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THE MULTIPOINT INITIAL-FINAL VALUE PROBLEMS FOR LINEAR SOBOLEV-TYPE EQUATIONS WITH RELATIVELY p-SECTORIAL OPERATOR AND ADDITIVE "NOISE"

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Abstract. The multipoint initial-final value problems for linear sobolev-type equations with relatively p-sectorial operator and additive "noise" are investigated. Abstract results in the space of "noises" are used to study the solvability of a boundary value problem for the linear system of the Navier – Stokes equations with initial-final conditions and additive "white noise". Researches are based on the notion of the Nelson – Gliklikh derivative of the Wiener process. The main result is to prove the unique solvability of the problem with multipoint initial-final conditions. The stochastic problem is considered as a generalization of determinate case.

1. Introduction

A linear operator differential equation of the form

$$L\dot{u} = Mu + f, \tag{1.1}$$

is called *the degenerate differential equation* [1] or *the Sobolev type equation* [2], if $\ker L \neq \{0\}$. One of the conditions of the equation (1.1) solvability is the requirement that operator M is the strong (L,p)-sectorial , $p \in \{0\} \cup \mathbf{N}$ (see [2], Chapter 3). Sufficient conditions for operator M to be strongly (L,p)-sectorial can be found, for example, in [3]. However, for our purposes the necessary conditions for operator M to be strongly (L,p)-sectorial are more suitable (see for example [4]). (Incidentally, these necessary and sufficient conditions are equivalent, see[2], p. 3.5). The equation is equipped with the set of terminal conditions [5]

$$\lim_{t\to 0+} P_0(u(t) - u_0) = 0, \quad P_j(u(\tau_j) - u_j) = 0, \quad j = \overline{1,n}, \tag{1.2}$$

where relatively spectral projectors P_j, $j = \overline{0,n}$, are determined by L-spectrum $\sigma^L(M)$ of the operator M; $0 < \tau_1 < \ldots < \tau_n < \tau$, $\tau \in \mathbf{R}_+$.

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In the context of our research, the consideration of determinate problem (1.1), (1.2) is propaedeutic in nature. Main attention will be paid to the stochastic problem

$$L \overset{\circ}{\eta} = M\eta + N\omega, \tag{1.3}$$

$$\lim_{t\to 0+} P_0(\eta(t) - \xi_0) = 0, \quad P_j(\eta(\tau_j) - \xi_j) = 0, \quad j = \overline{1,n}, \tag{1.4}$$

where $\omega = \omega(t)$ is specified and $\eta = \eta(t)$ are the desired stochastic processes, the operators L, M, P_j, $j = \overline{0,n}$, and the numbers $\tau_j \in \mathbf{R}_+$, $j = \overline{1,n}$, are the same as in (1.1), (1.2), ξ_j, $j = \overline{0,n}$, are pairwise independant (Gaussian) random variables, and the operator N will be defined below. The main difference between these results and the results of [3] is replacing the terms of the Showalter – Sidorov problem for more general conditions (1.4). Because both here and in [3] there is an approach to the study of stochastic processes, based on a new concept different from the approach Ito – Stratonovich – Skorokhod, the results of this article should be considered a continuation of studies [3], [4]. We will remind that the basis of this concept is the notion of the derivative of Nelson – Gliklikh of the stochastic process $\eta = \eta(t)$, which is denoted by $\overset{\circ}{\eta}$. The basis of this concept is established in [6], some directions of its development are in [3], [7].

Abstract results on the solvability of problem (1.3), (1.4) in the space of "noises" are used to study the solvability of a boundary value problem for the linear system of the Navier – Stokes equations with initial-final conditions and additive "white noise", as a derivative of Nelson – Gliklikh of Wiener K-process $\omega = \overset{\circ}{W}_K$. The stochastic problem is considered as a generalization of determinate case [4]. Pay attention to the reduction of this problem to problem (1.1), (1.2), which differs from the well-known approaches of O. A. Ladyzhenskaya [8] and R. Temam [9].

The article, in addition to the introduction and reference list contains three parts. In the first part we construct spaces of differentiable stochastic processes with values in separable Hilbert space. Moreover, the derivative is the derivative of Nelson – Gliklikh. We call stochastic processes with derivatives of Nelson – Gliklikh differentiable "noises" [3], [6], [7], [10]. The example of such "noise" in addition to the above mentioned "white noise" $\overset{\circ}{W}_K = \overset{\circ}{W}_K(t)$, $t \in \mathbf{R}_+$, is also "black noise" (i.e., "absolute" silence) a stochastic process, whose trajectories are a.s. (almost surely) equal to zero. In the second part of the paper we present results on the solvability of deterministic (1.1), (1.2) and stochastic (1.3), (1.4) problems provided a strong (L,p)-sectorial operator M, $p \in \{0\} \cup \mathbf{N}$, and one condition that guarantees the existence of relatively spectral projectors P_j, $j = \overline{0,n}$. These results generalize and develop abstract results of the work [3], [4]. The third part contains applications of the obtained abstract results. The list of references is not exhaustive and only reflects the tastes and preferences of the authors.

2. Space of a differentiable "noises"

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $\mathbf{R}$ is the set of real numbers, endowed with Boreal σ-algebra. The measurable mapping $\xi : \Omega \to \mathbf{R}$ is called *random variable*. The set of random variables with zero mathematical expectations and finite variances forms Hilbert space with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. This

Hilbert space will be denoted $\mathbf{L_2}$. Let's call $\xi \in \mathbf{L_2}$ *Gaussian values*, if they have a normal (Gaussian) distribution. Let $\mathcal{A}_0$ be σ-subalgebra of σ-algebra $\mathcal{A}$. Then we construct the space $\mathbf{L_2^0}$ of random variables, measurable relative to $\mathcal{A}_0$. It can be shown that $\mathbf{L_2^0}$ is a subspace of $\mathbf{L_2}$.

