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Exponential expansion of the solution of a half-linear differential equation*

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Abstract. In this paper, exponential expansions of the solution are investigated for a half-linear initial value problem $y'' |y'|^{p-1} - y|y|^{p-1} = 0$ with initial conditions $y(0) = 0,\ y'(0) = 1$ or $y(0) = 1,\ y'(0) = 0$, where $p > 0$. The exponential expansions of the solution of the two nonlinear problems are investigated. By using recursive formulas we determine the coefficients of the exponential series and we provide formulas between the solutions of the two nonlinear initial value problems.

AMS Subject Classifications: 4E05, 34A12, 34A25

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1. Preliminaries

Our goal is to study the asymptotic solution of the half-linear differential equation

$$y'' |y'|^{p-1} - y|y|^{p-1} = 0, \tag{1.1}$$

where p is a positive real number, and solutions are subjected to the initial conditions

$$y(0) = 0,\ y'(0) = 1 \tag{A}$$

or

$$y(0) = 1,\ y'(0) = 0. \tag{B}$$

We note that equation (1.1) is linear with $p = 1$. Equation (1.1) is used to call half-linear one since it preserves just half of the properties which characterize linearity.

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Its solution set is homogeneous but not additive (see [2]). If $y(x)$ is a solution of (1.1) then $Cy(x+x_0)$ is also a solution for any real constants C and x_0.

Multiplying equation (1.1) by y' and using the fact $\left[|y'|^{p+1}\right]' = (p+1)y''\,|y'|^{p-1}\,y'$ we obtain the identity $\left[|y'|^{p+1} - |y|^{p+1}\right]' = 0$, i.e., $|y'|^{p+1} - |y|^{p+1} = const$. Substituting the initial conditions (A) or (B) we obtain

$$|y'|^{p+1} - |y|^{p+1} = 1, \tag{1A}$$

or

$$|y'|^{p+1} - |y|^{p+1} = -1, \tag{1B}$$

respectively. We note that the case $p = 1$ is well-known: the solution of equation $y'' - y = 0$ with condition (A) is the hyperbolic sine function defined as

$$\sinh x = e^x \left(\frac{1}{2} - \frac{1}{2}e^{-2x}\right),$$

and the solution of $y'' - y = 0$ with condition (B) is the hyperbolic cosine function defined as

$$\cosh x = e^x \left(\frac{1}{2} + \frac{1}{2}e^{-2x}\right),$$

and equation (1A) is nothing else than $\cosh^2 x - \sinh^2 x = 1$.

In both cases, (1.1)-(A) or (1.1)-(B), y and y' is positive in some right neighborhood of $x = 0$. Using (1A), $y' = (1 + y^{p+1})^{1/(p+1)}$, i.e., $dx = dy/(1 + y^{p+1})^{1/(p+1)}$ in this neighborhood. Hence

$$x = \int_0^y \frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}}. \tag{1.2}$$

For $p = 1$, from (1.2) the sinh function is obtained. Generally, for $p \neq 1$ we denote y satisfying (1.2) by $y = Sh_p x$. Applying (1B), $y' = (y^{p+1} - 1)^{1/(p+1)}$, i.e., $dx = dy/(y^{p+1} - 1)^{1/(p+1)}$ in this neighborhood. Therefore

$$x = \int_0^y \frac{d\sigma}{(\sigma^{p+1}-1)^{1/(p+1)}}. \tag{1.3}$$

For $p = 1$, (1.3) gives the cosh function. When $p \neq 1$ we denote y satisfying (1.3) by $y = Ch_p x$. We remark that the asymptotic behaviors of solutions $Sh_p x$ and $Ch_p x$ were investigated by Elbert [4] as $x \to \infty$.

