MUK PUBLICATIONS
Open Access Publisher

International Journal of Qualitative Theory of Differential Equations and Applications
Vol. 4, No. 1 (January-June, 2018)

The differential equation with a power delay and its numerical discretizations*

Jan Čermák

Institute of Mathematics, Brno University of Technology, Brno, Czech Republic

Received: 15th April 2017 Revised : 10th May 2017 Accepted: 17th September 2017

Abstract. This paper deals with the asymptotic investigation of the exact and the numerical solution of a linear differential equation with a power delay. We pay a special attention to the application of the trapezoidal rule and derive the upper bound for the solutions of this discretization. Moreover, comparing the estimates for the exact and the numerical solutions we can observe the correspondence in the asymptotic behaviour of both types of solutions.

AMS Subject Classifications: 34K28, 39A11, 65L05

Keywords: Differential equation; Power delayed argument; Trapezoidal rule; Asymptotic behaviour.

1. Introduction

This paper discusses the qualitative behaviour of the exact and the numerical solution of the delay differential equation

$$y'(t) = a\,y(t) + b\,y(t^{\gamma}), \quad t \ge 1\,, \tag{1.1}$$

where a, $b \neq 0$ and $0 < \gamma < 1$ are real scalars (some additional assumptions on a, b will be imposed later). Because of the property $\limsup(t - t^{\gamma}) = \infty$ as $t \to \infty$ this equation belongs to the class of differential equations with an infinite time lag. A typical representative of this class is the pantograph equation

$$y'(t) = a\,y(t) + b\,y(\lambda t), \quad t \ge 0 \tag{1.2}$$

E-mail address: cermak.j@fme.vutbr.cz (J. Čermák)

*The research was supported by the research plan MSM 0021630518 "Simulation modelling of mechatronic systems" of the Ministry of Education, Youth and Sports of the Czech Republic and by the grant # 201/08/0469 of the Czech Grant Agency.

serving as a mathematical model of various problems ([14], [18]). Since its origin, the pantograph equation (1.2) has become the subject of many qualitative and numerical investigations (see, e.g. [9], [11], [12], [13] [15], [17] and the references cited therein). The important question connected with these investigations is the discussion on possible similarities or discrepancies in the behaviour of the exact and numerical solution of (1.2) ([2], [10], [16]). Our aim is to extend this discussion also to the equation (1.1) and present the correspondence between the asymptotic behaviour of the equation (1.1) and its trapezoidal rule discretization.

The paper is structured as follows: Section 2 presents an overview of the asymptotic behaviour of the equation (1.1). In Section 3, we mention some preliminaries on the discretization of the equation (1.1) with an emphasize to the application of the Euler method and the trapezoidal rule. Section 4 discusses the description of the asymptotics of the trapezoidal rule discretization of (1.1). Finally, some comparisons of the asymptotic bounds derived for the exact and the numerical solution of (1.1) are the subject of Section 5.

2. Asymptotic bounds for the exact solution of (1.1)

In this section, we recall some qualitative properties of the differential equation (1.1). As it might be expected, the asymptotics of its solution depends on the sign of a. The following asymptotic result, which is relevant with respect to our further investigations, was first proved in [8].

Theorem 2.1. *Let y be a solution of (1.1), where a, b, γ are real constants with $a < 0$, $b \neq 0$ and $0 < \gamma < 1$. Then there exists a continuous periodic function g of period $\log \gamma^{-1}$ such that*

$$y(t) = (\log t)^{\omega} g(\log \log t) + O((\log t)^{\mathrm{Re}\ \omega - 1}) \qquad \text{as } t \to \infty\,, \tag{2.1}$$

where ω is a root of $a + b\gamma^{\omega} = 0$.

It follows immediately from (2.1) that under the assumptions $a < 0$, $b \neq 0$ and $0 < \gamma < 1$ the upper bound

$$y(t) = O\Big((\log t)^{-\log_{\gamma}|b/a|}\Big) \qquad \text{as } t \to \infty \tag{2.2}$$

holds for any solution y of (1.1).