Let's introduce the designation of an orthoprojector $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$. If $\xi \in \mathbf{L_2}$, then $\Pi\xi$ is called *conditional mathematical expectation* of random variable ξ and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$. If $\mathcal{A}_0 = \{\emptyset, \Omega\}$, then $\mathbf{E}(\xi|\mathcal{A}_0) = \mathbf{E}\xi$; and if $\mathcal{A}_0 = \mathcal{A}$, then $\mathbf{E}(\xi|\mathcal{A}_0) = \xi$. Finally, we remind that the minimum σ-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, with respect to which the random variable ξ is measurable, is called *σ-algebra generated by ξ.*

Let further $\mathcal{I} \subset \mathbf{R}$ be some interval. Let us consider the mapping $f : \mathcal{I} \to \mathbf{L_2}$, which maps each $t \in \mathcal{I}$ to a random variable $\xi \in \mathbf{L_2}$ and the mapping $g : \mathbf{L_2} \times \Omega \to \mathbf{R}$, which maps each pair (ξ, ω) to the point $\xi(\omega) \in \mathbf{R}$. Let's call *a (one-dimensional)stochastic process* the mapping $\eta : \mathcal{I} \times \Omega \to \mathbf{R}$ having the form $\eta = \eta(t, \omega) = g(f(t), \omega)$.

Thus, for every fixed $t \in \mathcal{I}$ the stochastic process $\eta = \eta(t, \cdot)$ is a random variable, i.e. $\eta(t, \cdot) \in \mathbf{L_2}$, and for every fixed $\omega \in \Omega$ the stochastic process $\eta = \eta(\cdot, \omega)$ is called *a (selective) trajectory.* We will call the stochastic process η *continuous* if almost surely (a.s.) all its trajectories are continuous (i.e., when a. a. (almost all) $\omega \in \Omega$ trajectories $\eta(\cdot, \omega)$ are continuous). The set of continuous stochastic processes forms a Banach space which we denote as $\mathbf{CL_2}$. A continuous stochastic process, which (independent) random variables are Gaussian, is called *Gaussian.*

As an important example of a continuous Gaussian stochastic process we can present a (one-dimensional) Wiener process $\beta = \beta(t)$ simulating Brownian motion on a straight line in Einstein - Smolukhovsky theory. Let us formulate its properties.

(W1) a.s. $\beta(0) = 0$, a.s. all its trajectories $\beta(t)$ are continuous and for all $t \in \overline{\mathbf{R}}_+ (= \{0\} \cup \mathbf{R}_+)$ the random variable $\beta(t)$ is Gaussian;

(W2) the mathematical expectation $\mathbf{E}(\beta(t)) = 0$ and the autocorrelation function $\mathbf{E}\left((\beta(t) - \beta(s))^2\right) = |t - s|$ for all $s, t \in \overline{\mathbf{R}}_+$;

(W3) trajectories $\beta(t)$ are non-differentiable at any point of $t \in \overline{\mathbf{R}}_+$ and for any arbitrarily small interval have unlimited variation.

Further, we will call the stochastic process β satisfying the properties (W1) – (W3), *Brownian motion.*

Theorem 2.1. *With probability 1 there is only one Brownian motion β, and it can be represented in the form*

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t,$$

where ξ_k are pairwise independent Gaussian random variables, $\mathbf{E}\xi_k = 0$, $\mathbf{D}\xi_k = 4[\pi(2k+1)]^{-2}$.

Now we fix $\eta \in \mathbf{CL_2}$ and $t \in \mathcal{I} (= (\varepsilon, \tau) \subset \mathbf{R})$ and using $\mathcal{N}_t^\eta$ we denote σ-algebra generated by a random variable $\eta(t)$. We will redenote, for the sake of brevity, $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition 2.2. Let $\eta \in \mathbf{CL_2}$, and the random variable

$$D\eta\,(t,\cdot) = \lim_{\triangle t \to 0+} E_t^{\eta}\left(\frac{\eta\,(t+\triangle t,\cdot) - \eta(t,\cdot)}{\triangle t}\right)$$

$$\left(D_*\eta\,(t,\cdot) = \lim_{\triangle t \to 0-} E_t^{\eta}\left(\frac{\eta\,(t,\cdot) - \eta\,(t-\triangle t,\cdot)}{\triangle t}\right)\right),$$

is called a *forward* $D\eta(t,\cdot)$ *(a backward* $D_*\eta(t,\cdot)$*) mean derivative of the stochastic process* η *at the point* $t \in (\varepsilon,\tau)$ if the limit exists in the sense of uniform metric on $\mathbf{R}$. The stochastic process η is called *forward (backward) mean differentiable at* $t \in (\varepsilon,\tau)$, if for every point $t \in (\varepsilon,\tau)$ there exists the forward (backward) mean derivative.

Now, let the stochastic process $\eta \in \mathbf{CL_2}$ be forward (backward) mean differentiable on (ε,τ). Its forward (backward) mean derivative will also be a stochastic process, which we denote by the symbol $D\eta$ $(D_*\eta)$. If the stochastic process $\eta \in \mathbf{CL_2}$ is forward (backward) mean differentiable on (ε,τ), then we can define *the symmetric mean derivative* $D_S\eta = \frac{1}{2}\,(D + D_*)\,\eta$ *(the antisymmetric mean derivative* $D_A\eta = \frac{1}{2}\,(D_* - D)\,\eta$.) We note that the mean derivatives were introduced by E. Nelson, [11] and the theory of these derivatives was developed by Y. E. Gliklikh [12], so we shall call the symmetric mean derivative D_S of the stochastic process η a *Nelson – Gliklikh derivative* (denoted by $\overset{\circ}{\eta}$).

Using $\overset{\circ}{\eta}{}^{(l)}$ we denote the l-th derivative of Nelson – Gliklikh of a stochastic process η, $l \in \mathbf{N}$, and using $\mathbf{C}^l\mathbf{L}_2$ we denote space of stochastic processes having continuous derivatives of Nelson – Gliklikh to about $l \in \mathbf{N}$, inclusive.

Exactly $\mathbf{C}^l\mathbf{L}_2$, $l \in \mathbf{N}$, are called in [6] *spaces of differentiable "noises"* . Note that if the trajectories of a stochastic process η a.s. are continuously differentiable in the "usual sense" on (ε,τ), then their derivative of Nelson – Gliklikh coincides with the "usual" derivative.

