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Maximum Principles, Boundary Value Problems and Stability for First Order Delay Equations with Oscillating Coefficient*

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Abstract. On the basis of the maximum principles various assertions on unique solvability of boundary value problems, positivity of the Green's function of the generalized periodic problem and stability for delay equation with oscillating coefficient are proposed.

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1. Definitions of Maximum Principles

The theory of the delay differential equations had been started with the equation

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \in [0, \omega], \tag{1.1}$$

where

$$x(s) = \varphi(s) \quad \text{for} \quad s < 0, \tag{1.2}$$

and φ is a corresponding continuous function which is called an initial function. Note that we have to add the equality (1.2) to equation (1.1) in order to define what must be set instead of $x(t - \tau(t))$ when $t - \tau(t) < 0$. The problem to define a homogeneous object is the crucial one. If equation (1.1) is studied for all possible continuous initial functions φ, then the space of solutions of this equation becomes infinite-dimensional and there is no a direct connection between maximum principles and problems of

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existence and uniqueness of solutions to boundary value problems. The notion of Green's function does not appear in this case.

In the paper [2] a fully formed tradition to consider a solution of delay equation (1.1) as a continuously prolonged initial function $\varphi(t)$ was avoided and a homogeneous object was defined as equation (1.1) with the initial functions

$$x(\xi) = 0 \qquad \text{for } \xi < 0. \tag{1.3}$$

Precisely equation (1.1), (1.3) acts as a homogeneous equation in the theory of ordinary differential equations: the space of its solutions becomes one-dimensional and the formula for representation of the general solution of the nonhomogeneous equation

$$x'(t) + p(t)x(t - \tau(t)) = f(t), \qquad t \in [0, \omega], \tag{1.4}$$

with the initial function (1.3) is the following

$$x(t) = \int_0^t C(t, s)f(s)ds + C(t, 0)x(0), \tag{1.5}$$

where $C(t, s)$ is called the Cauchy function of equation (1.4). Note that $C(t, s)$ as the function of the argument t for each fixed s is a solution of the equation

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \in [s, \omega], \tag{1.6}$$

$$x(\xi) = 0 \qquad \text{for } \xi < s, \tag{1.7}$$

satisfying the condition $C(s, s) = 1$.

In the mathematical literature there are several definitions of the maximum principles. Talking about them, we mean assertions of the following three types.

1) **Maximum inequalities principle** can be formulated as follows**:** *solutions of inequalities are greater or less than the solution of the equation.*

First results about comparison of solutions for delay differential equations can be found in the well known book of A. D. Myshkis [10]. The integral representations of solutions to boundary value problems for delay differential equations (1.4), (1.3), (1.8),

$$lx = c, \tag{1.8}$$

where $l : D_{[0,\omega]} \to R^1$ is a linear bounded functional defined on the space of absolutely continuous functions $D_{[0,\omega]}$ and $c \in R^1$, were proposed by N. V. Azbelev [2]. The problem (1.4), (1.3), (1.8) is uniquely solvable in the space $D_{[0,\omega]}$ for each $f \in L_{[0,\omega]}$ and $c \in R^1$ if and only if the homogeneous problem

$$(Mx)(t) \equiv x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \in [0, \omega], \quad lx = 0, \tag{1.9}$$

has only the trivial solution [1]. In this case the solution of problem (1.4), (1.3), (1.8) can be written in the form

$$x(t) = \int_0^\omega G(t, s)f(s)ds + X(t), \tag{1.10}$$

where $X(t)$ is a solution of the homogeneous equation $(Mx)(t) = 0,\ t \in [0, \omega]$, satisfying the boundary condition $lx = c$. The kernel $G(t, s)$ is called the Green's function. On the basis of this representation the comparison results were first formulated in the form of positivity/negativity of corresponding Green's functions in the paper [2].

For first order functional differential equations various assertions about positivity of Green's functions of the Cauchy and periodic problems were obtained in the papers [3, 5, 7] and in the terminology of inequalities - in the book [9], where problems with the generalized periodic condition $\nu x(0)+\mu x(\omega) = c$ has been also considered. Results about positivity of Green's functions for more general boundary conditions can be found in the recent paper [6].

2) **Maximum principle as a boundedness of solutions:** *there exists a positive constant N such that* $|x| \le N(\|f\| + |c|)$, where $\|f\|$ is the norm in the spaces $L^{\infty}_{[0,\omega]}$ or $L_{[0,\omega]}$ respectively.

