Global and Stochastic Analysis
Vol. 5, No. 2, December (2018), 139-152

MUK PUBLICATIONS
Open Access Publisher

THE BARENBLATT—ZHELTOV—KOCHINA MODEL WITH THE SHOWALTER—SIDOROV CONDITION AND ADDITIVE "WHITE NOISE" IN SPACES OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS WITHOUT BOUNDARY

DMITRIY E. SHAFRANOV AND OLGA G. KITAEVA*

Abstract. The paper is devoted to the study of the Showalter—Sidorov problem for the stochastic Barenblatt—Zheltov—Kochina equation defined in spaces of smooth differential forms on a Riemannian smooth manifold without boundary and considered as a concrete interpretation of Sobolev type equation (with an uninvertible operator under the derivative). Together with this, we consider a more particular Cauchy problem for a homogeneous Barenblatt—Zheltov—Kochina equation. To solve the problem in the indicated spaces, we reduce the corresponding operators and spaces, in particular, instead of the Laplace operator, its generalization is used on the form of the Laplace—Beltrami operator. There is no differentiability in the usual sense in the considerate spaces and we use derivative in the sense of Nelson—Gliklikh.

Introduction

Consider the linear Barenblatt—Zheltov—Kochina equation

$$(\lambda - \Delta)\dot{u} = \alpha\Delta u + f, \tag{0.1}$$

describing the dynamics of the pressure of the fluid filtered in a fractured porous medium [1]. The coefficient λ corresponds to the ratio of cracks and pores in rock, and the coefficient α corresponds for the visco-elastic properties of the liquid. Later it was found out, that equation (0.1) also simulates the process of moisture transfer in soil [2] and the process of heat conduction with "two temperatures" [3].

The beginning of the investigation of equation (0.1) should be related to [5], where this equation was considered for the first time as a linear inhomogeneous Sobolev type equation

$$L\dot{u} = Mu + f. \tag{0.2}$$

Here the operators L and M are the operators $\lambda - \Delta$ and $\alpha\Delta$, given in some functional spaces $L, M : \mathfrak{U} \to \mathfrak{F}$, defined in the spaces of differential forms defined

---

Date: Date of Submission April 15, 2018; Date of Acceptance June 20, 2018 , Communicated by Yuri E. Gliklikh .

2000 *Mathematics Subject Classification.* Primary 58J65; Secondary 35R60 58A10.

Key words and phrases. stochastic K-process, Sobolev type equation, derivative in the sense of Nelson—Gliklikh, Laplace—Beltrami operator, differential forms, Riemannian manifolds.

* We express our sincere gratitude to Professor G.A. Sviridyuk for fruitful discussions and strict but constructive criticism.

on smooth Riemannian manifolds without boundary. Equation (0.2) is usually equipped with the initial Showalter—Sidorov condition [6]

$$P(u(0) - u_0) = 0, \tag{0.3}$$

where the projector P is constructed with the use of operators L and M. Note that in the case of existence operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$, condition (0.3) becomes the Cauchy condition

$$u(0) = u_0. \tag{0.4}$$

We also note [7], where the system of equations (0.1) is reduced to the form (0.2) (or system of linear Oskolkov equations) given on a geometric graph, and the Barenblatt—Zheltov—Kochina equation given on Riemannian manifolds [4].

We will be interested in the stochastic interpretation of the deterministic equation (0.2), namely:

$$L \overset{o}{\eta} = M\eta + N\Theta. \tag{0.5}$$

Here the operators L and M are the same as in (2), the operator $N \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, $\eta = \eta(t)$ is required, and $\Theta = \Theta(t)$ is a given stochastic process with values in the Hilbert space $\mathfrak{U}$. Through $\overset{o}{\eta}$ we denote the Nelson— Gliklikh derivative [8] of the stochastic process $\eta = \eta(t)$ (for details see [9]). This approach differs from the classical Ito—Stratonovich—Skorokhod approach (see, for example, [10]), however their method can be used in the study of certain Sobolev type equations [11]. It is necessary to mention the Melnikova—Filinkov—Alshansky approach [14], where the stochastic equations in Frechet spaces are studied.

So, in the focus of our attention there will be the stochastic Barenblatt—Zheltov—Kochina equation, given in the space of stochastic K-processes with coefficients in the form of differential forms defined on a Riemannian smooth manifold without boundary and presented in form (0.5). In addition to the introduction and the bibliography, the article also contains four parts. The first part describes the structure of the "noise" space. In the second part we describe a stochastic analogue of the Sobolev equation type in the spaces of stochastic K-processes. In the third part the properties of the spectrum of the Laplace—Beltrami operator on a manifold, taken from [14], are described, and the form of these eigenfunctions of the operator Laplace—Beltrami equations in the space of k-forms on a two-dimensional sphere, as a particular case of a manifold without boundary is given. In the fourth part the stochastic Barenblatt—Zheltov—Kochina equation is reduced to a stochastic Sobolev type equation and then the solvability of the Cauchy problem (0.4) for the homogeneous equation and the Showalter—Sidorov problem (0.3) for inhomogeneous equation, in spaces of stochastic K-processes with additive "white noise" is studied. Solutions are presented in the form of a Fourier series with the Laplace—Beltrami operator eigenfunctions in the space of k-forms defined on a smooth compact oriented Riemannian manifolds without boundary. At the end of the fourth part the remark on possible directions for further research based on other publications of the Chelyabinsk scientific school is formulated. Then follows the list of literature, which does not claim to be complete, but answers only to authors' preferences.

