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MAJORIZATION PROPERTIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE GENERALIZED DERIVATIVE OPERATOR

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ABSTRACT: Invoking generalize derivative operator $D^{n,m}_{\lambda_1,\lambda_2,b}$, we investigate the Majorization properties for analytic functions.

1. Introduction

Let f and g be Holomorphic in the open unit disc

$$\Delta = \{z : z \in \mathbb{C}, |z| < 1\} \tag{1.1}$$

We say that f is majorized by g in Δ (see[7]) and write

$$f(z) << g(z) \quad (z \in \Delta), \tag{1.2}$$

If there exist a function φ, analytic in Δ s.t.

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z) g(z) \quad (z \in \Delta), \tag{1.3}$$

It may be noted that (1.2) is closely related to the concept of quasi-subordination between holomorphic functions

Let A denote the class of holomorphic function of the form

$$f(z) = \sum_{k=2}^{\infty} a_k z^k, \tag{1.4}$$

where, a_k is the complex number, which is holomorphic in the open unit disc $\Delta = \{Z = \mathbb{C} : |Z| < 1\}$. An holomorphic function $f(z)$ is subordinate to the holomorphic function $g(z)$, written $f(z) \prec g(z)$, if there exist an holomorphic function w in Δ, such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. In particular, if $g(z)$ is univalent in Δ, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Now E.El.Yagubi and M.Darus [1] define the generalized derivative operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}$ is defined as follows:

Definitation 1. For $f(z) \in$ A, the operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}$ is defined by $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b} :$ A $\to$ A

$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b} f(z) = \mathcal{F}^{m}_{\lambda_1,\lambda_2,b} f(z) \times \mathcal{R}^n f(z), \; z \in \text{U} \tag{1.5}$$

where, $n, m, b \in \mathcal{N}_0 = \{0, 1, 2, 3, ...\}$, $\lambda_2 \geq \lambda_1 \geq 0$ and $\mathcal{R}^n f(z)$ denotes the Ruscheweyh derivative operator, given by

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} c(n, k) a_k z^k \tag{1.6}$$

where, $c(n, k) = \dfrac{(n+1)_{k-1}}{(1)_{k-1}}$ (1.7)

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and
$$\mathcal{F}^{m}_{\lambda_1,\lambda_2,b}f(z) = z + \sum_{k=1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(k-1)+b}{1+(\lambda_2)(k-1)+b}\right]^{m} z^k \tag{1.8}$$
where, $m,\ b \in \mathcal{N}_0 = \{0, 1, 2, 3, \ldots\}$ and $\lambda_2 \geq \lambda_1 \geq 0$.

If $f(z)$ is given by (1.4), then by (1.6) we can write
$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(k-1)+b}{1+(\lambda_2)(k-1)+b}\right]^{m} c(n,k)a_k z^k \tag{1.9}$$
where, $c(n, k)$ is given by (1.8).

In view of (1.9), it is clear that $\mathcal{D}^{n,0}_{\lambda_1,\lambda_2,b}$ is the Ruscheweyh Derivative operator (see [6]) and $\mathcal{D}^{n,1}_{\lambda_1,0,0} = \mathcal{D}^{\lambda_1}_{n}$ the generalized Ruscheweyh derivative operator (see[4]). $\mathcal{D}^{0,m}_{1,0,0} = s^n$ the Salageon derivative operator (see [2]). $\mathcal{D}^{0,m}_{1,0,b} = \mathcal{D}^{m}_{b}$ the Cho and Srivastava derivative operator (see[5]).

Definitation 2. A function $f(z)$ is said to be in the class $\mathcal{S}^{n,m}_{\lambda_1,\lambda_2,b}$ if and only if
$$\frac{z[\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)]'}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)} \prec \frac{1+\mathrm{A}z}{1+\mathrm{B}z} \tag{1.10}$$
where, $1 \geq \mathrm{A} > \mathrm{B} \geq -1$

For detail one can see [1].

Lemma 1. $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)$ Satisfiees the following:
$$z[\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)]' = \frac{(1+b)}{(\lambda_1+\lambda_2)}\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}f(z) - \frac{\{1-(\lambda_1+\lambda_2)(k-1)+b\}}{(\lambda_1+\lambda_2)}\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z) \tag{1.11}$$
it is three term recurrence relation.

