\rangle + \langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle = \tag{3.11}$$

$$\langle f(t) - f_\infty, u(t) - u_\infty\rangle.$$

By using the notation

$$\langle [A_u(u_\infty)](t), z\rangle = \int_\Omega \sum_{i=1}^n a_i(t,x,u_\infty(x), Du_\infty(x); u) D_i z dx +$$

$$\int_\Omega a_0(t,x,u_\infty(x), Du_\infty(x); u) z dx,$$

(3.5) and Young's inequality, we obtain for the second term in (3.11)

$$\langle [A(u)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle = \langle [A(u)](t) - [A_u(u_\infty)], u(t) - u_\infty\rangle + \tag{3.12}$$

$$\langle [A_u(u_\infty)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle \geq [g_2(u)](t) \parallel u(t) - u_\infty \parallel_V^p -$$

$$\varepsilon^p/p \parallel u(t) - u_\infty \parallel_V^p - 1/(q\varepsilon^q) \parallel [A_u(u_\infty)](t) - A_\infty(u_\infty) \parallel_{V^\star}^q$$

with arbitrary $\varepsilon > 0$. By Vitali's theorem we obtain from (A_2), (3.3), (3.4), (3.6)

$$\lim_{t\to\infty} \parallel [A_u(u_\infty)](t) - A_\infty(u_\infty) \parallel_{V^\star} = 0. \tag{3.13}$$

Finally, by Young's inequality we have for the right hand side of (3.11)

$$|\langle f(t) - f_\infty, u(t) - u_\infty\rangle| \leq \varepsilon^p/p \parallel u(t) - u_\infty \parallel_V^p - 1/(q\varepsilon^q) \parallel f(t) - f_\infty \parallel_{V^\star}^q .$$

Thus, choosing sufficiently small $\varepsilon > 0$, since $\int_\Omega u(t,x)^2 dx$ is bounded, (3.1), (3.7), (3.11) - (3.13) and Hölder's inequality yield for $y(t) = \parallel u(t) - u_\infty \parallel_{L^2(\Omega)}^2$

$$y'(t) + \tilde{c} y(t)^{p/2} \leq y'(t) + c^\star \parallel u(t) - u_\infty \parallel_V^p \leq \psi(t), \tag{3.14}$$

where $\lim_{t\to\infty} \psi(t) = 0$ and $c^\star, \tilde{c}$ are positive constants. It is not difficult to show that (3.14) implies the first part of (3.8) (see [3]). Combining the first part of (3.8) and the second part of (3.14) one obtains the second part of (3.8).

Theorem 3.3. *Assume that the conditions of Theorem 3.2 are fulfilled in the following modified form:*

$$a_0(t,x,\zeta_0,\zeta;u) = a_0^1(t,x,\zeta_0,\zeta;u) + a_0^2(t,x,\zeta_0;u) \tag{3.15}$$

where a_0^1 *satisfies* (A_1), (A_2) *and* a_0^2 *satisfies*

$$|a_0^2(t,x,\zeta_0;u) \leq g_1(u)|\zeta_0| + [\tilde{k}_1(u)](x)$$

with some $\tilde{k}_1(u) \in L^2(\Omega)$ *and for all* $u, u^\star, v \in L^p_{loc}(0,\infty;V) \cap L^\infty(0,\infty;L^2(\Omega))$, $T_2 > T_1 \geq 0$,

$$\int_{T_1}^{T_2} \int_\Omega |a_0^2(t,x,v;u) - a_0^2(t,x,v;u^\star)|^2 dt dx \leq c_3^2 \int_{\max\{0,T_1-a\}}^{T_2} \int_\Omega [u-u^\star]^2 dt dx \tag{3.16}$$

with some constants $a, c_3 > 0$ *and instead of (3.5) we have*

$$\sum_{i=1}^{n}[a_i(t,x,\zeta_0,\zeta;u) - a_i(t,x,\zeta_0^\star,\zeta^\star;u)](\zeta_i - \zeta_i^\star) + \tag{3.17}$$