1. The space of differential "noises" on smooth Riemannian manifolds without boundary

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $\mathbb{R}$ be a set of real numbers endowed with a Borel σ-algebra. Measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable.* A set of random variables with a zero mathematical expectation ($\mathbf{E}$) and finite variance ($\mathbf{D}$) forms a Hilbert space with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. The resulting Hilbert space is denoted by symbol $\mathbf{L_2}$. Next, we will be interested in the random variables $\xi \in \mathbf{L_2}$, which have a normal (Gaussian) distribution; they will be called *Gaussian* variables.

Let us denote by $\mathcal{A}_0$ *the* σ-subalgebra of the σ-algebra $\mathcal{A}$ and construct the space $\mathbf{L_2^0}$ of random variables measurable with respect to $\mathcal{A}_0$. Obviously, $\mathbf{L_2^0}$ is the subspace of $\mathbf{L_2}$; we denote by $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$ an orthoprojector. Let $\xi \in \mathbf{L_2}$ then $\Pi\xi$ is called *the conditional mathematical expectation* of a random variable ξ and denoted by the symbol $\mathbf{E}(\xi|\mathcal{A}_\mathbf{0})$. Note that $\mathbf{E}(\xi|\mathcal{A}_\mathbf{0}) = \mathbf{E}\xi$, for $\mathcal{A}_0 = \{\emptyset, \Omega\}$; and $\mathbf{E}(\xi|\mathcal{A}_\mathbf{0}) = \xi$, if $\mathcal{A}_0 = \mathcal{A}$. Recall also that the minimal σ-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, with respect to which the random variable ξ is measurable, is called the *σ-algebra generated by ξ.*

Let $\Im \subset \mathbb{R}$ be a certain interval. Consider two mappings: $f : \Im \to \mathbf{L_2}$, which puts to each $t \in \Im$ a random variable $\xi \in \mathbf{L_2}$, and $g : L_2 \times \Omega \to \mathbb{R}$, which puts to each pair (ξ, ω) the point $\xi(\omega) \in \mathbb{R}$. The map $\eta : \Im \times \Omega \to \mathbb{R}$, having the form $\eta = \eta(f(t), \omega)$, is called *a stochastic process.* The stochastic process η is called *continuous*, if a.s. (almost surely) all its trajectories are continuous (for almost all $\omega \in \Omega$ the trajectories $\eta(\cdot, \omega)$ are continuous). By the symbol $\mathbf{CL_2}$ we denote the set of the continuous stochastic processes. Let's call *Gaussian* continuous stochastic process the process, if its (independent) random variables are Gaussian.

A (one-dimensional) Wiener process $\beta = \beta(t)$, which simulates Brownian motion on a line in the Einstein-Smoluchowski theory, is an example of a continuous Gaussian stochastic process. It has the following properties:

(W1) a.s. $\beta(0) = 0$, a.s. all its trajectories $\beta(t)$ are continuous, and for all $t \in \overline{\mathbb{R}}_+ (= \{0\} \cup \mathbb{R}_+)$ the random variable $\beta(t)$ is Gaussian;

(W2) the mathematical expectation $\mathbf{E}(\beta(t)) = 0$ and the autocorrelation function $\mathbf{E}\left((\beta(t) - \beta(s))^2\right) = |t - s|$ for all $s, t \in \overline{\mathbb{R}}_+$;

(W3) trajectories $\beta(t)$ are non-differentiable at any point $t \in \overline{\mathbb{R}}_+$ and on any arbitrarily small interval has unbounded variation.

Theorem 1.1. *There exists a unique stochastic process β, satisfying the properties (W1) - (W2) with probability 1, and it can be presented in the form*

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t, \tag{1.1}$$

where ξ_k are independent Gaussian variables, $\mathbf{E}\xi_k = 0, \mathbf{D}\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$.

Further, the stochastic process β, satisfying the properties (W1) – (W3), we call *the Brownian motion.*

Fix $\eta \in \mathbf{CL_2}$ and $t \in \Im (= (\varepsilon, \tau) \subset \mathbb{R})$ and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by the random variable $\eta(t)$. Let us rename again, for brevity sake, $\mathbf{E}_t^\eta = \mathbf{E}(\cdot, \mathcal{N}_t^\eta)$.

Definition 1.2. Let $\eta \in \mathbf{CL_2}$. The random variable

$$D\eta(t,\cdot) = \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta \left(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t} \right),$$

$$\left(D_*\eta(t,\cdot) = \lim_{\Delta t \to 0-} \mathbf{E}_t^\eta \left(\frac{\eta(t,\cdot) - \eta(t-\Delta t,\cdot)}{\Delta t} \right) \right),$$

is called *the mean derivative on the right* $D\eta(t,\cdot)$ *(on the left* $D_*\eta(t,\cdot)$*) of the stochastic process* η *at the point* $t \in (\xi,\tau)$, if the limit exists in the sense of a uniform metric on $\mathbb{R}$. A stochastic process η is said to be *mean differentiable the right (on the left)* on (ε,τ), if at each point $t \in (\varepsilon,\tau)$ there exists a mean derivative on the right (on the left).

Suppose that the stochastic process $\eta \in \mathbf{CL_2}$ is mean differentiable on the right (on the left) on (ε,τ). Its mean derivative on the right (left) is a stochastic process, which we denote by $D\eta(D_*\eta)$. If the stochastic process $\eta \in \mathbf{CL_2}$ is mean differentiable by both on the right and on the left and on (ε,τ), then we can define a symmetric (antisymmetric) mean derivative $D_S\eta = \frac{1}{2}(D + D_*)\eta$ $\big(D_A\eta = \frac{1}{2}(D_* - D)\eta\big)$. Since mean derivatives was develops by E. Nelson, and the theory of such derivatives was developed by Yu.E. Gliklikh, then further, for brevity, the symmetric mean derivative D_S of a stochastic process η will be called the *Nelson—Gliklikh derivative* and denote by $\overset{o}{\eta}$, i.e. $D_S\eta \equiv \overset{o}{\eta}$.