If $a > 0$, then the solutions of the equation (1.1) admit quite a different type of asymptotics which can be described in terms of the function $\exp\{at\}$ (see [3]). The case where $a = 0$ is "something between". In particular, the equation

$$y'(t) = b\, y(t^{\gamma}), \quad t \geq 1\,, \tag{2.3}$$

where $b \neq 0$ and $0 < \gamma < 1$ are real scalars, can be via the substitution

$$s = \log t, \qquad z(s) = t^{\frac{1}{\gamma - 1}} y(t)$$

converted into the equation of the type (1.2) whose qualitative behaviour is fully described in [11], and [12]. Using these results (and the corresponding backward transformation) we can find that the upper bound

$$y(t) = O\Big(t^{\frac{1}{1-\gamma}}(\log t)^{-\log_\gamma |b(1-\gamma)|}\Big) \qquad \text{as } t \to \infty$$

holds for any solution y of (2.3). Moreover, this estimate is non-improvable in the sense that if

$$y(t) = o\Big(t^{\frac{1}{1-\gamma}}(\log t)^{-\log_\gamma |b(1-\gamma)|}\Big) \qquad \text{as } t \to \infty\,,$$

then y must be the zero solution of (2.3).

For the sake of completeness we recall that the above stated types of asymptotics describe the behaviour of the solutions of (1.1) also in the advanced case $\gamma > 1$ (see [7]) as well as in some generalizations of (1.1) (see [17] and [3]). The extensions for complex coefficients a, b are possible, too.

3. Some numerical preliminaries

The basic numerical methods for delay differential equations extend the standard ODE methods by use of the interpolation. We recall here the Euler method combined with the piecewise constant approximation of the delayed term and the trapezoidal rule discretization approximating the delayed term by the piecewise linear interpolation. This type of the discretization is the most important particular case of the so called θ-method which is frequently employed in the numerical analysis of delay differential equations. We apply the above stated methods directly to the differential equation (1.1).

Let $t_n = 1 + nh$, $n = 0, \ldots$ be the set of grid points with the constant stepsize $h > 0$ and let y_n denote the approximation of the exact solution y of (1.1) at t_n. Then the application of the forward Euler method and the trapezoidal rule to the equation (1.1) yields the recurrence relations

$$y_{n+1} = (1 + ah)y_n + bhy_{r_n}, \qquad n = 0, 1, \ldots \tag{3.1}$$

and

$$y_{n+1} = Ry_n + S\left(\beta_n y_{r_n} + \alpha_n y_{r_n+1}\right), \qquad n = 0, 1, \ldots\,, \tag{3.2}$$

respectively, with

$$
\begin{aligned}
r_n &= \Big\lfloor \frac{(t_n)^\gamma - 1}{h} \Big\rfloor, \qquad (\lfloor\ \rfloor \text{ means the floor function})\,, \\
R &= \frac{2+ha}{2-ha}, \qquad S = \frac{2hb}{2-ha}\,, \\
\alpha_n &= \frac{1}{2h\gamma}\Big((t_{n+1})^\gamma - (t_n)^\gamma\Big) \\
&\quad \times \Big\{ (t_n)^{1-\gamma}\Big(\frac{(t_n)^\gamma - 1}{h} - \Big\lfloor \frac{(t_n)^\gamma - 1}{h} \Big\rfloor\Big) \\
&\qquad + (t_{n+1})^{1-\gamma}\Big(\frac{(t_{n+1})^\gamma - 1}{h} - \Big\lfloor \frac{(t_n)^\gamma - 1}{h} \Big\rfloor\Big)\Big\}\,, \\
\beta_n &= \frac{1}{2h\gamma}\Big((t_{n+1})^\gamma - (t_n)^\gamma\Big)\Big((t_n)^{1-\gamma} + (t_{n+1})^{1-\gamma}\Big) - \alpha_n\,.
\end{aligned}
\tag{3.3}
$$

Both formulae (3.1) and (3.2) are linear delay difference equations of a variable order. While the method (3.1) can be derived quite straightforwardly, the introduction of the formula (3.2) requires some additional calculations connected especially with the approximation of the numerical solution at non-grid points. This procedure is described in detail in [2], where the equation (1.1) with a general (increasing and differentiable) delay has been considered. Some general properties of the formulae (3.1) and (3.2), such as the convergence, various types of stability, etc., can be found in the book [1]. We note that the qualitative investigation of delay difference equations is, in general, much less developed than in the continuous counterpart. For some relevant results we refer to [5], [6] and [19].

4. Asymptotic bounds for the numerical solution of (1.1)

This section presents the upper bound for numerical solutions of (1.1) by the trapezoidal rule (3.2). The corresponding asymptotic result for the forward Euler method (3.1) has been obtained only recently ([4]) and we recall it in the following assertion.