Such, for example, is the case with a stochastic process $\eta = \alpha\sin(\beta t)$, where α is a Gaussian random variable, $\beta \in \mathbf{R}_+$ is a fixed constant, and $t \in \mathbf{R}$ has the physical meaning of time.

It is easy to show that $(\alpha\eta \overset{\circ}{+} \beta\zeta) = \alpha\ \overset{\circ}{\eta}\ + \beta\ \overset{\circ}{\zeta}$ for all $\alpha,\ \beta\ \in\ \mathbf{R}$ and $(\overset{\circ}{\eta\zeta}) = \overset{\circ}{\eta}\,\zeta + \eta\,\overset{\circ}{\zeta}$ for any $\eta,\ \zeta \in \mathbf{C}^1\mathbf{L}_2$.

Theorem 2.3. (Y.E. Gliklikh, [13]) $\overset{\circ}{\beta}\,(t) = (2t)^{-1}\beta(t)$ *for all* $t \in \mathbf{R}_+$.

Let's consider $\mathcal{U} \equiv (\mathcal{U}, \langle\cdot,\cdot\rangle)$ is a real separable Hilbert space; we consider the operator $K \in \mathcal{L}(\mathcal{U})$, whose spectrum $\sigma(K)$ is non-negative, discrete finitely multiple and thickens only to zero. We denote using $\{\lambda_j\}$ a sequence of eigenvalues of the operator K, numbered in non-increasing order based on their multiplicity. Note that the linear span of the set $\{\varphi_j\}$ of the corresponding orthonormal eigenfunctions of the operator K is dense in $\mathcal{U}$. We also suppose that the operator K is nuclear (i.e., its trace is $\operatorname{Tr} K = \sum\limits_{j=1}^{\infty} \lambda_j < +\infty$).

Let's take the sequence of independent stochastic processes $\{\eta_j\}$ and define a stochastic K-process

$$\Theta_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\eta_j(t)\varphi_j \tag{2.1}$$

provided that the series (2.1) converges uniformly on any compact of $\mathcal{I}$. Note that if $\{\eta_j\} \subset \mathbf{CL}_2$, then the existence of stochastic K-process Θ_K means a.s. the continuity of its trajectories. Now we introduce derivatives of Nelson – Gliklikh of a stochastic K-process

$$\overset{\circ}{\Theta}{}_K^{(l)}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \overset{\circ}{\eta}{}_j^{(l)}(t)\varphi_j \tag{2.2}$$

provided that derivatives in the right part of (2.2) to l inclusive exist, and all series uniformly converge on any compact of $\mathcal{I}$. Let's consider the space $\mathbf{C}_K\mathbf{L}_2$ of stochastic K-processes whose trajectories are a.s. continuous, and the spaces $\mathbf{C}^l_K\mathbf{L}_2$ of stochastic K-processes whose trajectories are a.s. continuously differentiable by Nelson – Gliklikh to order $l \in \mathbf{N}$, inclusive.

As an example, let's consider the Wiener K-process

$$W_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)\varphi_j, \tag{2.3}$$

which obviously exists on $\overline{\mathbf{R}}_+$. Moreover, the following is also true:

Corollary 2.4. $\overset{\circ}{W}_K(t) = (2t)^{-1}W_K(t)$ *for all* $t \in \mathbf{R}_+$ *and nuclear operators* $K \in \mathcal{L}(\mathcal{U})$.

In addition, Wiener K-process (2.3) satisfies the conditions (W1) – (W3), if the symbol β is replaced by W_K. If this substitution is made, then the following is true:

Theorem 2.5. *With any nuclear operator $K \in \mathcal{L}(\mathcal{U})$ with probability 1 there is a unique Wiener K-process, and it can be represented in the form (2.3).*

3. Multipoint initial-final conditions

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$ (i.e. linear and continuous), and the operator $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$ (i.e. a linear, closed and densely defined). Consider the L-resolvent set $\rho^L(M) = \{\mu \in \mathbf{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\}$ and the L-spectrum $\sigma^L(M) = \mathbf{C} \setminus \rho^L(M)$ of the operator M. Let $\rho^L(M) \neq \emptyset$, then we can consider *right* and *left*

$$R^L_{(\mu,p)}(M) = \prod_{k=0}^{p} R^L_{\mu_k}(M) \quad \text{and} \quad L^L_{(\mu,p)}(M) = \prod_{k=1}^{L} p^L_{\mu_k}(M)$$

(L,p)-*resolvents* of the operator M. Here $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$, and points $\mu_k \in \rho^L(M)$, $k = \overline{0,p}$.

Definition 3.1. ([2], chapter 3) Operator M is called *p-sectorial relatively* of operator L with the number $p \in \{0\} \cup \mathbf{N}$ (in short, (L,p)-*sectorial*), if there exist constants $K \in \mathbf{R}_+$, $a \in \mathbf{R}$, $\Theta \in (\pi/2, \pi)$ such, that the sector

$$S^L_{a,\Theta}(M) = \{\mu \in \mathbf{C} : |\arg(\mu - a)| < \Theta,\ \mu \neq a\}, \quad S^L_{a,\Theta}(M) \subset \rho^L(M),$$

and

$$\max\left\{ \left\| R^L_{(\mu,p)}(M) \right\|_{\mathcal{L}(\mathcal{U})}, \left\| L^L_{(\mu,p)}(M) \right\|_{\mathcal{L}(\mathcal{F})} \right\} \leq \frac{K}{\prod\limits_{k=0}^{p} |\mu_k - a|} \qquad (*)$$

for all $\mu_k \in S^L_{a,\Theta}(M)$, $k = \overline{0,p}$.

Remark 3.2. It is clear that if inequality (*) is executed when any $p \in \{0\} \cup \mathbf{N}$, then it will be executed and if $q \in \mathbf{N}$ such that $q > p$. In the proof this fact does not matter, and in applications we take the smallest p for which (*) is executed.