We denote by $\overset{o}{\eta}{}^{(l)}$ the l-th Nelson—Gliklikh derivative of the stochastic process η, $l \in \mathbb{N}$. Note, that if the trajectories of the stochastic process η are a.s. continuously differentiable in the "ordinary sense" on (ε,τ), then their Nelson—Gliklikh derivative coincides with the "ordinary" derivative. For example, this is the case with the stochastic process $\eta = \alpha\cos(\beta t)$, where α is a Gaussian random variable, $\beta \in \mathbb{R}_+$ is some fixed constant, and $t \in \mathbb{R}$ has a physical sense of time.

Theorem 1.3. *(Yu. E. Gliklikh)* $\overset{o}{\beta}(t) = (2t)^{-1}\beta(t)$ *for all* $t \in \mathbb{R}_+$.

We introduce the space $\mathbf{C}^l\mathbf{L_2}$, $l \in \mathbb{N}$, of stochastic processes from $\mathbf{CL_2}$, which trajectories are a.s. differentiable with respect to Nelson—Gliklikh on $\mathfrak{I}$ up to order l inclusive. If $\mathfrak{I} \in \mathbb{R}_+$, then from theorem 1.3 follows the existence of the derivative $\overset{o}{\beta} \in \mathbf{C}^1\mathbf{L_2}$, which we call (*one-dimensional*) "*white noise*". The spaces $\mathbf{C}^l\mathbf{L_2}$ will be called *the spaces of differentiable "noises"*.

Further, let $\mathfrak{U} \equiv (\mathfrak{U}, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space; we consider the operator $K \in \mathcal{L}(\mathfrak{U})$, which spectrum $\sigma(K)$ is nonnegative, discrete, finitely multiple, and condense only to point zero. We denote by $\{\lambda_j\}$ the sequence of eigenvalues of the operator K, numbered by nonincreasing with allowance for multiplicity. Notice, that the linear span of the set $\{\varphi_j\}$ of the corresponding orthonormal eigenvectors of the operator K is dense in $\mathfrak{U}$. We also require that the operator K be nuclear (that is, its trace $\operatorname{Tr} K = \sum_{j=1}^{\infty} \lambda_j < +\infty$).

Consider a sequence of independent stochastic processes $\{\eta_j\}$ and define a stochastic K-process

$$\Theta_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\eta_j(t)\varphi_j \tag{1.2}$$

under the condition that the series (1.2) converges uniformly on any compact set in $\mathfrak{I}$. We point out that if $\{\eta_j\} \subset \mathbf{CL}_2$, then from the existence of a stochastic K-process Θ_K follows the a.s. continuity of its trajectories. We introduce the Nelson-Gliklikh derivatives of a stochastic K-process

$$\overset{o}{\Theta}{}_K^{(l)}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \overset{o}{\eta}{}_j^{(l)}(t)\varphi_j \tag{1.3}$$

provided that the derivatives on the right-hand side of (1.3) up to order l exist inclusively, and all the series converge uniformly on any compact set from $\mathfrak{I}$. Analogously to the finite-dimensional case, we consider the space $\mathbf{C}_K\mathbf{L}_2$ of stochastic K-processes, which trajectories are a.s. continuous, and the spaces $\mathbf{C}_K^l\mathbf{L}_2$ of stochastic K-processes, which trajectories are a.s. continuously differentiable with respect to Nelson—Gliklikh up to order $l \in \mathbb{N}$.

As an example, we consider a Wiener K-process

$$W_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)\varphi_j, \tag{1.4}$$

which obviously exists on $\overline{\mathbb{R}}_+$. In addition, the following assertion is fair.

Corollary 1.4. *$\overset{o}{W}_K(t) = (2t)^{-1}W_K(t)$ for all $t \in \mathbb{R}_+$ and for any nuclear operator $K \in \mathcal{L}(\mathfrak{U})$.*

In addition, the Wiener K-process (1.4) satisfies the conditions (W1) – (W3), if the symbol β in them is replaced by the symbol W_K. And if such a substitution done, it's fair the theorem.

Theorem 1.5. *For any nuclear operator $K \in \mathcal{L}(\mathfrak{U})$ with probability equal to 1, there exists a unique Wiener K-process satisfying the conditions (W1) – (W3), and it can be represented in the form (1.4).*

2. Stochastic Sobolev type equations with relatively p-bounded operators

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L, M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (i.e. are linear and continuous). Following [13], chapter 4 we introduce the L-resolvent set $\rho^L(M) = \left\{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\right\}$ and the L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of the operator M. If the L-spectrum $\sigma^L(M)$ of the operator M is bounded, then the operator M is said to be (L,σ)-bounded. If the operator M is (L,σ)-bounded, then there exist projectors

$$P = \frac{1}{2\pi i}\int_\gamma R_\mu^L(M)d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int_\gamma L_\mu^L(M)d\mu \in \mathcal{L}(\mathfrak{F}).$$

Let $R^L_\mu(M) = (\mu L - M)^{-1}L$ is the *right*, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ is the *left L-resolvent* of the operator M, and the closed contour $\gamma \subset \mathbb{C}$ bounds a domain containing $\sigma^L(M)$. We set $\mathfrak{U}^0(\mathfrak{U}^1) = \ker P(\mathrm{im}P)$, $\mathfrak{F}^0(\mathfrak{F}^1) = \ker Q(\mathrm{im}Q)$ and denote by $L_k(M_k)$ the restriction of the operator $L(M)$ on $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 2.1. *(Sviridyuk's theorem* [5] *on splitting) Let the operator M be (L,σ)-bounded, then*
(i) the operators $L_k(M_k) \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$;
(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$.