Theorem 4.1. *Let y_n be a solution of (3.1), where a, b, γ are real constants such that $a < 0$, $1 + ha > 0$, $b \neq 0$, $a + |b| \geq 0$ and $0 < \gamma < 1$. Then*

$$y_n = O\Big((\log n)^{-\log_\gamma |b/a|}\Big) \qquad \text{as } n \to \infty\,. \tag{4.1}$$

We aim at the obtaining of the analogical result for (3.2). Since this type of discretization is more difficult to analyze, we have to start with some auxiliary results.

We consider the inequality

$$|S|\,(|\beta_n|\varrho_{r_n} + |\alpha_n|\varrho_{r_n+1}) \leq (1 - |R|)\,\varrho_n, \qquad n = 0, 1, \ldots\,, \tag{4.2}$$

where r_n, α_n, β_n, R and S are given by (3.3). We are searching for a positive solution sequence ϱ_n of (4.2) and on this account we first state the following estimate.

Proposition 4.1. *Let* α_n *and* β_n *be given by (3.3). Then*

$$\alpha_n + |\beta_n| \le 1 + \frac{1+h}{(t_n)^{1-\gamma}}, \qquad n = 0, 1, \dots . \tag{4.3}$$

Proof. We distinguish two cases with respect to the sign of β_n. First let $\beta_n \ge 0$. Then

$$\begin{aligned}
\alpha_n + |\beta_n| &= \frac{1}{2h\gamma}\Big((t_{n+1})^\gamma - (t_n)^\gamma\Big)\Big((t_n)^{1-\gamma} + (t_{n+1})^{1-\gamma}\Big) \\
&= \frac{1}{2h\gamma}(t_n)^\gamma\Big((1+\frac{h}{t_n})^\gamma - 1\Big)\Big((t_n)^{1-\gamma} + (t_{n+1})^{1-\gamma}\Big) \\
&\le \frac{1}{2}(t_n)^{\gamma-1}\Big((t_n)^{1-\gamma} + (t_{n+1})^{1-\gamma}\Big) \\
&= \frac{1}{2} + \frac{1}{2}\Big(1+\frac{h}{t_n}\Big)^{1-\gamma} \le \frac{1}{2} + \frac{1}{2}\Big(1 + \frac{(1-\gamma)h}{t_n}\Big) = 1 + \frac{(1-\gamma)h}{2t_n}
\end{aligned}$$

by use of the binomial formula. Now consider the case $\beta_n < 0$. Then

$$\begin{aligned}
\alpha_n + |\beta_n| &= \frac{1}{h\gamma}\Big((t_{n+1})^\gamma - (t_n)^\gamma\Big) \\
&\quad \times\Big\{(t_n)^{1-\gamma}\Big(\frac{(t_n)^\gamma - 1}{h} - \Big\lfloor\frac{(t_n)^\gamma - 1}{h}\Big\rfloor\Big) \\
&\qquad +(t_{n+1})^{1-\gamma}\Big(\frac{(t_{n+1})^\gamma - 1}{h} - \Big\lfloor\frac{(t_n)^\gamma - 1}{h}\Big\rfloor\Big)\Big\} \\
&\quad -\frac{1}{2h\gamma}\Big((t_{n+1})^\gamma - (t_n)^\gamma\Big)\Big((t_n)^{1-\gamma} + (t_{n+1})^{1-\gamma}\Big) \\
&\le (t_n)^{\gamma-1}\Big\{(t_n)^{1-\gamma}\frac{1}{2} + (t_{n+1})^{1-\gamma}\Big(\gamma(t_n)^{\gamma-1} + 1 - \frac{1}{2}\Big)\Big\} \\
&= \frac{1}{2} + \Big(1+\frac{h}{t_n}\Big)^{1-\gamma}\Big(\gamma(t_n)^{\gamma-1} + \frac{1}{2}\Big) \\
&\le \frac{1}{2} + \frac{1}{2}\Big(1 + \frac{(1-\gamma)h}{t_n}\Big) + \gamma(t_n)^{\gamma-1}\Big(1 + \frac{(1-\gamma)h}{t_n}\Big) \\
&\le 1 + \frac{(1-\gamma)h}{2(t_n)^{1-\gamma}} + \frac{\gamma}{(t_n)^{1-\gamma}} + \frac{\gamma(1-\gamma)h}{(t_n)^{1-\gamma}} \le 1 + \frac{1+h}{(t_n)^{1-\gamma}}\,.
\end{aligned}$$

□

Now let $t^* \ge 1$ be a (unique) real root of the equation $t - t^\gamma = h$ and further let $k^* = \lfloor (t^*-1)/h \rfloor + 1$. Then using this notation we can present the following explicit form of a solution of (4.2).