This is actually a problem of continuous dependence of solutions on the right hand side f and the boundary condition c. The formula of the integral representation of solution (1.10) reduces the maximum boundedness principle to the fact of the unique solvability of boundary value problems.

3) **Maximum boundaries principle** usually means that maximal and minimal values of the solution can be only at the points 0 or ω.

For the first order ordinary differential equation $x'(t)+p(t)x(t) = 0$ with oscillating coefficient this situation is impossible. If the coefficient $p(t)$ changes its sign at the points t_k $(k = 1, 2, 3, ...)$, we can define the generalized maximum principle as follows: *solutions have their maximums and minimums only at these points t_k.*

2. Generalized Maximum Principle for Equations with Oscillating Coefficient

Consider the delay equation

$$x'(t) + p(t)x(h(t)) = f(t), \qquad t \in [0, +\infty), \tag{2.1}$$

$$x(\xi) = 0, \qquad \xi < 0, \tag{2.2}$$

with oscillating coefficient $p(t)$ changing its sign at the points t_k $(k = 1, 2, 3, ...)$. We set $t_0 = 0$.

Theorem 2.1. *Let one of two conditions a) or b) be satisfied:*

a) the coefficient $p(t)$ satisfies the inequalities: $p(t) \ge 0$ for $[t_{2k}, t_{2k+1}]$, $p(t) \le 0$ for $[t_{2k+1}, t_{2k+2}]$ for $k = 0, 1, 2, ...,$ and

$$\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1, \quad k = 0, 1, 2, ..., \tag{2.3}$$

and the deviating argument $h(t)$ *satisfies the inequalities:* $t_{2k-1} \leq h(t)$ *for* $t \in [t_{2k}, t_{2k+1}]$.

b) the coefficient $p(t)$ *satisfies the inequalities:* $p(t) \leq 0$ *for* $[t_{2k}, t_{2k+1}]$, $p(t) \geq 0$ *for* $[t_{2k+1}, t_{2k+2}]$ *for* $k = 0, 1, 2, ...$, *and*

$$\int_{t_{2k+1}}^{t_{2k+2}} p(t)dt < 1, \quad k = 0, 1, 2, ..., \tag{2.4}$$

and the deviating argument $h(t)$ *satisfies the inequalities:* $t_{2k} \leq h(t)$ *for* $t \in [t_{2k+1}, t_{2k+2}]$.

Then $C(t, s) > 0$ *for* $0 \leq s \leq t < +\infty$.

Proof. Let us prove that the condition *a)* implies $C(t, s) > 0$ for $0 \leq s \leq t < +\infty$. Denote $p(t) = p^+(t) - p^-(t)$, where $p^+(t) \geq 0$, $p^-(t) \geq 0$, and consider the equation

$$x'(t) + p^+(t)x(h(t)) = 0, \qquad t \in [0, +\infty), \tag{2.5}$$

$$x(\xi) = 0, \qquad \xi < 0. \tag{2.6}$$

Denote by $C^+(t, s)$ its Cauchy function. The function $x(t) = C^+(t, s)$ for each fixed s as a function of the argument t is a solution of the equation

$$x'(t) + p^+(t)x(h(t)) = 0, \qquad t \in [s, +\infty), \tag{2.7}$$

$$x(\xi) = 0, \qquad \xi < 0.$$

It is clear that the solution $x(t) = C^+(t, s)$ is equal to the constants on each of the intervals $[t_{2k+1}, t_{2k+2}]$, $k = 0, 1, 2, ...$. The inequality (2.3) implies, according to Corollary 1.1 [7], positivity of $x(t) = C^+(t, s)$ for $[t_{2k}, t_{2k+1}]$, $k = 0, 1, 2, ...$. The positivity of the Cauchy function $C^+(t, s)$ of equation (2.5), (2.6) for $0 \leq s \leq t < +\infty$ implies, according to Theorem 2 of the paper [7], the positivity of the Cauchy function $C(t, s)$ of equation (2.1),(2.2) for $0 \leq s \leq t < +\infty$.

Analogously one can prove that the condition *b)* implies $C(t, s) > 0$ for $0 \leq s \leq t < +\infty$. □

Inequalities (2.3) and (2.4) cannot be improved as the following examples demonstrate.