Lemma 3.3. *Let operator M be (L,p)-sectorial. Then in the sector $\Sigma = \{\tau \in \mathbf{C} : |\arg\tau| < \Theta - \pi/2,\ \tau \neq 0\}$, where Θ is taken from definition 3.1, there exists an analytic and uniformly bounded resolving semigroup $\{U^t : t > 0\}$ ($\{F^t : t > 0\}$) of the equation (1.1), $f \equiv 0$, and it is represented by Dunford – Taylor type integrals*

$$U^t = \frac{1}{2\pi i} \int\limits_{\Gamma} R^L_\mu(M) e^{\mu t} d\mu \qquad \left(F^t = \frac{1}{2\pi i} \int\limits_{\Gamma} L^L_\mu(M) e^{\mu t} d\mu \right),$$

where $t \in \mathbf{R}_+$, countour $\Gamma \subset S^L_{a,\Theta}(M)$ is such that $|\arg\mu| \to \Theta$ with $\mu \to \infty$, $\mu \in \Gamma$.

Lemma 3.4. *Let operator M be (L,p)-sectorial. Then $\lim\limits_{t\to 0+} U^t u = u$ for any $u \in \mathrm{im} R^L_{(\mu,p)}(M)$ and $\lim\limits_{t\to 0+} F^t f = f$ for any $f \in \mathrm{im} L^L_{(\mu,p)}(M)$).*

Consider kernels $\ker U^{\cdot} = \mathcal{U}^0$, $\ker F^{\cdot} = \mathcal{F}^0$ and images $\mathrm{im} U^{\cdot} = \mathcal{U}^1$, $\mathrm{im} F^{\cdot} = \mathcal{F}^1$ of these semigroups.

It is easy to show that $\overline{\mathcal{U}^0 \oplus \mathcal{U}^1} = \overline{\mathcal{U}^0} \oplus \overline{\mathcal{U}^1} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\overline{\mathcal{F}^0 \oplus \mathcal{F}^1} = \overline{\mathcal{F}^0} \oplus \overline{\mathcal{F}^1} = \mathcal{F}^0 \oplus \mathcal{F}^1$.

We need a stronger statement

$$\mathcal{U}^0 \oplus \mathcal{U}^1 = \mathcal{U} \quad (\mathcal{F}^0 \oplus \mathcal{F}^1 = \mathcal{F}), \qquad (A1)$$

which is fulfilled either in the case when the operator M is strongly (L,p)-sectorial on the right (left), $p \in \{0\} \cup \mathbf{N}$, or when the space $\mathcal{U}$ ($\mathcal{F}$) is reflexive [3].

We denote by L_k (M_k)the restriction of the operator L (M) on $\mathcal{U}^k$ ($\mathrm{dom} M \cap \mathcal{U}^k$), $k = 0, 1$.

Lemma 3.5. *Let operator M be (L,p)-sectorial. Then*

(i) $L_0 \in \mathcal{L}(\mathcal{U}^0; \mathcal{F}^0)$, $M_0 \in \mathcal{C}l(\mathcal{U}^0; \mathcal{F}^0)$, and there exists the operator $M_0^{-1} \in \mathcal{L}(\mathcal{F}^0; \mathcal{U}^0)$,

(ii) operators $L_1 \in \mathcal{L}(\mathcal{U}^1; \mathcal{F}^1)$, $M_1 \in \mathcal{C}l(\mathcal{U}^1; \mathcal{F}^1)$.

And if the operator M is strongly (L,p)-sectorial on the right and on the left, $p \in \{0\} \cup \mathbf{N}$, then $L_k \in \mathcal{L}(\mathcal{U}^k; \mathcal{F}^k)$, $M_k \in \mathcal{C}l(\mathcal{U}^k; \mathcal{F}^k)$, $k = 0, 1$, and there exists the operator $M_0^{-1} \in \mathcal{L}(\mathcal{F}^0; \mathcal{U}^0)$, and also the projector $P = s - \lim\limits_{t\to 0+} U^t$

($Q = s - \lim_{t\to 0+} F^t$) splitting the space $\mathcal{U}$ ($\mathcal{F}$) according (A1), and $\mathcal{U}^1 = \mathrm{im}P$ ($\mathcal{F}^1 = \mathrm{im}Q$).

Let us introduce one more condition

$$\text{there exists the operator} \quad L_1^{-1} \in \mathcal{L}(\mathcal{F}^1;\mathcal{U}^1), \tag{A2}$$

which occurs in the case when the operator M is strongly (L,p)-sectorial , $p \in \{0\}\cup\mathbf{N}$. (Previously, it was shown that (A1) together with the condition of (L,p)-sectoriality of the operator M, $p \in \{0\}\cup\mathbf{N}$, gives a strongly (L,p)-sectoriality of the operator M on the right (left), $p \in \{0\}\cup\mathbf{N}$, and if we add the condition (A2), we obtain the strongly (L,p)-sectoriality of the operator M, $p \in \{0\}\cup\mathbf{N}$).

Finally, we introduce another important condition on the L-spectrum of the operator M [5] is following form

$$\begin{cases} \sigma^L(M) = \bigcup\limits_{j=0}^{n} \sigma_j^L(M),\ n \in \mathbf{N},\ \text{and}\ \sigma_j^L(M) \neq \emptyset \text{ is contained in bounded} \\ \text{domain } D_j \subset \mathbf{C} \text{ with piecewise smooth boundary } \partial D_j = \Gamma_j \subset \mathbf{C}. \text{ Also,} \\ \overline{D_j} \cap \sigma_0^L(M) = \emptyset \text{ and } \overline{D_k} \cap \overline{D_l} = \emptyset \text{ for all } j,k,l = \overline{1,n}, k \neq l. \end{cases} \tag{A3}$$