Proposition 4.2. *Let* $0 < 1 - |R| \le |S|$ *and* $0 < \gamma < 1$. *Then the sequence*

$$\varrho_n = \Big(1 - 2(1+h)(t_{n-1})^{\frac{\gamma-1}{\gamma}}\Big)\Big(\log(t_n - k^*h)\Big)^{-\log_\gamma(|S|/(1-|R|))} \tag{4.4}$$

defines a positive solution sequence of (4.2) for all n *large enough.*

Proof. Using (4.3) and the eventual monotonicity of ϱ_n we can estimate the left-hand side of the inequality (4.2) as

$$\begin{aligned} |S|\left(|\beta_n|\varrho_{r_n}+|\alpha_n|\varrho_{r_n+1}\right) &\leq |S|\left(|\beta_n|+|\alpha_n|\right)\varrho_{r_n+1} \\ &\leq |S|\left(1+\frac{1+h}{(t_n)^{1-\gamma}}\right)\varrho_{r_n+1}\,. \end{aligned}$$

After the substitution of ϱ_n from (4.4) we can continue in our calculations:

$$\begin{aligned} &|S|\left(1+\frac{1+h}{(t_n)^{1-\gamma}}\right)\left(1-\frac{2(1+h)}{(1+r_nh)^{\frac{1-\gamma}{\gamma}}}\right)\Big(\log(1+(r_n+1-k^*)h)\Big)^{-\log_\gamma(|S|/(1-|R|))} \\ &\leq |S|\left(1-\frac{1+h}{(t_n)^{1-\gamma}}\right)\Big(\log((t_n)^\gamma+h-k^*h)\Big)^{-\log_\gamma(|S|/(1-|R|))} \\ &\leq |S|\left(1-\frac{2(1+h)}{(t_{n-1})^{\frac{1-\gamma}{\gamma}}}\right)\Big(\log(t_n-k^*h)^\gamma)\Big)^{-\log_\gamma(|S|/(1-|R|))} \\ &= |S|\left(1-\frac{2(1+h)}{(t_{n-1})^{\frac{1-\gamma}{\gamma}}}\right)\frac{1-|R|}{|S|}\Big(\log(t_n-k^*h)\Big)^{-\log_\gamma(|S|/(1-|R|))} \\ &= (1-|R|)\varrho_n \end{aligned}$$

by use of the definition of k^*. □

Now we are ready to formulate the main result of this section.

Theorem 4.2. *Let y_n be a solution sequence of (3.2), where*

$$0<1-|R|\leq|S| \qquad \text{and} \qquad 0<\gamma<1\,.$$

Then

$$y_n=O\Big((\log n)^{-\log_\gamma(|S|/(1-|R|))}\Big) \qquad \text{as } n\to\infty\,. \tag{4.5}$$

Proof. First we introduce the substitution $z_n=y_n/\varrho_n$, where ϱ_n is given by (4.4). Then (3.2) becomes

$$\varrho_{n+1}z_{n+1}=R\varrho_nz_n+S\left(\varrho_{r_n}z_{r_n}\beta_n+\varrho_{r_n+1}z_{r_n+1}\alpha_n\right)\,. \tag{4.6}$$

Now let $\sigma_0\in\mathbb{Z}^+$ be the sufficiently large starting point of our asymptotic estimation. Further, let $\sigma_{m+1}:=\left\lfloor\frac{(1+(\sigma_m-1)h)^{\frac{1}{\gamma}}-1}{h}\right\rfloor$, $m=0,1,\ldots$, $I_0:=[r_{\sigma_0},\sigma_0]\cap\mathbb{Z}^+$, $I_{m+1}:=[\sigma_m,\sigma_{m+1}]\cap\mathbb{Z}^+$ and $B_m:=\sup(|z_k|,\,k\in\cup_{j=0}^m I_j)$, $m=0,1,\ldots$.