In this paper, our aim is to study the exponential series form of solutions $Sh_p x$ and $Ch_p x$ in the right neighborhood of $x = 0$ for the half-linear differential equation (1.1) with initial conditions (A) or (B), respectively. We show the existence of that type of solution and we give recursive formulas for the determination of the coefficients. On the base of the structure of the exponential expansions for the two nonlinear problems connections are proved between the two solutions.

2. Results

First we show the existence of the solution for differential equation (1.1) of exponential series form

$$y(x) = Ae^x \sum_{n=0}^{\infty} a_n \left(Ae^x\right)^{-n\ (p+1)} \tag{2.1}$$

for $y' > 0,\ y > 0$.

Theorem 2.1. *For any $p > 0$ there exists a solution of the exponential form $y(x) = e^x \sum_{n=0}^{\infty} \gamma_n e^{-n\ (p+1)x}$ near zero for problems (1.1)-(A) and (1.1)-(B).*

Proof. Let us substitute $u = e^{-(p+1)x}$, i.e., $du = -(p+1)u\, dx$ into (1.1) and introduce $y(x) = u^{-1/(p+1)} w(u)$. Hence $y'(x) = u^{-1/(p+1)}(w - (p+1)uw')$ and rewrite (1A) and (1B)

$$(w - (p+1)uw')^{p+1} - w^{p+1} = \pm u,$$

where the "+" sign is valid for condition (A) and the "-" sign for (B) and

$$w'' = u^{-1}\left[\frac{-p}{p+1}w' - \frac{(p+1)w^p w' \pm 1}{(p+1)^2(w-(p+1)uw')^p}\right]. \tag{2.2}$$

Introduce function w as follows

$$w(u) = \gamma_0 + \gamma_1 u + z(u), \tag{2.3}$$

where $z \in C^2(0,a)$, $z(0) = 0$, $z'(0) = 0$. Therefore w fulfills the properties $w(0) = \gamma_0$, $w'(0) = \gamma_1$, corresponding to (A) or (B), and $w'(u) = \gamma_1 + z'(u)$, $w''(u) = z''(u)$. From initial condition $w(0) = A$ we have that $\gamma_0 = A$. We restate (2.2) as a system of equations

$$\left.\begin{array}{l} z_1(u) = w(u) \\ z_2(u) = w'(u) \end{array}\right\} \text{ with } \left.\begin{array}{l} z_1(0) = 0 \\ z_2(0) = 0 \end{array}\right\}.$$

Generate the system of Briot-Bouquet equations

$$\left.\begin{array}{l} u_1(u, z_1(u), z_2(u)) = u\ z_1'(u) \\ u_2(u, z_1(u), z_2(u)) = u\ z_2'(u) \end{array}\right\}$$

as follows

$$\left.\begin{array}{l} u_1(u, z_1(u), z_2(u)) = u\ z_2, \\ u_2(u, z_1(u), z_2(u)) = \left[\frac{-p}{p+1} z_2 - \frac{(p+1)z_1^p z_2 \pm 1}{(p+1)^2(z_1-(p+1)uz_2)^p}\right]. \end{array}\right\} \tag{2.4}$$

Such systems were first studied by C. C. Briot and J. C. Bouquet [3]. We can show by the Briot-Bouquet theorem (see [3], [7]) that there exists a right neighborhood of zero such that there is an analytic solution

$$w(u) = \sum_{k=0}^{\infty} \gamma_k u^k \tag{2.5}$$

of (2.2). Conditions $u_1(0,0,0) = 0$ and $u_2(0,0,0) = 0$ are fulfilled with $\gamma_1 = \mp\frac{1}{(p+1)^2\gamma_0^p}$. A holomorphic solution of (2.4) satisfying the initial conditions $z_1(0) = 0$, $z_2(0) = 0$ exists if none of the eigenvalues of the matrix

$$\begin{bmatrix} \frac{\partial u_1}{\partial z_1}\Big|_{(0,0,0)} & \frac{\partial u_1}{\partial z_2}\Big|_{(0,0,0)} \\ \frac{\partial u_2}{\partial z_1}\Big|_{(0,0,0)} & \frac{\partial u_2}{\partial z_2}\Big|_{(0,0,0)} \end{bmatrix}$$