$$[a_0^1(t,x,\zeta_0,\zeta;u) - a_0^1(t,x,\zeta_0^\star,\zeta^\star;u)](\zeta_0 - \zeta_0^\star)$$
$$\geq [g_2(u)](t)[|\zeta - \zeta^\star|^p + |\zeta_0 - \zeta_0^\star|^p] + c_2|\zeta_0 - \zeta_0^\star|^2$$

where the constant c_2 *satisfies* $c_2 > c_3$*. Further, assume that for any fixed* $u \in L^p_{loc}(0,\infty;V) \cap L^\infty(0,\infty;L^2(\Omega))$*,* $w \in V$

$$|a_i(t,x,\zeta_0,\zeta;u) - a_{i,\infty}(x,\zeta_0,\zeta)| \leq \Phi(t)[|\zeta_0|^{p-1} + |\zeta|^{p-1}], \quad i = 1,...,n, \tag{3.18}$$

$$|a_0^1(t,x,\zeta_0,\zeta;u) - a_{0,\infty}^1(x,\zeta_0,\zeta)| \leq \Phi(t)[|\zeta_0|^{p-1} + |\zeta|^{p-1}],$$
$$|a_0^2(t,x,\zeta_0;w) - a_{0,\infty}^2(x;w)| \leq \Phi(t)(1 + |\zeta_0|), \quad \| f(t) - f_\infty \|_{V^\star} \leq \Phi(t)$$

with limit functions $a_{0,\infty}^1 : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$*,* $a_{0,\infty}^2 : \Omega \times V \to \mathbb{R}$ *where*

$$\lim_\infty \Phi = 0, \quad \int_0^\infty \Phi(t)^q dt < \infty. \tag{3.19}$$

Then for a solution of (2.4) in $(0,\infty)$ *we have*

$$\int_0^\infty \| u(t) - u_\infty \|_V^p \, dt < \infty, \quad \int_0^\infty \| u(t) - u_\infty \|_{L^2(\Omega)}^2 \, dt < \infty, \tag{3.20}$$

$$\lim_{t\to\infty} \| u(t) - u_\infty \|_{L^2(\Omega)} = 0, \tag{3.21}$$

$$\int_T^\infty \| u(t) - u_\infty \|_{L^2(\Omega)}^2 \, dt \leq const \left\{ e^{-\gamma T} + \int_0^T \left[e^{-\gamma(T-t)} \int_t^\infty \Phi(\tau)^q d\tau \right] dt \right\} \tag{3.22}$$

with some constant $\gamma > 0$ *where* $u_\infty \in V$ *is the unique solution to*

$$A_\infty(u_\infty) = f_\infty \tag{3.23}$$

and the operator $A_\infty : V \to V^\star$ *is defined for* $z, v \in V$ *by*

$$\langle A_\infty(z), v\rangle = \sum_{i=1}^{n} \int_\Omega a_{i,\infty}(x,z,Dz) D_i v dx + \int_\Omega a_{0,\infty}^1(x,z,Dz) v dx + \int_\Omega a_{0,\infty}^2(x;z) v dx.$$

Proof Integrating (3.11) over (T_1, T_2), one obtains by (3.16) for $y(t) = \int_\Omega [u(t,x) - u_\infty]^2 dx$ (by using Young's inequality, similarly to (3.14))

$$y(T_2) - y(T_1) + c^\star \int_{T_1}^{T_2} \| u(t) - u_\infty \|_V^p \, dt + c_2 \int_{T_1}^{T_2} y(t) dt - \tag{3.24}$$

$$\left[\int_{T_1}^{T_2}\int_\Omega |a_0^2(t,x,u;u)-a_0^2(t,x,u;u_\infty)|^2 dtdx\right]^{1/2}\left[\int_{T_1}^{T_2} ydt\right]^{1/2} \le \int_{T_1}^{T_2}\Phi^q dt.$$