We choose arbitrary $n^*\in I_{m+1}$, $n^*>\sigma_m$ and first assume that $R=0$, i.e. $2+ha=0$. Then (4.6) can be simplified as

$$z_{n^\star}=\frac{1}{\varrho_{n^\star}}S(\varrho_{r_{n^*-1}}z_{r_{n^*-1}}\beta_{n^\star-1}+\varrho_{r_{n^*-1}+1}z_{r_{n^*-1}+1}\alpha_{n^\star-1})\,,$$

i.e.,

$$|z_{n^\star}| \le B_m \frac{1}{\varrho_{n^\star}} |S| (\varrho_{r_{n^\star-1}} |\beta_{n^\star-1}| + \varrho_{r_{n^\star-1}+1} |\alpha_{n^\star-1}|).$$

Now we can employ (4.2) to obtain

$$|z_{n^\star}| \le \frac{\varrho_{n^\star-1}}{\varrho_{n^\star}} B_m, \qquad \text{i.e.} \qquad B_{m+1} \le B_m$$

by use of the eventual monotonicity of ρ and arbitrariness of $n^* \in I_{m+1}$.

If $R \neq 0$, then multiplication of (4.6) by R^{-n-1} yields

$$\Delta \left(\varrho_n z_n R^{-n} \right) = S \left(\varrho_{r_n} z_{r_n} \beta_n + \varrho_{r_n+1} z_{r_n+1} \alpha_n \right) R^{-n-1} .$$

To express explicitly the value z_{n^*} we sum the previous relation from σ_m to $n^* - 1$ and obtain

$$\varrho_{n^\star} z_{n^\star} R^{-n^*} - \varrho_{\sigma_m} z_{\sigma_m} R^{-\sigma_m} = S \sum_{p=\sigma_m}^{n^\star-1} (\varrho_{r_p} z_{r_p} \beta_p + \varrho_{r_p+1} z_{r_p+1} \alpha_p) R^{-p-1} ,$$

consequently

$$z_{n^\star} = \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} z_{\sigma_m} R^{n^\star-\sigma_m} + \frac{S}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} (\varrho_{r_p} z_{r_p} \beta_p + \varrho_{r_p+1} z_{r_p+1} \alpha_p) R^{n^*-p-1} .$$

Considering (4.2) we arrive at

$$\begin{aligned} |z_{n^\star}| &\le B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} + \frac{1-|R|}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \varrho_p |R|^{n^*-p-1} \right) \\ &\le B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} + \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \varrho_p \Delta |R|^{n^*-p} \right) . \end{aligned}$$

Now using the discrete formula of the integration by parts we can continue in our estimation:

$$\begin{aligned} |z_{n^\star}| &\le B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^*-\sigma_m} + 1 - |R|^{n^*-\sigma_m} \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} - \sum_{p=\sigma_m}^{n^\star-1} |R|^{n^*-p-1} \frac{\Delta \varrho_p}{\varrho_{n^\star}} \right) \\ &= B_m \left(1 - \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \Delta \varrho_p |R|^{n^*-p-1} \right) \\ &\le B_m \end{aligned}$$

by use of the property $\Delta \varrho_p \ge 0$. Hence the sequence B_m is bounded as $m \to \infty$ and ϱ_n is the upper bound sequence for y_n. Since ϱ_n and $(\log n)^{-\log_\gamma(|S|/(1-|R|))}$ are asymptotically equivalent as $n \to \infty$, the property (4.5) is proved. □

5. Some comparisons and remarks

The important theoretical question in the numerical analysis of differential equations is the problem whether the numerical solution can retain the main qualitative properties of the exact solution. This question is investigated especially in the frame of the stability analysis and, in a more general sense, in the asymptotic investigation of the exact and discretized equation. Among papers closely related to this problem we refer to [2] and [16] discussing the asymptotics of the numerical solutions of the pantograph equation (1.2). While the paper [2] formulates the upper bound of the trapeziodal rule discretization which does not quite coincide with the upper bound of the exact solution, the paper [16] reports that the numerical solution of (1.2) by the backward Euler method has asymptotically the same decay rate as the exact solution.

Our intention is to perform these comparisons for the equation (1.1) and its numerical solutions. We emphasize that, by (2.2) and (4.1), the Euler method (3.1) has under some restrictions on a, b and h the same upper bound of the solutions as the exact solution of (1.1) admits. We show that the similar correspondence is guaranteed by Theorem 4.2 also for the trapezoidal rule (3.2).