Example 2.1. Consider the equation

$$x'(t) + x([t]) = 0, \qquad t \in [0, +\infty), \tag{2.8}$$

where $[t]$ is the integer part of t. The solution of this equation is the following

$$x(t) = C(t, 0) = \begin{cases} 1 - t, & 0 \leq t < 1, \\ 0, & 1 \leq t. \end{cases} \tag{2.9}$$

Example 2.2. Consider the equation

$$x'(t) + p(t)x([t]) = 0, \qquad t \in [0, +\infty), \tag{2.10}$$

where

$$p(t) = \begin{cases} b(t), & t_{2k} \leq t \leq t_{2k+1}, \\ -a(t), & t_{2k+1} < t < t_{2k+2}. \end{cases} \tag{2.11}$$

$a(t) \geq 0,\ b(t) > 1 + \varepsilon,\ \varepsilon > 0$. The solution $x(t) = C(t, 0)$ changes its sign in every interval $[t_{2k}, t_{2k+1}]$, $k = 0, 1, 2, \ldots$.

Theorem 2.2. *Let the condition a) (b)) of Theorem 2.1 be fulfilled, then the modulus of every solution x of the homogeneous equation*

$$x'(t) + p(t)x(h(t)) = 0, \quad t \in [0, +\infty), \tag{2.12}$$

where $x(\xi) = 0$ for $\xi < 0$, has its maximums only at the points t_{2k} (t_{2k+1}) and its minimums only at the points t_{2k+1} (t_{2k}), $k = 0, 1, 2, \ldots$.

In order to prove it, let us note that according to Theorem 2.1, solutions of the homogeneous equation (2.12) do not change their signs. This implies that $|x(t)|$ does not increase when $p(t) \leq 0$ and does not decrease when $p(t) \leq 0$.

3. Applications of Maximum Principle to Boundary Value Problems

The maximum principle obtained in Theorem 2.2 implies various assertions about unique solvability of boundary value problems for equation (1.4).

Theorem 3.1. *Let the condition a) of Theorem 2.1 be satisfied. Then the following assertions are true:*

1) *If $l : C_{[0,\omega]} \to R^1$ is a linear nonzero positive functional, then boundary value problem (1.4), (1.3), (1.8) is uniquely solvable for each $f \in L_{[0,\omega]},\ c \in R^1$;*
2) *the boundary value problem (1.4), (1.3), (3.1), where*

$$lx \equiv \sum_{k=1}^{n} \{x(t_{2k}) - m_k x\} = c, \tag{3.1}$$

and the norm of every linear functional $m_k : C_{[t_{2k-1}, t_{2k}]} \to R^1$ is less than one, is uniquely solvable for each $f \in L_{[0,\omega]},\ c \in R^1$;

3) *the boundary value problem (1.4),(1.3), (3.2)*

$$\sum_{k=0}^{n}\alpha_k x(s_{2k})=\sum_{k=0}^{n}\beta_k x(s_{2k+1})+c, \tag{3.2}$$

where the inequalities $t_{2k}\le s_{2k}<s_{2k+1}\le t_{2k+1}$ *and* $\alpha_k\ge\beta_k\ge 0$ *are satisfied for* $k=0,1,2,...,n$, *and there exists* j *such that* $\alpha_j>\beta_j$, *is uniquely solvable for each* $f\in L_{[0,\omega]},\ c\in R^1$;

4) *the boundary value problem (1.4),(1.3),(3.3), where*

$$\sum_{j=1}^{n}\left\{\int_{t_{2j-1}}^{t_{2j}}\alpha(t)x(t)dt\right\}=c,\quad 0=t_0\le t_1<t_2<...<t_{2n-1}<t_{2n}\le\omega, \tag{3.3}$$

in the case when $\alpha(t)\le 0$ *for* $t\in[t_{2j-1},s_j]$, $\alpha(t)\ge 0$ *for* $t\in[s_j,t_{2j}]$, *where* $t_1<s_1<t_2,...,\ t_{2n-1}<s_n<t_{2n}$, $\int_{t_{2j-1}}^{t_{2j}}\alpha(t)dt\ge 0$, $j=1,...,k$, *and there exists* j *such that* $\int_{t_{2j-1}}^{t_{2j}}\alpha(t)dt>0$, *is uniquely solvable for each* $f\in L_{[0,\omega]},\ c\in R^1$.

Example 3.1. The periodic problem for the equation

$$x'(t)=0,\qquad t\in[0,\omega],$$

has the nontrivial solution $x(t)\equiv 1,\ t\in[0,\omega]$. This demonstrates that the condition about existence of such j that $\alpha_j>\beta_j$ is essential.