is a positive integer. Evaluating the elements of the matrix we obtain

$$\begin{array}{ll} \frac{\partial u_1}{\partial z_1}\Big|_{(0,0,0)} = 0, & \frac{\partial u_1}{\partial z_2}\Big|_{(0,0,0)} = 0, \\ \frac{\partial u_2}{\partial z_1}\Big|_{(0,0,0)} = \mp\frac{p}{p+1}\frac{1}{\gamma_0^{p+1}}, & \frac{\partial u_2}{\partial z_2}\Big|_{(0,0,0)} = -1, \end{array}$$

hence the eigenvalues of matrix at $(0,0,0)$ are 0 and -1. According to the Briot-Bouquet theorem the existence of unique analytic solutions z_1 and z_2 at zero follows. On the other hand, by (2.5)

$$u^{-1/(p+1)}w(u) = u^{-1/(p+1)}\left(\gamma_0 + \gamma_1 u + \gamma_2 u^2 + \ldots\right)$$

and y is of the form

$$y(x) = e^{-x}\left(\gamma_0 + \gamma_1 e^{-(p+1)x} + \gamma_2 e^{-2(p+1)x} + \ldots\right),$$

which is is equivalent to (2.1) with $\gamma_k = a_k A^{1-(p+1)k}$ for $k \geq 0$. □

Theorem 2.2. *For exponential series*

$$y(x) = Ae^x \sum_{n=0}^{\infty} a_n \left(Ae^x\right)^{-n\ (p+1)}, \tag{2.6}$$

of the solution for problems (1.1)-(A) or (1.1)-(B) the coefficients a_n $(n = 0, 1, 2, \ldots)$ *are given by* $a_0 = 1$,

$$a_1 = \pm\left(-\frac{1}{(p+1)^2)}\right) \tag{2.7}$$

$$a_n = \pm\frac{1}{n^2\left(p+1\right)^2}\left[\left(p+1\right)\left(n-1\right)-1\right]a_{n-1}\left(p+1-np\right) - \frac{1}{n^2(p+1)}\sum_{k=2}^{n-1}\left[\left(p+1\right)\left(n-k\right)-k\right]\alpha_k a_{n-k}\left(n-k\right) \text{ for } n > 1, \tag{2.8}$$

where the "+" sign is valid for (A) and the "-" sign for (B).

Proof. In case of the exponential expansion

$$y = Ae^x \left(1 + a_1 (Ae^x)^{-(p+1)} + a_2 (Ae^x)^{-2(p+1)} + \dots\right)$$

of the solution for (1.1) we have that

$$y' = Ae^x \left(1 + (-p)a_1 (Ae^x)^{-(p+1)} + (-1-2p)a_2 (Ae^x)^{-2(p+1)} + \dots\right).$$

Applying substitution $(Ae^x)^{-(p+1)} = z$ we get

$$y^{p+1} = z^{-1} \left(1 + a_1 z + a_2 z^2 + \dots\right)^{p+1} = z^{-1} \left(\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots\right),$$

and

$$\begin{aligned} y'^{p+1} &= z^{-1} \left(1 + (-p)a_1 z + (-1-2p)a_2 z^2 + \dots\right)^{p+1} \\ &= z^{-1} \left(\beta_0 + \beta_1 z + \beta_2 z^2 + \dots\right) \end{aligned}$$

with some coefficients α_k, β_k, $k = 0, 1, \dots$ depending on a_j, $j = 1, 2, \dots$. Substituting them to the differential equation (1A) or (1B) we get for the coefficients that $\alpha_0 = \beta_0 = 1$, and