First we rewrite the assumption $0 < 1 - |R| \le |S|$ of Theorem 4.2 in terms of the coefficients a, b of the exact equation. Using (3.3) we can easily verify that the inequality $0 < 1 - |R|$ occurs if and only if $a < 0$. Similarly,

$$1 - |R| \le |S| \quad \Leftrightarrow \quad \begin{array}{ll} |b| \ge |a| & \text{for } h|a| \le 2, \\ |b| \ge 2/h & \text{for } h|a| > 2\,. \end{array}$$

Now we can reformulate Theorem 4.2 and compare the estimates (4.5) and (2.2). If $a < 0$, $2 + ah \ge 0$ and $a + |b| \ge 0$, then (4.5) becomes (4.1). In other words, under a modest restriction on the stepsize h we can observe the same upper bound for the exact solution of (1.1) as well as for its numerical solution (3.2) by the trapezoidal rule.

The natural question is the necessity of the assumptions $2+ah \ge 0$ and $a+|b| \ge 0$ (the assumption $a < 0$ is consistent with the continuous case). While the legitimacy of the restriction on the stepsize h is quite open, we can put a conjecture that the assumption $a + |b| \ge 0$ is superfluous. However, the rigorous justification of this conjecture seems to be complicated and exceeding the range of the paper.

References

[1] A. Bellen and M. Zennaro, Numerical Methods For Delay Differential Equations, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 2003.

[2] M. D. Buhmann and A. Iserles, Stability of the discretized pantograph differential equation, Math. Comp. 60 (1993) 575–589.

[3] J. Čermák, On a linear differential equation with a proportional delay, Math. Nachr. 280, No. 5-6 (2007) 495–504.

[4] J. Čermák, On the differential and difference equation with a power delayed argument, Electron. J. Qual. Theory Differ. Equ., Proc. 8th Coll. QTDE, 2008, No. 5, 1–8.

[5] S. Elaydi and I. Győri, Asymptotic theory for delay difference equations, J. Difference Equ. Appl. 1 (1995) 99–116.

[6] I. Győri and M. Pituk, Asymptotic formulae for the solutions of a linear delay difference equations, J. Math. Anal. Appl. 195 (1995) 376–392.

[7] M. L. Heard, Asymptotic behavior of solutions of the functional differential equation $x'(t) = ax(t) + bx(t^\alpha), \alpha > 1$, J. Math. Anal. Appl. 44 (1973) 745–757.

[8] M. L. Heard, A change of variables for functional differential equations, J. Differential Equations 18 (1975) 1–10.

[9] A. Iserles, On the generalized pantograph functional-differential equation, European J. Appl. Math. 4 (1993) 1–38.

[10] A. Iserles, Numerical analysis of delay differential equations with variable delays, Ann. Numer. Math. 1 (1994) 133–152.

[11] T. Kato and J. B. McLeod, The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. 77 (1971) 891–937.

[12] T. Kato, Asymptotic behaviour of solutions of the functional-differential equations $y'(t) = ay(\lambda t) + by(t)$, in: K. Schmitt, Ed., Delay and Functional Differential Equations and Their Applications, Academic Press, New York, 1972, 197–218.

[13] T. Koto, Stability of Runge-Kutta methods for the generalized pantograph equation, Numer. Math. 84 (1999) 870–884.

[14] H. Lehninger and Y. Liu, The functional-differential equation $y'(t) = Ay(t) + By(\lambda t) + Cy'(qt) + f(t)$, European J. Appl. Math. 9 (1998) 81–91.

[15] M. Z. Liu, Z. W. Yang and Y. Xu, The stability of the modified Runge-Kutta methods for the pantograph equation, Math. Comp. 75 (2006) 1201–1215.

[16] Y. Liu, On the θ-method for delay differential equations with infinite lag, J. Comput. Appl. Math. 71 (1996) 177–190.

[17] G. Makay and J. Terjéki, On the asymptotic behavior of the pantograph equations, Electron. J. Qual. Theory Differ. Equ. 2 (1998) 1–12.

[18] J. R. Ockendon, A. B. Tayler, The dynamics of a current collection system for an electric locomotive, Proc. Roy. Soc. Edinburg Sect. A, 322 (1971) 468–477.

[19] H. Péics, On the asymptotic behaviour of difference equations with continuous argument, Ser. A. Math. Anal. 9 (2002) 257–273.