We construct the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Corollary 2.2. *Suppose that the operator M is (L,σ)-bounded, then for all $\mu \in \rho^L(M)$*

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k H^k M_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} L_1^{-1} Q.$$

The operator M is called *(L,p)-bounded,* $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point (that is, $H \equiv \mathbb{O}, p = 0$) or a pole of order $p \in \mathbb{N}$ (that is $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of the *$L$-resolvent* $(\mu L - M)^{-1}$ of the operator M.

Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$, and consider $\mathfrak{U}$ be a real separable Hilbert space. Consider a linear stochastic equation of Sobolev type

$$L \overset{o}{\eta} = M\eta + N\Theta, \tag{2.1}$$

where the free term will be determined later. We supplement equation (2.1) with the initial Showalter-Sidorov condition

$$\left[R^L_\alpha(M)\right]^{p+1} (\eta(0) - \eta_0) = 0, \tag{2.2}$$

on the advantages of which in comparison with the Cauchy condition

$$\eta(0) = \eta_0 \tag{2.3}$$

we spoke above. Here

$$\eta_0 = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k\varphi_k, \tag{2.4}$$

where $\{\varphi_k\}$i s an orthonormal basis of the space $\mathfrak{U}$, and pairwise independent random Gaussian variables $\xi_k \in \mathbf{L}_2$ are such that $D\xi_k \leq C_0$, and $\{\lambda_k\}$ is the spectrum of some nuclear operator $K \in \mathcal{L}(\mathfrak{U})$.

We further assume that $\mathfrak{I} = [0, \tau)$. Let $\mathfrak{U}$ be a real separable Hilbert space, $K \in \mathcal{L}(\mathfrak{U})$ is a nuclear operator which eigenvalues $\{\lambda_j\} \subset \mathbb{R}_+$. We call a stochastic K-process $\eta = \eta(t)$ *a classical solution of equation* (2.1), if a.s. all its trajectories satisfy equation (2.1) for some stochastic K-process $\Theta = \Theta(t)$ of the form

$$\Theta = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\alpha_k\varphi_k, \qquad (2.5),$$

where $\alpha_k(t)$ are one-dimensional continuous Gaussian stochastic processes such that the series (2.5) converges uniformly on $\mathfrak{I}$ and the operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, for all $t \in (0, \tau)$. The solution $\eta = \eta(t)$ of equation (2.1) is called a (*classical*) *solution* of problem (2.1), (2.2), if in addition condition (2.2) is satisfied. The classical solution of problem (2.1), (2.3) is defined similarly. Note, that the fulfillment of (2.2) follows from the fulfillment of (2.3).

We consider the problem (2.3) for the homogeneous equation

$$L \overset{o}{\eta} = M\eta. \tag{2.6}$$

In this (and only in this) case we assume $\mathfrak{I} = \mathbb{R}$.

Definition 2.3. A set $\mathfrak{P} \subset \mathfrak{U}$ is called *a phase space* of equation (2.5), if

(i) a.s. any trajectory of the solution $\eta = \eta(t)$ lies in $\mathfrak{P}$ pointwise, i.e. $\eta(t) \in \mathfrak{P}$ for all $t \in \mathbb{R}$;

(ii) for any random variable $\eta_0 \in \mathfrak{P}$ of the form (2.4) there exists a unique classical solution $\eta = \eta(t)$ of problem (2.6), (2.3).

Theorem 2.4. *Let the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the phase space of equation (2.5) is the subspace $\mathfrak{U}^1$.*

Indeed, by virtue of theorem 2.1, equation (2.5) is reduced to an equivalent system

$$H \overset{o}{\eta}_0 = \eta_0, \ \overset{o}{\eta}_1 = S\eta_1, \tag{2.7}$$

where $\eta^0 = (\mathbb{I} - P)\eta$, $\eta^1 = P\eta$. Differentiating by Nelson-Gliklikh the first equation in (2.7) and multiplying it from the left by H, we obtain successively

$$0 = H^{p+1} \overset{o}{\eta}^{0(p+1)} = ... = H^2 \overset{o}{\eta}^{0(2)} = ... = H \overset{o}{\eta}^{0} = \eta^0.$$

Condition (i) of Definition 2.3 thus is satisfied. For execution condition (ii) we note that if $\eta_0 \in \mathfrak{U}^1$ and has the form (2.4), then the unique solution of problem (2.7), (2.3) exists and has the form $\eta^1 = \eta^1(t) = e^{tS}\eta_0$, where $e^{tS} = \sum\limits_{k=0}^{\infty} \frac{t^k S^k}{k!}$. Then the unique solution of problem (2.6), (2.3) for $\eta_0 \in \mathfrak{U}^1$ will have the form $\eta(t) = (\mathbb{O}(\mathbb{I} - P) + e^{tS}P)\eta_0$.

Corollary 2.5. *Under the conditions of Theorem 2.4, the solution of problem (2.6), (2.3) is a Gaussian stochastic K-process if the random variable η_0 has form (2.4).*

Definition 2.6. The map $U^\bullet \in C^\infty(\mathbb{R}; \mathcal{L}(\mathfrak{U}))$ is called *the group of resolving operators* if

$$U^s U^t = U^{s+t} \text{ for all } s, t \in \mathbb{R}. \tag{$*$}$$

A group $\{U^t : t \in \mathbb{R}\}$ is said to be *holomorphic*, if it is analytically extended to the whole complex plane retaining its property (*); and degenerate if its unit $U^0 \in \mathcal{L}(\mathfrak{U})$ is a projection.

For example, a holomorphic degenerate group is the resolving group $U^t = \mathbb{O}(\mathbb{I} - P) + e^{tS}P$ of the equation (2.6). If $\{U^t : t \in \mathbb{R}\}$ is a holomorphic degenerate group, then its image $\operatorname{im} U^\bullet = \operatorname{im} U^0$ and kernel $\ker U^\bullet = \ker U^0$.