We construct relatively spectral projectors [5]

$$\begin{aligned} P_j &= \frac{1}{2\pi i}\int_{\Gamma_j} R_\mu^L(M)d\mu \in \mathcal{L}(\mathcal{U}), \\ Q_j &= \frac{1}{2\pi i}\int_{\Gamma_j} L_\mu^L(M)d\mu \in \mathcal{L}(\mathcal{F}),\ j = \overline{1,n}. \end{aligned} \tag{3.1}$$

and it turns out that when the operator M is strongly (L,p)-sectorial, then $P_jP = PP_j = P_j$ and $Q_jQ = QQ_j = Q_j$, $j = \overline{1,n}$. So, in this case, there is a projector $P_0 = P - \sum\limits_{j=1}^{n} P_j$, $P_0 \in \mathcal{L}(\mathcal{U})$. So, let the conditions (A1) – (A3) be fulfilled. We fix $\tau_j \in \mathbf{R}_+$ ($\tau_j < \tau_{j+1}$), $u_j \in \mathcal{U}$, $j = \overline{0,n}$, and consider a multipoint initial-final condition (1.2) for a linear Sobolev type equation (1.1). Vector function $u \in C^1((0,\tau);\mathcal{U}) \cap C([0,\tau];\mathcal{U})$ satisfying the equation (1.1), is called its *solution*; the solution $u = u(t)$ of equation (1.1) we call *the solution of the multipoint initial-final value problem* (1.1), (1.2), if the condition (1.2)is fulfilled.

Lemma 3.6. *Let the operator M be (L,p)-sectorial, and conditions (A1) – (A3) are fulfilled. Then $U^t = \sum\limits_{j=0}^{n} P_jU^t = \sum\limits_{j=0}^{n} U_j^t$, $F^t = \sum\limits_{j=0}^{n} Q_jF^t = \sum\limits_{j=0}^{n} F_j^t$, and U_j^t and F_j^t can be represented in the form*

$$\begin{aligned} U_j^t &= \frac{1}{2\pi i}\int_{\Gamma_j} (\mu L - M)^{-1}Le^{\mu t}d\mu, \\ F_j^t &= \frac{1}{2\pi i}\int_{\Gamma_j} L(\mu L - M)^{-1}e^{\mu t}d\mu,\ j = \overline{1,n}. \end{aligned} \tag{3.2}$$

So we consider im $P_j = \mathcal{U}^{1j}$, im $Q_j = \mathcal{F}^{1j}$, $j = \overline{0,n}$. By construction $\mathcal{U}^1 = \bigoplus_{j=0}^{n} \mathcal{U}^{1j}$ and $\mathcal{F}^1 = \bigoplus_{j=0}^{n} \mathcal{F}^{1j}$. We denote by L_j (M_j) the restriction of the operator L (M) on $\mathcal{U}^{1j}$ $(\text{dom} M \cap \mathcal{U}^{1j})$, $j = \overline{0,n}$. It is easy to show that the operators $L_j \in \mathcal{L}(\mathcal{U}^{1j};\mathcal{F}^{1j})$, $M_j \in \mathcal{C}l(\mathcal{U}^{1j};\mathcal{F}^{1j})$, $j = \overline{0,n}$, moreover, due to (A2) there exists the operator $L_j^{-1} \in \mathcal{L}(\mathcal{F}^{1j};\mathcal{U}^{1j})$, $j = \overline{0,n}$. It is also easy to show that the operator $S_0 = L_0^{-1}M_0 \in \mathcal{C}l(\mathcal{U}_0)$ is sectorial, and the operator $S_j = L_j^{-1}M_j : \mathcal{U}^{1j} \to \mathcal{U}^{1j}$, $j = \overline{1,n}$ is bounded.

Now we are ready to prove the unique solvability of the problem (1.2) for the equation (1.1), which due to (L,p)-sectoriality of the operator M and conditions (A1) – (A3), is reduced to the form

$$G\dot{u}^0 = u^0 + M_0^{-1} f^0, \tag{3.3}$$

$$\dot{u}^{1j} = S_j u^{1j} + L_{1j}^{-1} f^{1j},\ j = \overline{0,n} \tag{3.4}$$

where $f^0 = (\mathbf{I} - Q)f$, $f^{1j} = Q_j f$, $u^0 = (\mathbf{I} - P)u$, $u^{1j} = P_j u$, $j = \overline{0,n}$, operator $G = M_0^{-1}L_0 \in \mathcal{L}(\mathcal{U}^0)$.

Theorem 3.7. *Let the operator M be (L,p)-sectorial, $p \in \{0\} \cup \mathbf{N}$, moreover, the conditions (3.4), (A1) – (A3) are fulfilled. Then for any vector-function $f^0 \in C([0,\tau];\mathcal{F}^0) \cap C^{p+1}((0,\tau);\mathcal{F}^0)$, $f^1 \in C([0,\tau];\mathcal{F}^1)$ and for all $u_j \in \mathcal{U}$, $j = \overline{0,n}$ there exists the unique solution of the problem (1.1), (1.2), which also has the form*

$$u(t) = -\sum_{q=0}^{p} G^q M_0^{-1} f^{0(q)}(t) + \sum_{j=0}^{n} \left(U_j^{t-\tau_j} u_j + \int_{\tau_j}^{t} U_j^{t-s} L_{1j}^{-1} Q_j f(s) ds \right). \tag{3.5}$$

The proof of the existence is carried out by substitution of (3.5) in (1.2), (3.3), (3.4). The uniqueness is proved as usual (see e.g. [2], Ch. 3; [5]). Let us consider the stochastic problem (1.3), (1.4).

Now suppose that $\mathcal{U}$ is a real separable Hilbert space. Let the operator $K \in \mathcal{L}(\mathcal{U})$ be nuclear, and construct *random K-variables*

$$\xi_j = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \xi_{jk} \phi_k, \quad j = \overline{0,n}, \tag{3.6}$$

where $\{\xi_{jk}\} \subset \mathbf{L}^2$ is a sequence of pairwise independent random variables such that $\mathbf{D}\xi_{jk} \leq C_j, k \in \mathbf{N}$; $\{\lambda_k\}$ $(\{\phi_k\})$ is the sequence of eigenvalues (vectors) of the operator K. Stochastic K-process

$$\eta = \eta(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \eta_k(t) \phi_k,$$

where $\{\eta_k(t)\} \subset \mathbf{C}^1\mathbf{L}^2$ is a sequence of differentiable "noises" will be called *the solution of equation (1.3)* if the a. e. of its trajectory. a.s. satisfy (1.3) for a stochastic K- process $\omega = \omega(t)$ on the interval $(0,\tau)$.The solution $\eta = \eta(t)$ of equation (1.3) will be called *the solution of the problem (1.3), (1.4)*, if in addition it satisfies the initial-final conditions (1.4), where the random K-variables ξ_j, $j = \overline{0,n}$ are of the form (3.6).