$$\alpha_1 = \beta_1 - 1 \quad \text{for} \quad (1A), \tag{2.9}$$

$$\alpha_1 = \beta_1 + 1 \quad \text{for} \quad (1B), \tag{2.10}$$

and for α_k and β_k $(k \in \mathbf{N})$

$$\alpha_k = \beta_k, \quad \text{for} \quad k > 1. \tag{2.11}$$

For the determination of coefficients α_n and β_n for $n = 0, 1, 2, \dots$ we use the J.C.P. Miller formula (see [5], [6]) which states that the coefficients of x^k in $P(x)^m$, where $P(x) = \sum_{k=0}^{L} p_k x^k$ $(p_0 \neq 0)$ and $P(x)^m = \sum_{k=0}^{mL} a(m,k) x^k$, are the following

$$a(m,k) = \frac{1}{kp_0} \sum_{i=1}^{L} p_i \left[(m+1)i - k\right] a(m, k-i).$$

Hence

$$\alpha_0 = 1, \quad \alpha_n = \frac{1}{n} \sum_{k=0}^{n-1} \left[(p+1)(n-k) - k\right] \alpha_k a_{n-k}, \qquad \text{for} \quad n > 1,$$

$$\beta_0 = 1, \quad \beta_n = \frac{1}{n} \sum_{k=0}^{n-1} \left[(p+1)(n-k) - k\right] \beta_k a_{n-k} \left[1 - (p+1)(n-k)\right], \qquad \text{for} \quad n > 1.$$

On the basis of (2.9)-(2.11) we get coefficients a_n in the form of (2.7) and (2.8). $\square$

Proposition 2.1. *Let p be a positive real number, then*

$$\Phi(p) = -\int_0^\infty \frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}} + \int_1^\infty \frac{d\sigma}{\sigma} = \frac{1}{p+1}\left(\psi\left(\frac{1}{p+1}\right) - \psi(1)\right),$$

where ψ denotes the digamma function.

Proof. By substitution $\sigma = z^{\frac{1}{p+1}}$, i.e., $d\sigma = dz/\left[(p+1)z^{p/(p+1)}\right]$ we have

$$\Phi(p) = \frac{1}{p+1}\left(\int_1^\infty \frac{dz}{z} - \int_1^\infty \frac{z^{-p/(p+1)}\, dz}{(1+z)^{1/(p+1)}}\right). \tag{2.12}$$

The second integral in (2.12) is a special case of $B\,(a,b) = \int_1^\infty \frac{z^{a-1}\, dz}{(1+z)^{a+b}}$, $a > 0$, $b \geq 0$, with $a = \frac{1}{p+1}$ and $b = 0$ where $B(.,.)$ denotes the Beta function. Let us consider a more general version of $\Phi(p)$ by

$$\overline{\Phi}(p) = \frac{1}{p+1}\left(\int_1^\infty \frac{dz}{z^{\vartheta+1}} - \int_1^\infty \frac{z^{-p/(p+1)}\, dz}{(1+z)^{\vartheta+1/(p+1)}}\right), \quad \vartheta \geq 0, \tag{2.13}$$

and for any positive ϑ tending to zero we get $\Phi(p) = \lim_{\vartheta\to 0} \overline{\Phi}(p)$. Evaluating the integrals in (2.13) we have

$$\overline{\Phi}(p) = \frac{1}{p+1}\left(\frac{1}{\vartheta} - B\left(\frac{1}{p+1}, \vartheta\right)\right),$$

and applying identities $B\,(a,b) = \frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(\vartheta+1) = \vartheta\Gamma(\vartheta)$ (see [1]) one can obtain

$$\overline{\Phi}(p) = \frac{1}{p+1}\left(\frac{1}{\vartheta} - \frac{1}{\vartheta}\frac{\Gamma(\frac{1}{p+1})\ \Gamma(\vartheta+1)}{\Gamma(\frac{1}{p+1}+\vartheta)}\right).$$