Since y is bounded, (3.16), (3.24) with $T_1 = 0$ and $c_2 > c_3$ and (3.19) imply (3.20). Further, by (3.20), (3.16)

$$\int_0^\infty \int_\Omega |a_0^2(t,x,u;u)-a_0^2(t,x,u;u_\infty)|^2 dtdx < \infty$$

thus (3.20), (3.24) imply (3.21). Because, first observe that by (3.20)

$$\liminf_{t\to\infty} y(t) = 0$$

Hence there exist

$$T_1 < T_2 < ... < T_k < ... \to +\infty \text{ such that } \lim_{k\to\infty} y(T_k) = 0.$$

Applying (3.24) to $T_1 = T_k$ and $T_2 = T$ with $T > T_k$, we obtain

$$0 \le y(T) \le y(T_k) + a_k \text{ where } \lim_{k\to\infty} a_k = 0$$

and so $\lim_\infty y = 0$.

Finally, from (3.16), (3.21), (3.24) we obtain (for $T_1 > a$) as $T_2 \to \infty$

$$-y(T_1) + c^\star \int_{T_1}^\infty \| u(t) - u_\infty \|_V^p \, dt + c_2 \int_{T_1}^\infty ydt -$$

$$c_3 \left[\int_{T_1-a}^\infty ydt\right]^{1/2}\left[\int_{T_1}^\infty ydt\right]^{1/2} \le \text{const} \int_{T_1}^\infty \Phi(t)^q dt.$$

Hence, by using the notation $Y(T) = \int_T^\infty y(t)dt$,

$$Y'(T_1) + (c_2 - c_3/2)Y(T_1) - (c_3/2)Y(T_1 - a) \le \tag{3.25}$$

$$Y'(T_1) + c_2 Y(T_1) - c_3 Y(T_1 - a)^{1/2} Y(T_1)^{1/2} \le \text{const} \int_{T_1}^\infty \Phi^q dt.$$

Since the real part of the roots of the characteristic equation

$$\lambda + (c_2 - c_3/2) - (c_3/2)\mathrm{e}^{-\lambda} = 0$$

is negative, we obtain for the solution of (3.25) the inequality (3.22).

Examples In [4] examples of the following type were considered:

$$a_i(t,x,\zeta_0,\zeta;u) = b(t,x,[H(u)](t,x))\zeta_i|\zeta|^{p-2}, \quad i = 1,...,n,$$

$$a_0(t,x,\zeta_0,\zeta;u) = b_0(t,x,[H_0(u)](t,x))\zeta_0|\zeta_0|^{p-2} +$$
$$\hat{b}_0(t,x,[F_0(u)](t,x))\hat{\alpha}_0(t,x,\zeta_0,\zeta)$$

where $b, b_0, \hat{b}_0, \hat{\alpha}_0$ are Carathéodory functions satisfying

$$b(t,x,\theta) \geq \frac{c_2}{1+|\theta|^{\sigma^\star}}, \quad b_0(t,x,\theta) \geq \frac{c_2}{1+|\theta|^{\sigma^\star}}$$

with some positive constants $c_2, \sigma^\star < p-1$,

$$|\hat{b}_0(t,x,\theta)| \leq 1 + |\theta|^{p-1-\rho^\star} \text{ with } \rho^\star < p-1$$

and

$$|\hat{\alpha}_0(t,x,\zeta_0,\zeta)| \leq c_1(|\zeta_0|^{\hat{\rho}} + |\zeta|^{\hat{\rho}}), \quad \sigma^\star + \hat{\rho} < \rho^\star, \quad \hat{\rho} \geq 0.$$

Finally,

$$H, H_0 : L^p(0,T;V) \to C(\overline{Q_T}), \quad F_0 : L^p(0,T;V) \to L^p(Q_T)$$

are linear continuous operators of Volterra type.

One can show that the examples of the above type satisfy the conditions of the above existence theorems in the case when

$$H, H_0 : L^p(Q_T) \to L^p(Q_T)$$

are continuous linear operators (for a fixed $T > 0$ or arbitrary finite $T > 0$, respectively) and b, b_0 are bounded. Thus, H and H_0 may have more general forms: also the forms, formulated in [4] for F_0. It is not difficult to formulate conditions on H, H_0, F_0 and $\hat{b}_0, \hat{\alpha}_0$ which imply the conditions of theorems on boundedness and stabilization of solutions.