The location of the points s_k in assertion 3) is essential. If instead of the inequality $t_{2k}\le s_{2k}<s_{2k+1}\le t_{2k+1}$, we assume that $t_{2k}\le s_{2k}<s_{2k+1}\le t_{2k+2}$, then the assertion about unique solvability is not true as the following example demonstrates.

Example 3.2. Consider the generalized periodic problem

$$x'(t)+p(t)x(0)=0,\qquad 2x\left(\frac{1}{2}\right)=x(1),\quad t\in[0,1],$$

where

$$p(t)=\begin{cases}1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t\le 1.\end{cases}$$

This generalized periodic problem has the nontrivial solution

$$x(t)=\begin{cases}1-t, & 0\le t<\frac{1}{2},\\ t, & \frac{1}{2}\le t\le 1.\end{cases}$$

4. Exponential Stability of Equation with Oscillating Coefficient

Consider the equation

$$(Mx)(t) \equiv x'(t) + p(t)x(h(t)) = f(t), \qquad t \in [0, +\infty), \tag{4.1}$$

$$x(\xi) = 0, \qquad \xi < 0, \tag{4.2}$$

with oscillating coefficient $p(t)$ changing its sign at the points t_k $(k = 1, 2, 3, ...)$. Let us assume that $t_0 = 0$ and there exist positive numbers c_1 and c_2 such that $c_1 < t_{k+1} - t_k < c_2$ for every k.

Definition 4.1 [1]. *We say that equation (4.1) is exponentially stable, if for each solution x of the corresponding homogeneous equation*

$$x'(t) + p(t)x(h(t)) = 0, \qquad t \in [0, +\infty), \tag{4.3}$$

$$x(\xi) = 0, \qquad \xi < 0,$$

there exist positive constants α and N such that $|x(t)| \le Ne^{-\alpha t}$ for $t \in [0, +\infty)$.

Theorem 4.1. *Let the following conditions be satisfied:*

a) the coefficient $p(t)$ satisfies the inequalities: $p(t) \ge 0$ for (t_{2k}, t_{2k+1}), $p(t) \le 0$ for (t_{2k+1}, t_{2k+2}) and $\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1$ for $k = 0, 1, 2, ...$;

b) the deviating argument $h(t)$ satisfies the inequalities: $t_{k-1} \le h(t)$ for $t \in [t_k, t_{k+1}]$, $h(t) \le t_{2k-1}$ for $t \in [t_{2k-1}, t_{2k}]$;

c) there exists a number γ such that

$$\gamma_{k+1} \equiv \exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(t)\chi(h(t), t_{2k})dt\right] + \int_{t_{2k+1}}^{t_{2k+2}} |p(t)|\, dt \le \gamma < 1, \tag{4.4}$$

$k = 0, 1, 2, ...,$ *where*

$$\chi(t, s) = \begin{cases} 1, & t \ge s, \\ 0, & t < s. \end{cases} \tag{4.5}$$

Then equation (4.1) is exponentially stable.

Proof. According to Theorem 2.1, the conditions *a*) and *b*) imply positivity of the Cauchy function $C(t, s)$ of equation (4.1),(4.2) for $0 \le s \le t < \infty$. The function

$$u(t) = \begin{cases} \gamma_0 ... \gamma_{k-1} \exp\left[-\int_{t_{2k}}^{t} p(t)\chi(h(s), t_{2k})ds\right], & t_{2k-2} \le t < t_{2k-1}, \\ \gamma_0 ... \gamma_{k-1} \exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(s)\chi(h(s), t_{2k})ds\right] \\ \quad + \int_{t_{2k+1}}^{t} |p(s)|\, ds, & t_{2k-1} \le t < t_{2k}, \end{cases} \quad k = 1, 2, ... \tag{4.6}$$

where $\gamma_0 = 1$, satisfies the inequality $(Mu)(t) \geq 0$. The positivity of $C(t,s)$ implies that $u(t) \geq x(t)$ for $t \in [0,+\infty)$, where the function x is a solution of the initial problem $(Mx)(t) = 0,\ t \in [0,+\infty),\ x(0) = 1$. □

Remark 4.1. The inequality

$$\exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(t)\chi(h(t),t_{2k})dt\right] + \int_{t_{2k+1}}^{t_{2k+2}} |p(t)|\,dt < 1, \qquad k = 0,1,2,...., \tag{4.7}$$

cannot be set instead of the condition c) as the following example demonstrates.