Taking the limit of $\overline{\Phi}(p)$ we get

$$\begin{aligned}
\Phi(p) &= \lim_{\vartheta\to 0}\overline{\Phi}(p) = \frac{1}{p+1}\lim_{\vartheta\to 0}\frac{1}{\vartheta}\left(1-\frac{\Gamma(\frac{1}{p+1})\,\Gamma(\vartheta+1)}{\Gamma(\frac{1}{p+1}+\vartheta)}\right)\\
&= \frac{1}{p+1}\lim_{\vartheta\to 0}\frac{\Gamma(\frac{1}{p+1}+\vartheta)-\Gamma(\frac{1}{p+1}+\vartheta)\Gamma(\vartheta+1)}{\vartheta\Gamma(\frac{1}{p+1}+\vartheta)}\\
&\quad+\frac{1}{p+1}\lim_{\vartheta\to 0}\frac{\Gamma(\frac{1}{p+1}+\vartheta)\Gamma(\vartheta+1)-\Gamma(\frac{1}{p+1})\Gamma(\vartheta+1)}{\vartheta\Gamma(\frac{1}{p+1}+\vartheta)}\\
&= \frac{1}{p+1}\frac{1}{\Gamma(1)}\lim_{\vartheta\to 0}\frac{\Gamma(1)-\Gamma(\vartheta+1)}{\vartheta}\\
&\quad+\frac{1}{p+1}\lim_{\vartheta\to 0}\frac{\Gamma(\vartheta+1)}{\Gamma(\frac{1}{p+1}+\vartheta)}\frac{\Gamma(\frac{1}{p+1}+\vartheta)-\Gamma(\frac{1}{p+1})}{\vartheta}\\
&= \frac{1}{p+1}\left(-\frac{\Gamma'(1)}{\Gamma(1)}+\frac{\Gamma'(\frac{1}{p+1})}{\Gamma(\frac{1}{p+1})}\right).
\end{aligned}$$

With the notation of the digamma function $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ the statement of Proposition 2.1 follows. Note that $-\frac{\Gamma'(1)}{\Gamma(1)}=-\psi(1)=0.577215664901532...$ is called the Euler-Mascheroni constant [1]. □

Proposition 2.2. *Let p be a positive real number then*

$$\Theta(p) = -\int_1^\infty\left[\frac{d\sigma}{(\sigma^{p+1}-1)^{1/(p+1)}}-\frac{d\sigma}{\sigma}\right]=\frac{1}{p+1}\left(\psi\left(\frac{p}{p+1}\right)-\psi(1)\right).$$

Proof. By substitution $\sigma=(1+z)^{\frac{1}{p+1}}$, i.e., $d\sigma = dz/\left[(p+1)\,(1+z)^{p/(p+1)}\right]$ we have

$$\Theta(p)=\frac{1}{p+1}\left[\int_0^\infty\frac{z^{-\frac{1}{p+1}}\,dz}{(1+z)^{p/(p+1)}}-\int_1^\infty\frac{dz}{z}\right] \tag{2.14}$$

Similarly as in the proof of Proposition 2.1 for $\vartheta\ge 0$ we take

$$\overline{\Theta}(p)=\frac{1}{p+1}\left[\int_0^\infty\frac{z^{-\frac{1}{p+1}}\,dz}{(1+z)^{\vartheta+p/(p+1)}}-\int_1^\infty\frac{dz}{z^{1+\vartheta}}\right]=\frac{1}{p+1}\left(\frac{1}{\vartheta}-B\left(\frac{p}{p+1},\vartheta\right)\right)$$

and

$$\Theta(p)=\lim_{\vartheta\to 0}\overline{\Theta}(p)=\frac{1}{p+1}\left(-\frac{\Gamma'(1)}{\Gamma(1)}+\frac{\Gamma'(\frac{p}{p+1})}{\Gamma(\frac{p}{p+1})}\right).$$