Theorem 3.8. *Let*

(i) $\mathcal{U}$ is a real separable Hilbert space;

(ii) operators $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$, and operator M is (L,p)-sectorial, $p \in \{0\} \cup \mathbf{N}$;

(iii) the conditions (A1) – (A3) are satisfied.

Then for any operator $N \in \mathcal{L}(\mathcal{U};\mathcal{F})$, nuclear operator $K \in \mathcal{L}(\mathcal{U})$, random K-variables ξ_j, $j = \overline{0,n}$ of form (3.6), and stochastic K-process $\Theta_K = \Theta_K(t)$, $t \in [0,\tau)$, such that $(\mathbf{I}-Q)N\Theta_K \in \mathbf{CL}_2([0,\tau);\mathcal{F}^0) \cap \mathbf{C}^{p+1}\mathbf{L}_2((0,\tau);\mathcal{F}^0)$, and $QNQ_k \in \mathbf{CL}_2([0,\tau);\mathcal{F}^1)$, there exists the unique solution $\eta = \eta(t)$ of the problem (1.3), (1.4), which also has the form

$$\begin{aligned}\eta(t) = &-\sum_{q=0}^{p} G^q M_0^{-1}(\mathbf{I}-Q)N \overset{\circ}{\Theta}{}_K^{(q)}(t) + \\ &+\sum_{j=0}^{n}\left(U_j^{t-\tau_j}\xi_j + \int_{\tau_j}^{t} U_j^{t-s} L_{1j}^{-1} Q_j N \Theta_K(s)ds\right).\end{aligned} \tag{3.7}$$

Corollary 3.9. *Let the conditions of theorem 3.8 are fulfilled. Then for any operator N, nuclear operator $K \in \mathcal{L}(\mathcal{U})$, random K-variables ξ_j, $j = \overline{0,n}$, of form (3.6), there exists the unique solution $\eta = \eta(t)$ of the problem (1.3), (1.4), where $\omega = \overset{\circ}{W}_K$ is white "noise" , and moreover, it has the form*

$$\begin{aligned}\eta(t) = &-\sum_{q=0}^{p} G^q M_0^{-1}(\mathbf{I}-Q)N \overset{\circ}{W}{}_K^{(q+1)}(t) + \\ &+\sum_{j=0}^{n}\left(U_j^{t-\tau_j}\xi_j + \int_{\tau_j}^{t} U_j^{t-s} L_{1j}^{-1} Q_j N \overset{\circ}{W}_K(s)ds\right).\end{aligned} \tag{3.8}$$

Proof of the last two statements is not fundamentally different from the proof of the theorem 3.7. We will note only two things. Firstly, the first term (the subtrahend) in the formula (3.8) satisfies (1.4) despite the corollary 2.4, as

$$\sum_{q=0}^{p} G^q M_0^{-1}(\mathbf{I}-Q)N \overset{\circ}{W}{}_K^{(q+1)}(t) \in \ker P_0$$

for all $t \in (0,\tau)$. Hence the first of the conditions (1.4) is fulfilled, the others are obvious. Secondly, for any $\varepsilon \in (0,\tau)$ we have

$$\begin{aligned}\int_{\varepsilon}^{t} U_0^{t-s} L_{10}^{-1} Q_0 N \overset{\circ}{W}_K(s)ds = L_{10}^{-1}Q_0 N W_K(t) - \\ -U_0^{t-\varepsilon} L_{10}^{-1} Q_0 N W_K(\varepsilon) - \int_{\varepsilon}^{t} L_{10}^{-1} M_{10} U_0^{t-s} L_{10}^{-1} Q_0 N W_K(s)ds.\end{aligned}$$

Hence, by (W1) – (W3) there are

$$\int_{0}^{t} U_0^{t-s} L_{10}^{-1} Q_0 N \overset{\circ}{W}_K(s)ds = L_{10}^{-1}Q_0 N W_K(t) - \int_{0}^{t} L_{10}^{-1} M_{10} U_0^{t-s} L_{10}^{-1} N W_K(s)ds.$$

The existence of other integral terms in equation (3.8) is doubtless.

Remark 3.10. Since the solutions of Sobolev type equations (1.1) and (1.3) are received not only by integration, but differentiation of the right side , then the use of the traditional concept of white noise is hardly possible. However, in some cases, [14], [15] it is possible to use the approach of Ito – Stratonovich – Skorokhod.

4. Linear Navier – Stokes system

Let $\Omega \subset \mathbf{R}^d$, $d \in \mathbf{N} \setminus \{1\}$ be a bounded domain with the boundary $\partial\Omega$ of the class C^∞. In the cylinder $\Omega \times \mathbf{R}_+$ we consider the Dirichlet problem

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbf{R}_+ \tag{4.1}$$

for the system of equations

$$v_t = \nu\nabla^2 v - p + f, \quad \nabla(\nabla \cdot v) = 0. \tag{4.2}$$

Here vector-functions $v = \mathrm{col}(v_1, v_2, \ldots, v_d)$, $v_k = v_k(x,t)$, $p = \mathrm{col}(p_1, p_2, \ldots, p_d)$, $p_k = p_k(x,t)$, and $f = \mathrm{col}(f_1, f_2, \ldots, f_d)$, $f_k = f_k(x,t)$ respond to speed, the pressure gradient and external load of a viscous incompressible fluid in the linear approximation, respectively, $(x,t) \in \Omega \times \mathbf{R}_+$, $k = \overline{1,d}$. Note that the system of equations (4.2) can be obtained from the linear system of Navier – Stokes (see e.g. [9], p. 8, or [16])

$$v_t = \nu\nabla^2 v - \nabla p + f, \quad \varepsilon p_t = \nabla \cdot v$$

for "weak" compressible fluid after taking the gradient of both parts of the last equation, replacing $\nabla p \to p$ and aspiration $\varepsilon \to 0$.