We call the group *the resolving group of equation* (2.6), if its image coincides with the phase space of the given equation

$$U^t = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)e^{\mu t}d\mu, \quad t \in \mathbb{R}.$$

Let us return to equation (2.1) and take the interval $\mathfrak{I} = [0,\tau)$. Let the stochastic K-process $\Theta = \Theta(t), t \in [0,\tau)$ has the form (2.5) and is such that

$$\begin{cases} \text{(i) almost surely all trajectories of the stochastic process } N\Theta \\ \text{ are continuous on } \mathfrak{I}; \\ \text{(ii) almost surely all trajectories of the stochastic process } (\mathbb{I}-Q)N\Theta \\ \text{are continuously differentiable with respect to Nelson—Gliklikh} \\ \text{up to order } (p+1) \text{ inclusive on the interval } (0,\tau) \end{cases} \quad (2.8)$$

We note that the K-process, satisfying conditions (2.8) is the Wiener K-process (1.4).

Theorem 2.7. *Let the operator M is (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for any $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, any stochastic K-process $\Theta = \Theta(t)$ of the form (2.5) and for any random variable η_0 of the form (2.4), there exists a unique classical solution $\eta = \eta(t)$ of the problem (2.1), (2.2), which also has the form*

$$\eta(t) = U^t\eta_0 + \int\limits_0^t U^{t-s}L_1^{-1}QN\Theta(s)ds - \sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I}-Q)N \overset{o}{\Theta}{}^{(q)}(t). \quad (2.9)$$

If, in addition, the random variable η_0 is such that

$$(P-\mathbb{I})\eta_0 = \sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I}-Q)N \overset{o}{\Theta}{}^{(q)}(0),$$

then the solution (2.9) is also the unique solution of problem (2.1), (2.2).

The proof of Theorem 2.7 consists in substituting solution (2.9) into equation (2.1). The uniqueness of this solution follows from Theorem 2.4. A complete proof can be found in [9].

3. The Laplace—Beltrami operator spectrum

We consider a n-dimensional smooth compact oriented connected Riemannian manifold without boundary Ω and the space of differential q-forms on Ω we denote by $E^q = E^q(\Omega), 0 \le q \le n$. In particular $E^0(\mathbb{R}^n)$ is the space of functions of n variables. Note that there exists a linear Hodge operator $* : E^q \to E^{n-q}$, which associates the q-form with Ω $(n-q)$-form. In the double application of the Hodge operator, the equality $** = (-1)^{q(n-q)}$ holds. In addition, there is an operator for taking the external differential $d : E^q \to E^{q+1}$. We define the operator $\delta : E^q \to E^{q-1}$, setting $\delta = (-1)^{n(q+1)+1} * d\, *$. On 0-forms the operator δ is simply a zero linear functional.

Definition 3.1. The Laplace-Beltrami operator $\Delta : E^q \to E^q$ is defined by the equality $\Delta = \delta d + d\delta$, and it is a linear operator on space $E^q, 0 \leq q \leq n$.

We introduce the space of harmonic q-form $H^q = \{\omega \in E^q : \Delta\omega = 0\}$. We denote by $H(\alpha)$ the projection onto the space of harmonic forms.

Theorem 3.2. *(Hodge–Kodaira decomposition theorem) For any integer $q, 0 \leq q \leq n$, space H^q is finite-dimensional and there is the following decomposition of the space of smooth q-forms on Ω into an orthogonal direct sum*

$$E^q = \Delta(E^q) \oplus H^q = d\delta(E^q) \oplus \delta d(E^q) \oplus H^q. \tag{3.1}$$

Consequently, the equation $\Delta\omega = \alpha$ has solution $\omega \in E^q$ precisely then, when q-form α is orthogonal to the space of harmonic forms H^q.

By the formula

$$(\xi, \eta)_0 = \int\limits_{\Omega_n} \xi \wedge *\eta,\ \xi, \eta \in E^q \tag{3.2}$$

where $*$ is the Hodge operator, we define a scalar product in the space E^q, $q = 0, 1, \ldots, n$, and denote the corresponding norm by $||\cdot||_0$. We continue the scalar product (3.2) by a direct sum $\overset{n}{\underset{q=0}{\oplus}} E^q$, requiring that different spaces E^q were orthogonal. Completion of space E^q in the norm $||\cdot||_0$ we denote by $\mathfrak{H}_0^q$. We denote by $P_{q\Delta}$ the orthoprojector on $\mathfrak{H}_\Delta^q$.

By the formulas

$$(\xi, \eta)_1 = (\Delta\xi, \eta)_0 + (\xi_\Delta, \eta_\Delta)_0, \tag{3.3}$$

$$(\xi, \eta)_2 = (\Delta\xi, \Delta\eta)_0 + (\xi, \eta)_1, \tag{3.4}$$

we introduce the scalar product on E^q , where $\omega_\Delta = P_{q\Delta}\omega$. Completion of the lineal E^q according to the corresponding norms $||\cdot||_1$ and $||\cdot||_2$ we denote by $\mathfrak{H}_1^q$ and $\mathfrak{H}_2^q$ respectively. Actually upper index means how many times differentiable in the generalized sense of the k-form in the corresponding spaces.

The spaces $\mathfrak{H}_l^q$,$l = 1, 2$ are Banach spaces (further their Hilbert structure does not interest us), and we have continuous and dense embeddings $\mathfrak{H}_2^q \subset \mathfrak{H}_1^q \subset \mathfrak{H}_0^q$. The following assertion is true.