□

The reflection formula for the digamma function has the form $\psi(1-x)-\psi(x) = \pi\cot(\pi x)$ (see [1]). Hence for any positive p we get

$$\begin{aligned}\Phi(p)-\Theta(p) &= \frac{1}{p+1}\left[\frac{\Gamma'(\frac{1}{p+1})}{\Gamma(\frac{1}{p+1})}-\frac{\Gamma'(\frac{p}{p+1})}{\Gamma(\frac{p}{p+1})}\right]\\ &= \frac{1}{p+1}\left[\Psi\left(\frac{1}{p+1}\right)-\Psi\left(\frac{p}{p+1}\right)\right]\\ &= \frac{\pi}{p+1}\cot\frac{\pi p}{p+1}.\end{aligned}$$

It implies the following assertion:

Remark 2.1. $\Phi(p)=\Theta(p)$ *if and only if* $p=1$.

Theorem 2.3. *For solution Sh_p of (1.1)-(A) and solution Ch_p of (1.1)-(B) the following formulas hold*

$$\begin{aligned}Ch_p(x) &= e^{-i\frac{\pi}{p+1}}Sh_p(x-\Psi(p)+i\tfrac{\pi}{p+1}),\\ Sh_p(x) &= e^{i\frac{\pi}{p+1}}Ch_p(x+\Psi(p)-i\tfrac{\pi}{p+1}),\end{aligned} \tag{2.15}$$

where $\Psi(p)=\Phi(p)-\Theta(p)$.

Proof. From (2.7) and (2.8) it follows that

$$y(x) = Ae^x\left(1+a_1\left(Ae^x\right)^{-(p+1)}+a_2\left(Ae^x\right)^{-2(p+1)}+\ldots\right) \tag{2.16}$$

is the solution of (1A), and

$$\tilde{y}(x) = Ae^x\left(1-a_1\left(Ae^x\right)^{-(p+1)}+a_2\left(Ae^x\right)^{-2(p+1)}-\ldots\right)$$

is the solution of (1B) for some constant A. By (1.2) we have

$$x=\int_0^y\frac{d\sigma}{(1+\sigma^{p+1})^{1/(p+1)}}-\int_1^y\frac{d\sigma}{\sigma}+\int_1^y\frac{d\sigma}{\sigma},$$

and extending for $y\to\infty$ by Proposition 2.1 we obtain $x=\ln y-\Phi(p)+o(1)$. Proposition 2.2 and (1.3) with

$$x=\int_1^y\left[\frac{d\sigma}{(\sigma^{p+1}-1)^{1/(p+1)}}-\frac{d\sigma}{\sigma}\right]+\int_1^y\frac{d\sigma}{\sigma}$$

gives $x=\ln y-\Theta(p)+o(1)$. In the two cases, (1.1)-(A) and (1.1)-(B) constant A can be obtained by $A=e^{\Phi(p)}$ and $A=e^{\Theta(p)}$, respectively. □

Note that the formulas in (2.15) for the case $p = 1$ are equivalent to the known formulas

$$\cosh x = \frac{1}{i}\sinh(x + i\frac{\pi}{2}),$$
$$\sinh x = i\sinh(x - i\frac{\pi}{2}).$$

We point out that solution (2.16) with constant A and solution $y(x + x_0)$ with constant A_0 are equivalent when $A = A_0 e^{x_0}$.

For the convergence of the series expansion we have the following remark:

Remark 2.2. *For the radius of convergence r_u for u in (2.5)*

$$r_u \le e^{(p+1)\Phi(p)} \text{ for (1.1)-(A) and } r_u \le e^{(p+1)\Theta(p)} \text{ for (1.1)-(B)}$$

hold. For the radius of convergence r_x for x in (2.6)

$$r_x \ge \Phi(p) \text{ for (1.1)-(A) and } r_x \ge \Theta(p) \text{ for (1.1)-(B)}$$

hold, moreover both corresponding series of Sh_p and Ch_p converge for $x \ge \min(\Phi(p), \Theta(p))$.

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