We reduce the boundary value problem (4.1) for the system of equations (4.2) to the abstract equation (1.1). Consider $\mathbf{H}^2_\sigma$ and $\mathbf{H}^2_\pi$ ($\mathbf{H}_\sigma$ and $\mathbf{H}_\pi$) are subspaces of solenoidal and potential vector functions of the space $\mathbf{H}^2 = (W^2_2(\Omega) \cap \overset{\circ}{W}{}^1_2(\Omega))^d$ ($\mathbf{L}^2 = (L^2(\Omega))^d$). It is obvious that the attachments $\mathbf{H}^2_{\sigma(\pi)} \hookrightarrow \mathbf{H}_{\sigma(\pi)}$ are continuous (and even compact). We will show that $\mathbf{H}_\sigma \perp \mathbf{H}_\pi$. Indeed, let $\mathcal{S}^d_{\sigma(\pi)}$ be lineal of solenoidal (potential) vector functions whose components are infinitely differentiable in Ω functions with compact support. Let the vector-functions $\varphi \in \mathcal{S}^d_\sigma$ and $\psi \in \mathcal{S}^d_\pi$; then $\psi = \nabla\xi$, where ξ is some infinitely differentiable in Ω function with compact support, and

$$[\varphi, \psi]_{\mathbf{L}^2} = -(\nabla \cdot \varphi, \xi)_{L^2(\Omega)} = 0. \tag{4.3}$$

The equality (4.3) is true for all $\varphi \in \mathbf{H}_\sigma$ and $\psi \in \mathbf{H}_\pi$ due to the density of imbedding $\mathcal{S}^d_{\sigma(\pi)} \subset \mathbf{H}^2_{\sigma(\pi)}$. We will denote by $\Sigma : \mathbf{L}^2 \to \mathbf{H}_\sigma$ the projector along $\mathbf{H}_\pi$. Therefore, Σ, $\Pi \in \mathcal{L}(\mathbf{L}^2)$, where $\Pi = \mathbf{I} - \Sigma$ are orthoprojectors. Next, by a square matrix of order d we will define a closed linear densely defined operator $A = \mathrm{diag}\,\{\nabla^2, \nabla^2, \ldots, \nabla^2\}$ with domain of definition $\mathrm{dom}A = \mathbf{H}^2$. We will denote by $A_{\sigma(\pi)}$ the restriction of the operator A on $\mathbf{H}^2_{\sigma(\pi)}$. There are

Lemma 4.1. *(i) The spectrum $\sigma(A)$ of the operator A is negative discrete finitely multiple and thickens just to the point $-\infty$, and $\sigma(A) = \sigma(A_\sigma) = \sigma(A_\pi)$.*

(ii) The operator $A_{\sigma(\pi)} \in \mathcal{C}l(\mathbf{H}_{\sigma(\pi)})$ (i.e. linear, closed and densely defined), $\mathrm{dom}(A_{\sigma(\pi)}) = \mathbf{H}^2_{\sigma(\pi)}$, *and $A = A_\sigma\Sigma + A_\pi\Pi$.*

The proof of this result is based on the Cattabriga – Solonnikov – Vorovich – Yudovich theorem (see for example [17])

Lemma 4.2. ([18]). *Formula* $B : u \to \nabla(\nabla \cdot u)$ *sets the operator* $B \in \mathcal{L}(\mathbf{H}^2, \mathbf{H}_\pi)$, *and* $\ker B = \mathbf{H}^2_\sigma$.

Let $\mathcal{U} = \mathcal{F} \equiv \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \mathbf{H}_p$, $\mathbf{H}_p = \mathbf{H}_\pi$. Vector functions $u = u(t)$ and $f = f(t)$ are $u = \operatorname{col}(u_\sigma, u_\pi, u_p)$, and $f = \operatorname{col}(f_\sigma, f_\pi, 0)$, respectively. Formulas

$$L = \begin{pmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad M = \begin{pmatrix} \nu A_\sigma & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \nu A_\pi & -\mathbf{I} \\ \mathbf{O} & B & \mathbf{O} \end{pmatrix}$$

set operators $L \in \mathcal{L}(\mathcal{U}; \mathcal{F})$, $\operatorname{im} L = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \{0\}$, $\ker L = \{0\} \times \{0\} \times \mathbf{H}_p$, and $M \in \mathcal{C}l(\mathcal{U}; \mathcal{F})$, dom $M = \mathbf{H}^2_\sigma \times \mathbf{H}^2_\pi \times \mathbf{H}_p$. Thus, the reduction of the problem (4.1), (4.2) to the equation (1.1) is complete.

Lemma 4.3. *For any* $\nu \in \mathbf{R}_+$ *the operator* M *is strongly* $(L, 1)$*-sectorial.*

Let us sketch the proof, which is in itself verification of the requirements of the definition 2.2 and the conditions (A1) and (A2). We denote by $\{\lambda_k\}$ a sequence of eigenvalues of the operator A indexed in non-increasing order and taking into account their multiplicity. Without loss of generality we can identify $\sigma(A) = \{\lambda_k\}$. Then L-spectrum $\sigma^L(M)$ of the operator M can be identified with the sequence $\{\nu\lambda_k\}$, i.e. we believe $\sigma^L(M) = \{\nu\lambda_k\}$. Indeed, L-resolvent of the operator M

$$(\mu L - M)^{-1} = \begin{pmatrix} (\mu\mathbf{I} - \nu A_\sigma)^{-1} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & -B_\pi^{-1} \\ \mathbf{O} & \mathbf{I} & -(\mu\mathbf{I} - \nu A_\pi)B_\pi^{-1} \end{pmatrix}$$

exists and is continuous (even holomorphic) for all $\mu \in \rho^L(M)$. Here B_π is the restriction of the operator B on $\mathbf{H}^2_\pi$. By Lemma 4.2 the operator $B_\pi : \mathbf{H}^2_\pi \to \mathbf{H}_\pi$ is a top-linear isomorphism). Here right and left

$$R^L_\mu(M) = \begin{pmatrix} (\mu\mathbf{I} - \nu A_\sigma)^{-1} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \end{pmatrix}, \quad L^L_\mu(M) = \begin{pmatrix} (\mu\mathbf{I} - \nu A_\sigma)^{-1} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}$$

L-resolvents of the operator M, respectively. So, by definition 2.2 and sectoriality of the operator A_σ the operator M is $(L, 1)$-sectorial.