Corollary 3.3. *For any $q = 0, 1, \ldots, n$ there are splitting spaces*

$$\mathfrak{H}_l^q = \mathfrak{H}_{l\Delta}^{q1} \oplus \mathfrak{H}_\Delta^q,$$

where $\mathfrak{H}_{l\Delta}^{q1} = (\mathbb{I} - P_\Delta)[\mathfrak{H}_l^q],\ l = 0, 1, 2.$

We define the Green's operator $G : E^q \to (H^q)^\perp$, setting $G(\alpha)$ equal to the unique solution of the equation $\Delta\omega = \alpha - H(\alpha)$.

A real number λ, for which there exists non-zero q-form u, such that $\Delta u = \lambda u$, is called the eigenvalue of operator Δ. If λ is an eigenvalue, then any q-form u, such that $\Delta u = \lambda u$, is called an eigenfunction corresponding to eigenvalue λ.The eigenfunctions corresponding to a fixed λ, form a subspace of E^q, called the eigenspace corresponding to the eigenvalue.

Proposition 3.4. *The eigenvalues of the Laplace—Beltrami operator are nonnegative and do not have a finite limit point.*

Consider the restriction of operator Δ on $(E^q)^\perp$. Then $\Delta : (E^q)^\perp \to (E^q)^\perp$ and we have the Green's operator $G : (E^q)^\perp \to (E^q)^\perp$ and $\Delta G\alpha = \alpha$, $G\Delta\alpha = \alpha$ for all $\alpha \in (E^q)^\perp$. In view of this, the eigenvalues of the operator $G|(E^q)^\perp$ are inverse to eigenvalues $\Delta|(E^q)^\perp$.

Let

$$\eta = \sup_{\varphi \in (E^q)^\perp,\ \|\varphi\|=1} \|G\varphi\|.$$

Then $\eta > 0$ and $\lambda = 1/\eta$ is the eigenvalue of Laplace—Beltrami operator Δ.

Suppose that operator $\Delta|(H^q)^\perp$ contains eigenvalues
$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and eigenfunctions $u_1, u_2, ..., u_n$. Subspace R_n spanned by $\{u_1, u_2, ..., u_n\}$. Let's find

$$\eta_{n+1} = \sup_{\varphi \in (E^q \oplus R_n)^\perp,\ \|\varphi\|=1} \|G\varphi\|.$$

Acting as above, we obtain that $\lambda_{n+1} = 1/\eta_{n+1}$ the following proper value of the Laplace—Beltrami operator, while $\lambda_{n+1} \geq \lambda_n$.

Proposition 3.5. *Eigenfunctions corresponding to different eigenvalues of the Laplace-Beltrami operator are orthogonal and form a complete system in L_2.*

Let $\{\lambda_j\}$ be a sequence of eigenvalues of Δ on E^q with multiplicity taken into account, and $\{u_j\}$ is the corresponding orthonormal sequence of eigenfunctions. Let $\alpha \in E^q$. Then

$$\lim_{n \mapsto \infty} \|\alpha - \sum_{i=1}^{n} \langle \alpha, u_i \rangle u_i\| = 0.$$

Now we show a concrete form of eigenvalues and functions in the space of q-form on a two-dimensional sphere, which is a particular case of a smooth compact oriented Riemannian manifold without boundary. We note that for a two-dimensional manifold, only q-forms with $q = 0, 1, 2$. We consider a two-dimensional unit sphere given in spherical coordinates

$$\{x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta\} \tag{3.5}$$

with the conditions for the periodicity of the coefficients of the q-form

$$a(\theta,\varphi) = a(\theta,\varphi + 2\pi); a(0,\varphi) \text{ "}\textit{and}\text{" } a(\pi,\varphi) \text{ "}\textit{do not depend on}\text{" } \varphi. \tag{3.6}$$

It is evidently, the cases with $q = 0, 1, 2$ are possible.

Let us show the view of eigenfunctions φ_l. Firstly, let's look at the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on sphere for 0-form. The eigenvalues of the Laplace-Beltrami operator in the space of 0-forms on a two-dimensional sphere have the form $\nu_l = l(l + 1)$. Functions $\varphi_l = Y_l$ are the eigenfunctions for the eigenvalue ν_l . They can be representees as the Fourier series

$$Y_l(\theta,\varphi) = \sum_{m=-\infty}^{+\infty} Y_{l,m}(\theta,\varphi) = \sum_{m=-\infty}^{+\infty} Y_l^m(\theta)e^{im\varphi} \tag{3.7}$$

with respect to the functions Y_l^m, which are the solutions of equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dY_l^m}{d\theta}\right\} - \frac{m^2}{\sin\theta}Y_l^m = -Y_l^m. \tag{3.8}$$

The solutions (5.4) are the Legendre polynomials of the cosines $P_l^{|m|}(\cos\theta)$. The functions $Y_{l,m}$ for $l = 0, 1, 2, ...$ and $m = -l, -l+1, ..., 0, ..., l-1, l$ take the form

$$Y_{l,m}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\cdot\frac{(l-|m|)!}{(l+|m|)!}}\sin^{|m|}\theta P_l^{|m|}(\cos\theta)e^{im\varphi}. \tag{3.9}$$

By commuting the Hodge operator and the Laplace—Beltrami operator

$$\Delta * u = *\Delta u$$

the eigenvalues of 0-forms and 2-forms coincide, and the eigenfunctions differ by the presence of both differentials for each term for 2-forms.

If we consider 1-form on a two-dimensional sphere, then one of the ways of finding eigenvalues and eigenfunctions is to take the differential from the 0-form being an eigenfunction itself.