2. Majorization Problem for the Class $\mathcal{S}^{n,m}_{\lambda_1,\lambda_2,b}$

Theorem (1). Let the function $f(z) \in \mathrm{A}$ and suppose that $g(z) \in \mathcal{S}^{n,m}_{\lambda_1,\lambda_2,b}(\mathrm{A},\mathrm{B})$. If
$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z) \ll \mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}g(z) \tag{2.1}$$
Then,
$$\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)\right| \leq \left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}g(z)\right| \text{ for } |z| \leq r_o \tag{2.2}$$
$$r^3\left(\frac{1+b}{\lambda_1+\lambda_2}\right)[\{(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+\mathrm{b})\}] - r^2\left[\left(\frac{1+b}{\lambda_1+\lambda_2}\right)\left\{(1+b)+2\mathrm{B}(1+b)\right\}\right]$$
$$- r\left[2(1+b)+\left(\frac{1+b}{\lambda_1+\lambda_2}\right)\left\{(\lambda_1+\lambda_2)(A-B)+B(1+b)\right\}\right]+\left(\frac{1+b}{\lambda_1+\lambda_2}\right)(1+b) = 0$$
$$k,\ \eta \in \mathbb{C},\ p(0)=1,\ \lambda_2 \geq \lambda_1 \geq 0,\ n,\ m,\ b \in \mathcal{N}_0 = \{0, 1, 2, 3, \ldots\}$$
$$|1+b| \geq |(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)|,\ 1 \geq \mathrm{A} > \mathrm{B} \geq -1$$

Proof. Since $f(z) \in \mathcal{S}^{n,m}_{\lambda_1,\lambda_2,b}$, we find from (1.11) that
$$\frac{z(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z))}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)} = \frac{1+\mathrm{A}|z|}{1+\mathrm{B}|z|} \tag{2.3}$$
where, $\omega(z) = c_1 z + c_2 z^2 + \cdots$, $\omega \in \mathcal{P}$, $\mathcal{P}$ denotes the well known class of the bounded analytic functions in $\mathbb{U}$ and satisfies the condition $\omega(0) = 0$, and $|\omega(z)| \leq |z|$ $(z \in \mathbb{U})$

$$\left|\frac{z(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z))'}{\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)}\right| \le |p(z)| + \frac{|z||p'(z)|}{k|p(z)|+\eta} \tag{2.4}$$

By using (1.11) in (2.4)

$$\left|\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)\right| \le \frac{(1+b)(1+\mathrm{B}|z|)}{(1+b)-|z|\left\{\left|(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)\right|\right\}}\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)\right| \tag{2.5}$$

Since $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z)$ is majorized by $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}g(z)$ in the unit disk $\mathbb{U}$, from (1.3) we have

$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z) = \varphi(z)\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}g(z) \tag{2.6}$$

where, $\varphi(z) \le 1$. Differentiating (2.6) w.r. to z and multiplying by z, we get

$$z(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}f(z))' = z\varphi(z)'(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}g(z)) + z\varphi(z)(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}g(z))' \tag{2.7}$$

Which on using (1.11), gives

$$\left(\frac{1+b}{\lambda_1+\lambda_2}\right)\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)\right| = |z||\varphi(z)'|\left|\left(\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b}g(z)\right)\right| + |\varphi(z)|\left|\left(\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}g(z)\right)\right|\left(\frac{1+b}{\lambda_1+\lambda_2}\right) \tag{2.8}$$

Noting that $\varphi(z) \in \mathcal{P}$ satisfying the inequality (see[8])

$$|\varphi(z)'| \le \frac{1-|\varphi(z)|^2}{1-|z|^2} \tag{2.9}$$

Now making use of (2.5) and (2.9) in (2.8)

$$\left(\frac{1+b}{\lambda_1+\lambda_2}\right)\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)\right| \le \left[|\varphi(z)|\left(\frac{1+b}{\lambda_1+\lambda_2}\right) + \left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\right)|z| \right.$$
$$\left.\left\{\frac{(1+b)(1+\mathrm{B}|z|)}{(1+b)-|z|\left\{\left|(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)\right|\right\}}\right\}\right]\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}g(z)\right| \tag{2.11}$$