Example 4.1. Consider equation (4.3), where $h(t) \equiv t$,

$$p(t) = \begin{cases} \frac{1}{t^2+1}, & 2k \leq t < 2k+1, \\ 0, & 2k+1 \leq t < 2k+2, \end{cases} \qquad k = 0,1,2,... \tag{4.8}$$

Its nontrivial solutions tend to constants when $t \to +\infty$. Note that the condition c) avoids the possibility that $\lim_{k\to\infty} \int_{t_{2k}}^{t_{2k+1}} p(t)\chi(h(t),t_{2k})\,dt = 0$.

Theorem 4.2. *Let the conditions a) and b) of Theorem 4.1 be fulfilled, the deviating argument* $h(t)$ satisfy the inequality $t - h(t) \leq \tau$ for $t \in [0,\infty)$ *and a number* γ *exist such that*

$$\exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(t)\chi(h(t),t_{2k})dt\right]\left\{1 + \exp\left[\int_{t_{2k+1}-\tau}^{t_{2k+1}} p(\xi)d\xi\right]\int_{t_{2k+1}}^{t_{2k+2}} |p(t)|\,dt\right\} \leq \gamma < 1, \tag{4.9}$$

$k = 0,1,2,....$ *Then equation (4.1) is exponentially stable.*

Example 4.2. Assume that $h(t) \geq 0,\ t_{2k-1} \leq h(t)$ for $t \in [t_{2k}, t_{2k+1}],\ p(t+2) = p(t)$, where

$$p(t) = \begin{cases} \ln(1+t), & 0 \leq t \leq 1, \\ -\mu, & 1 < t < 2. \end{cases}$$

If $\mu < \frac{4-e}{4}$, *then equation (4.1),(4.2) with this coefficient* $p(t)$ *is exponentially stable.*

5. Positivity of Green's Function of Generalized Periodic Problems

Consider the equation

$$(Mx)(t) \equiv x'(t) + p(t)x(h(t)) = f(t), \qquad t \in [0,\omega], \tag{5.1}$$

$$x(\xi) = 0, \qquad \xi < 0, \tag{5.2}$$

with oscillating coefficient $p(t)$ changing its sign at the points t_k $(k = 1, 2, 3, ..., 2m-1)$ of the interval $[0, \omega]$. Denote $t_0 = 0$ and $t_{2m} = \omega$. For this equation we consider the generalized periodic condition

$$x(0) = \beta x(\omega). \tag{5.3}$$

Theorem 5.1. *Let the following conditions be satisfied:*

a) the coefficient $p(t)$ satisfies the inequalities: $p(t) \geq 0$ for (t_{2k}, t_{2k+1}), $p(t) \leq 0$ for (t_{2k+1}, t_{2k+2}) and $\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1$, for $k = 0, 1, 2, ..., m-1$;

b) the deviating argument $h(t)$ satisfies the inequalities: $t_{k-1} \leq h(t)$ for $t \in [t_k, t_{k+1}]$, $h(t) \leq t_{2k-1}$ for $t \in [t_{2k-1}, t_{2k}]$;

c) the inequality $\gamma_1 ... \gamma_m < \frac{1}{\beta}$ is satisfied, where γ_k are defined by formulas (4.4).

Then the Green's function $G(t, s)$ of the generalized periodic problem (5.1), (5.2), (5.3) is positive for $t, s \in [0, \omega]$.

Proof. According to Theorem 2.1, the conditions $a)$ and $b)$ implies the positivity of the Cauchy function $C(t, s)$ for $0 \leq s \leq t \leq \omega$. Now it is clear that the function $u(t)$ defined by formula (4.6) satisfies the inequality $u(t) \geq C(t, 0)$ for $t \in [0, \omega]$. The condition $c)$ implies the inequality $C(\omega, 0) < \frac{1}{\beta}$ and consequently the unique solvability of the problem (5.1), (5.2), (5.3). The Green's function of problem (5.1), (5.2), (5.3) has the representation

$$G(t, s) = C(t, s) + \frac{\beta C(\omega, s)}{1 - \beta C(\omega, 0)} C(t, 0), \tag{5.4}$$

where $C(t, s) = 0$ for $t < s$. The inequalities $C(t, s) > 0$ for $0 \leq s \leq t \leq \omega$ and $C(\omega, 0) < \frac{1}{\beta}$ imply that $G(t, s) > 0$ for $t, s \in [0, \omega]$. □

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