Let the 0-form u_l be an eigenfunction corresponding to eigenvalue λ_l, i.e.

$$(d\delta + \delta d)u_l = \delta du_l = \lambda_l u_l.$$

Then 1-form $v_l = du_l$ is an eigenfunction of the corresponding eigenvalue λ_l, since

$$(d\delta + \delta d)v_l = d\delta du_l + \delta ddu_l = d\delta du_l + 0 = d\lambda_l u_l = \lambda_l du_l = \lambda_l v_l.$$

Let 2-form u_l be an eigenfunction corresponding to eigenvalue λ_l, i.e.

$$(d\delta + \delta d)u_l = d\delta u_l = \lambda_l u_l.$$

Then 1-form $w_l = \delta u_l$ is an eigenfunction of corresponding eigenvalue λ_l, since

$$(d\delta + \delta d)w_l = d\delta\delta u_l + \delta d\delta u_l = 0 + \delta d\delta u_l = \delta\lambda u_l = \lambda_l\delta u_l = \lambda_l w_l.$$

4. "Stochastic" Barenblatt—Zheltov—Kochina equation

Let Ω_n be a n-dimensional oriented compact connected Riemannian manifold without boundary. We define, using the theory of smooth differential q-forms presented in points 1 and 3 with coefficients that are stochastic K-processes lying in $\mathbf{C^1_K}\mathfrak{H}^q_\mathbf{0}$

$$\omega(t, x_1, x_2, ..., x_n) = \sum_{|i_1,i_2,...,i_q|=q} \chi_{i_1,i_2,...,i_q}(t, x_1, x_2, ..., x_n)dx_{i_1}\wedge dx_{i_2}\wedge ... \wedge dx_{i_n},$$

where $|i_1, i_2, ..., i_q|$ is a multi-index, and the coefficients have the form

$$\chi_{i_1,i_2,...,i_q}(t, x_1, x_2, ..., x_n) = \sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_{k,i_1,i_2,...,i_q}(t)\varphi_k.$$

We assume that the phenomenon that we investigate occurs with velocities much lower than the speed of light and there is no time dependence and coordinates. We share time and local coordinates. Here at all points of the manifold the coefficients of differential forms depend only on the unified time t.

In our investigations we use stochastic K-processes that are continuous, but nondifferentiable at every point in the usual sense. We use differentiability in the sense of Nelson—Gliklikh. For fixed $\alpha \in \mathbb{R}, \lambda \in \mathbb{R}$ we introduce operators

$$L = (\lambda + \Delta),\ \ M = \alpha\Delta, \tag{4.1}$$

where Δ is the Laplace-Beltrami operator. Consider a "stochastic" equation with differential forms

$$L\overset{o}{\eta}= M\eta. \tag{4.2}$$

Remark 4.1. For $q = 0$ equation (4.2) coincides with (2.6).

The initial Cauchy condition will have the form

$$\eta\,(0) = \eta_0. \tag{4.3}$$

Operator L is Fredholm operator by the Atiyah—Singer theorem and the following assertion is valid.

Lemma 4.2. *For any $\alpha \in \mathbb{R} \setminus \{0\}, \lambda \in \mathbb{R} \setminus \{0\}$ the operator M is $(L, 0)$-bounded.*

Let $\{\nu_l\}$ be a sequence of eigenvalues of the Laplace-Beltrami numbered by nonincreasing with allowance for their multiplicity, $\{\varphi_l\}$ are the corresponding orthonormal (in the sense of $\mathfrak{U}$) eigenfunctions. We construct the projector $P \in \mathcal{L}(\mathfrak{U})$,

$$P = \begin{cases} \mathbb{I},\ \lambda \neq \nu_l \text{ for all } l \in \mathbb{N}; \\ \mathbb{I} - \sum\limits_{\lambda=\nu_l} \langle \cdot, \varphi_j\rangle\,\varphi_l, \text{if } \lambda = \nu_l. \end{cases}$$

We consider differential q-forms with coefficients that are $\mathfrak{U}$-valued stochastic K-processes.

By Theorem 2.7, in the homogeneous case, for $\Theta = 0$ we have an assertion.

Theorem 4.3. *For any $\lambda \in \mathbb{R} \setminus \{0\}$, $\alpha \in \mathbb{R} \setminus \{0\}$, $N \in \mathcal{L}(\mathfrak{F})$ and $\eta_0 \in \mathbf{L}_2$, which does not depend on Θ there exists a unique solution $\eta = \eta(t)$ of problem (4.3), (4.4), which has the form*

$$\eta(t) = {\sum_{l=1}^{\infty}}' \left[exp\left(\frac{\alpha\nu_l}{\lambda - \nu_l} t\right) \left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k(\varphi_k, \varphi_l)_0\varphi_l \right)\right] \tag{4.5}$$

Now let us consider inhomogeneous equation

$$L\overset{o}{\eta}= M\eta + N\Theta. \tag{4.6}$$

with initial Showalter—Sidorov condition

$$\left[R^L_\alpha(M)\right]^{p+1}(\eta\,(0) - \eta_0) = 0, \tag{4.7}$$

where η_0 decomposes into a series (2.4). By Theorem 2.7, the following theorem is true.