Which upon setting $|z| = r$, $|\varphi(z)| = \rho \quad (0 \le \rho \le 1)$

Leads us to the inequality

$$\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)\right| \le \frac{\vartheta(\rho)\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b}g(z)\right|}{(1-r^2)\left[(1+b)+r\left\{\left|(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)\right|\right\}\right]\left(\frac{1+b}{\lambda_1+\lambda_2}\right)} \tag{2.12}$$

where, $\vartheta(\rho) = \rho\left(\frac{1+b}{\lambda_1+\lambda_2}\right)(1-r^2)[(1+b)-r\{|(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)|\}]$
$+ r(1+b)(1+|\mathrm{B}|r)(1-\rho^2)$

Takes the maximum value at $\rho = 1$, with $r_0 = r_0(n, m, \lambda_1, \lambda_2, b)$. Where r_0 is the smallest positive root of equation (2.2). Furthermore if $0 \le \sigma \le r_0$, then the function $\phi(\rho)$ defined by

$$\phi(\rho) = \rho\left(\frac{1+b}{\lambda_1+\lambda_2}\right)(1-\sigma^2)[(1+b)-\sigma\{|(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)|\}]$$
$$+ \sigma(1+b)(1+|\mathrm{B}|\sigma)(1-\rho^2)$$

Is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\phi(\rho) = \phi(1) = \left(\frac{1+b}{\lambda_1+\lambda_2}\right)(1-\sigma^2)[(1+b)-\sigma\{|(\lambda_1+\lambda_2)(\mathrm{A}-\mathrm{B})+\mathrm{B}(1+b)|\}]$$
$$(0 \le \rho \le 1,\ 0 \le \sigma \le r_0) \tag{2.11}$$

Hence upon setting $\rho = 1$, in (2.14), we conclude that (2.1) of Theorem (1) holds true for $|z| \le r_0$.

Where, $r_0 = r_0$ (A, B, n, m, λ_1, λ_2, b) is the smallest positive root of equation (2.2). This completes the proof of Theorem 1.

Setting A = 1, B = −1 in Theorem (1), we have

Corollary (1): Let the function $f(z) \in$ A and suppose that $g(z) \in \mathcal{S}^{n,m}_{\lambda_1,\lambda_2,b}$. If

$$\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b} f(z) \ll \mathcal{D}^{n,m}_{\lambda_1,\lambda_2,b} g(z)$$

Then, $\left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b} f(z)\right| \le \left|\mathcal{D}^{n,m+1}_{\lambda_1,\lambda_2,b} g(z)\right|$ for $|z| \le r_1$

where, $$r_1(n, m, \lambda_1, \lambda_2, b) = \frac{\delta \pm \sqrt{\delta^2 - 4(1+b)\left(\frac{1+b}{\lambda_1+\lambda_2}\right)^2 \left|2(\lambda_1+\lambda_2) - (1+b)\right|}}{2\left(\frac{1+b}{\lambda_1+\lambda_2}\right)\left|2(\lambda_1+\lambda_2) - (1+b)\right|}$$

where, $\delta = \left(\frac{1+b}{\lambda_1+\lambda_2}\right)\{|2(\lambda_1+\lambda_2) - (1+b)| + (1+b)\} + 2(1+b)$

Setting $\xi = \left(\frac{1+b}{\lambda_1+\lambda_2}\right)$ in Corollary (1), we have

Corollary (2): Let the function $f(z) \in$ A and suppose that $g(z) \in \mathcal{S}^{n,m}_{\lambda_1,\lambda_2,b}$. If

$$\left(\mathcal{S}^{\lambda,\mu,\eta}_{0,z} f(z)\right)^j \ll \left(\mathcal{S}^{\lambda,\mu,\eta}_{0,z} g(z)\right)^j$$

Then, $\left|\left(\mathcal{S}^{\lambda+1,\mu,\eta}_{0,z} f(z)\right)^j\right| \le \left|\left(\mathcal{S}^{\lambda+1,\mu,\eta}_{0,z} g(z)\right)^j\right|$ for $|z| \le r_2(n, m, \lambda_1, \lambda_2, b)$

where, $$r_2(j, \lambda, \mu, \eta) = \frac{\left[|2-\xi| + (2+\xi)\right] \pm \sqrt{\left[|2-\xi| + (2+\xi)\right]^2 - 4|2-\xi|\xi}}{2|2-\xi|}$$

For detail one can see [3].

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