4. Periodic solutions

Now consider equation (1.1) in the following modified form:

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u,Du;u_t)] + a_0(t,x,u,Du;u_t) = f \text{ with } u_0 = \psi \tag{4.1}$$

where $u_t(s) = u(t+s)$, $s \in (-a,0)$ and $a_i : Q_T \times \mathbb{R}^{n+1} \times L^p(-a,0;V) \to \mathbb{R}$, $\psi \in L^p(-a,0;V)$ are given functions and we want to find $u \in L^p_{loc}(-a,\infty;V)$ satisfying (4.1) in weak sense. We shall show that for some ψ there exists a T-periodic solution.

Theorem 4.1. *Assume that functions a_i satisfy (A_1) - (A_5) in the following modified form: the last term in a_i is $u_t \in L^p(-a,0;V)$ instead of $u \in L^p(0,T;V)$ and in (A_5) instead of weak convergence in $L^p(0,T;V)$ and strong convergence in $L^p(0,T;W^{1-\delta,p}(\Omega))$ we write weak convergence in $L^p(-a,0;V)$ and strong convergence in $L^p(-a,0;W^{1-\delta,p}(\Omega))$, respectively.*

Then there exists $u \in L^p(-a,T;V)$ such that $D_t u \in L^q(-a,T;V^\star)$,

$$D_t u + A(u) = f \text{ for } t \in (0,T), \quad u(t) = u(t+T) \text{ for } t \in [-a,0] \tag{4.2}$$

where operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ *is defined by (2.3) such that the terms* u *after ";" are substituted by* u_t*, i.e.*

$$[A(u), v] = \int_{Q_T} [\sum_{i=1}^{n} a_i(t, x, u, Du; u_t) D_i v + a_0(t, x, u, Du; u_t) v] dt dx.$$

Proof Define operator $\tilde{A}$ for $u, v \in L^p(0,T;V)$ by

$$[\tilde{A}(u), v] = \int_{Q_T} [\sum_{i=1}^{n} a_i(t, x, u, Du; (Pu)_t) D_i v + a_0(t, x, u, Du; (Pu)_t) v] dt dx$$

where

$$(Pu)(t) = u(t + kT) \text{ if } t + kT \in (0,T) \text{ for some } k = 0, 1, ...$$

Then $\tilde{A}$ is bounded, demicontinuous, pseudomonotone with respect to

$$D(L) = \{u \in L^p(0,T;V) : D_t u \in L^q(0,T;V^\star), u(T) = u(0)\}$$

where $D(L)$ is the domain of the maximal monotone closed densely defined operator $L = D_t$. Consequently, for any $f \in L^q(0,T;V^\star)$ there exists $u \in D(L)$ satisfying $D_t u + \tilde{A}(u) = f$ for $t \in (0,T)$. (See [2], [6].). Then, clearly, $Pu \in L^p(-a,T;V)$ satisfies $D_t u \in L^q(-a,T;V^\star)$ and (4.2).

From Theorem 4.1 immediately follows

Theorem 4.2. *Assume that*

$$a_i : Q_\infty \times \mathbb{R}^{n+1} \times L^p(-a, 0; V) \to \mathbb{R}$$

satisfy the conditions of Theorem 2.2 and a_i*,* f *are* T*-periodic in* t*:*

$$a_i(t + T, x, \zeta_0, \zeta; w) = a_i(t, x, \zeta_0, \zeta; w), \quad f(t + T) = f(t)$$

for a.e. $(t,x) \in Q_\infty$*, all* $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$*,* $w \in L^p(-a,0;V)$*.*

Then there exists $u \in L^p_{loc}(-a,\infty;V)$ *such that* $D_t u \in L^q_{loc}(-a,\infty;V^\star)$*,*

$$D_t u + A(u) = f, \quad u(t) = u(t + T) \text{ for } t \in (0, \infty).$$

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