Theorem 4.4. *For any $\lambda \in \mathbb{R} \setminus \{0\}$, $\alpha \in \mathbb{R} \setminus \{0\}$ and any operator $N \in \mathcal{L}(\mathfrak{F})$ and $\eta_0 \in \mathbf{L}_2$, that does not depend on Θ there exists a unique classical solution $\eta = \eta(t)$ of problem (4.6), (4.7), which also has the form*

$$\eta(t) = {\sum_{l=1}^{\infty}}' \left[exp\left(\frac{\alpha\nu_l}{\lambda - \nu_l} t\right) \left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k(\varphi_k, \varphi_l)_0\varphi_l \right)\right] +$$

$$+ \int s_0^t U^{t-s} L_1^{-1} Q N \sum_{k=1}^{\infty} \sqrt{\lambda_k}\alpha_k(s)\varphi_k ds -$$

$$-\sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I}-Q)N\sum_{j=1}^{\infty}\sqrt{\lambda_j}\overset{o}{\alpha}{}_j^{(q)}(t)\varphi_j. \tag{4.8}$$

If in addition η_0 is such that

$$(P-\mathbb{I})\left(\sum_{k=1}^{\infty}\sqrt{\lambda_k}\xi_k(\varphi_k,\varphi_l)_0\varphi_l\right)=\sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I}-Q)N\left(\sum_{j=1}^{\infty}\sqrt{\lambda_j}\overset{o}{\alpha}{}_j^{(q)}(t)\varphi_j\right), \tag{4.9}$$

then the solution (4.8) is also the unique solution of problem (4.6), (4.7).

Remark 4.5. In the future, we plan to continue these researches to study the so-called "white noise" (in inverted commas) work on which (differs from that considered in this article) have GA. Sviridyuk, A. Favini, A.A. Zamyshshlyaeva [15], M.A. Sagadeeva [16]. The other way to generalize these results lies in the study of the generalizations of Showalter-Sidorov problem considered for example by S.A. Zagrebina and A.V. Keller ([17]).

References

1. Barenblatt, G.I.,Zheltov, Yu.P. and Kochina, I.N.: Basic concepts in the theory of seepage of homogeneous fluids in fissurized rocks, *J. Applied Mathematics and Mechanics. Moscow* **24**, N5 (1960) 1286–1303.
2. Hallaire, M.: On a theory of moisture-transfer, *Inst. Rech. Agronom. Paris* N3. (1964) 60–72.
3. Chen, P.J. and Gurtin, M.E.: On a theory of heat conduction involving two temperatures, *Z. Angew. Math. Physics. Basel* **19** (1968) 614–627.
4. Sviridyuk, G.A. and Shafranov D.E.: The Caushy problem for the Barenblatt—Zheltov—Kochina equation on a smooth manifold, *Bulletin of Chelyabinsk State University. Ser. Mathematics, mechanics and informatics. Chelyabinsk* N3 (2003) 171–177.
5. Sviridyuk, G.A.: On the general theory of operator semigroups, *Russian Mathematical Surveys. Moscow* **49**, N4 (1994) 45–74.
6. Sviridiuk, G.A. and Zagrebina, S.A.: The Showalter – Sidorov problem as a phenomena of the Sobolev-type equations, *The Bulletin of Irkutsk State University. Ser. Mathematics. Irkutsk* N1 (2010) 104–125.
7. Sviridyuk, G.A., Zagrebina, S.A. and Konkina, A.S.: The Oskolkov equations on the geometric graphs as a mathematical model of the traffic flow, *Bulletin of the South Ural State University. Ser. Mathematical Modeling, Programming and Computer Software. Chelyabinsk* **8**, N3 (2015) 148–154.
8. Gliklikh, Yu.E.: *Global and stochastic analysis with applications to mathematical physics*, Springer, London, Dordrecht, Heidelberg, New York, 2011.
9. Sviridyuk, G.A. and Manakova, N.A. : The dynamical models of Sobolev type with Showalter–Sidorov condition and additive "noise", *Bulletin of the South Ural State University. Ser. Mathematical Modeling, Programming and Computer Software. Chelyabinsk* **7**, N1 (2014) 90–103.
10. Kovacs, C. and Larsson S.: Introduction to stochastic partial differential equation, in: *Proc. New Direction in the Mathematical and Computer Scinces. Publications of ICMCS.* **4** (2007) 159–232, National Universities Commision, Abuja, Nigeria.
11. Zagrebina, S.A., Soldatova, E.A. and Sviridyuk, G.A.: The stochastic linear Oskolkov model of the oil transportation by the pipeline, in: *Springer Proc. in Mathematics and Statistics. Semigroups of Operators – Theory and Applications.* (2013) 317–325, Springer, Bedlewo, Poland.

12. Melnikova, I.V., Filinkov, A.I. and Alshansky, M.A. : Absract stochastic equations II solutions spaces of abstract stochastic distributions, *Journal of Mathematical Sciences. New York* **116**, N5 (2003) 3620–3656.
13. Sviridyuk, G.A. and Fedorov, V.E.: *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*, VSP, Utrecht etc., 2003.
14. Warner, F.: *Foundations of Differentiable Manifolds and Lie Groups Ser.: Graduate Texts in Mathematics.* **94** Springer-Verlag, New York, 1983.
15. Favini, A., Sviridyuk, G.A., and Zamyshlyaeva, A.A.: One class of sobolev type equations of higher order with additive white noise, *Communications on Pure and Applied Analysis.*, **15**, N1 (2016) 185–196.
16. Favini, A., Sviridyuk G.A., and Sagadeeva, M.A.: Linear sobolev type equations with relatively p-radial operators in space of "noises", *Mediterranean Journal of Mathematics.*, **13**, N6 (2016) 4607-4621.
17. Keller, A.V. and Zagrebina S.A.: Some generalizations of the Showalter – Sidorov problem for Sobolev-type models, *Bulletin of the South Ural State University, Ser. Mathematical Modelling, Programming and Computer Software. Chelyabinsk* **8**, N2 (2015) 5–23.

D.E. Shafranov: Department of Mathematics, Mechanics and Computer Technologies, South Ural State University (NRU), 454080, 76, Lenin avenue, Chelyabinsk, Russia
E-mail address: shafranovde@susu.ru

O.G. Kitaeva: Department of Mathematics, Mechanics and Computer Technologies, South Ural State University (NRU), 454080, 76, Lenin avenue, Chelyabinsk, Russia
E-mail address: kitaevaog@susu.ru