We will construct subspaces $\mathcal{U}^0 = \mathcal{F}^0 = \{0\} \times \mathbf{H}_\pi \times \mathbf{H}_p$ $\mathcal{U}^1 = \mathcal{F}^1 = \mathbf{H}_\sigma \times \{0\} \times \{0\}$. The fulfilment of conditions (A1) and (A2) is obvious, and

$$M_0^{-1}(\mathbf{I} - Q) = \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & B_\pi^{-1} \\ \mathbf{O} & -\mathbf{I} & \nu A_\pi B_\pi^{-1} \end{pmatrix}.$$

Also it is easy to check that

$$M_0^{-1}L_0(\mathbf{I}-P)=\begin{pmatrix}\mathbf{O} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & -\mathbf{I} & \mathbf{O}\end{pmatrix}$$

is a nilpotent operator of degree 1.

So the validity of Lemma 4.3 is established. Let us now consider the conditions (A3). As shown above, L-spectrum of the operator M has the form $\sigma^L(M)=\{\mu_k\in\mathbf{C}:\mu_k=\nu^{-1}\lambda_k, k\in\mathbf{N}\}$, where $\{\lambda_k\}$ is spectrum of the operator A, $\nu\in\mathbf{R}_+$. Therefore, we can choose finite subsets $\sigma_j^L(M)\subset\sigma^L(M)$, and construct groups of operators $U_j^t = \operatorname{diag}\left\{\sum\limits_{\mu_k\in\sigma_j^L(M)}\exp(\mu_k t)\langle\cdot,\varphi_k\rangle_\sigma\varphi_k,\mathbf{O},\mathbf{O}\right\}$, $j=\overline{1,n}$. Here $\langle\cdot,\cdot\rangle_\sigma$ is the scalar product in $\mathbf{H}_\sigma$ ($\langle\cdot,\cdot\rangle_\sigma=[\cdot,\cdot]_{\mathbf{L_2}}$); $\{\varphi_k\}$ is the orthonormal sequence of eigenvectors of the operator A_σ, corresponding eigenvalues $\{\lambda_k\}$. By the way, if we set $\sigma_0^L(M)=\sigma^L(M)\setminus\left(\bigcup\limits_{j=1}^n\sigma_j^L(M)\right)$, then $U_0^t=\operatorname{diag}\left\{\sum\limits_{\mu_k\in\sigma_0^L(M)}\exp(\mu_k t)\langle\cdot,\varphi_k\rangle_\sigma\varphi_k,\mathbf{O},\mathbf{O}\right\}$. Thus, from theorem 3.7 and lemmas 4.1 – 4.3 follows

Theorem 4.4. *For any* ν, $\tau\in\mathbf{R}_+$, $0<\tau_1<\ldots<\tau_n<\tau$, $f_\sigma\in C^0([0,\tau];\mathbf{H}_\sigma)$, $f_\pi\in C^0([0,\tau];\mathcal{F}_\pi)\cap C^1((0,\tau);\mathbf{H}_\pi)$ *and* $u_{\sigma j}\in\mathbf{H}_\sigma$, $j=\overline{0,n}$, $\tau\in\mathbf{R}_+$, *there exists the unique solution of the problem*

$$\lim_{t\to 0+}\sum_{\mu_k\in\sigma_0^L(M)}\langle u_\sigma(t)-u_{\sigma 0},\varphi_k\rangle\varphi_k=0,\qquad \sum_{\mu_k\in\sigma_j^L(M)}\langle u_\sigma(\tau_j)-u_{\sigma j},\varphi_k\rangle\varphi_k=0,$$

$j=\overline{1,n}$, *which also has the form*

$$u_\sigma(t)=\sum_{j=0}^n\left(\sum_{\mu_k\in\sigma_j^L(M)}\exp(\mu_k t)\langle u_{\sigma j},\varphi_k\rangle\varphi_k+\right.$$
$$\left.+\int_{\tau_j}^t\sum_{\mu_k\in\sigma_j^L(M)}\exp(\mu_k(t-s))\langle f_\sigma(s),\varphi_k\rangle\varphi_k ds\right),\ j=\overline{0,n},\ \tau_0=0;$$
$$u_\pi\equiv 0;\quad u_p=f_\pi(t),\quad t\in(0,\tau).$$

Let us now consider the problem (1.3), (1.4) in the context of the linear Navier – Stokes system(4.1), (4.2). First let's construct the nuclear operator $K\in\mathcal{L}(\mathbf{L}_2)$. As the first step we consider the operator $A^m=\operatorname{diag}\{(-\Delta)^m,(-\Delta)^m,\ldots,(-\Delta)^m\}$ with domain of definition $\operatorname{dom}A^m=\{v\in(W_2^{2m}(\Omega))^d:\Delta^{m-1}v(x)=\ldots=\Delta v(x)=v(x)=0,\,x\in\partial\Omega\}$, $m\in\mathbf{N}\setminus\{1\}$. The operator $A^m\in\mathcal{C}l(\mathbf{L}^2)$ (i.e. linear, closed and densely defined.) It is also self-adjoint and positively defined. Its spectrum $\sigma(A^m)$ is positive, discrete and thickens just to the point $+\infty$. If we set

$\sigma(A^m) = \{\lambda_{mk}\}$, where $\{\lambda_{mk}\}$ is the sequence of eigenvalues of the operator A^m, numerated in non-decreasing order taking into account their multiplicity, we find that $\lambda_{mk} = |\lambda_k|^m$. Moreover, as the eigenfunctions of the operator A^m we can take the eigenfunctions of the operator A. As the second (and the last step) we notice that for any $m \in \mathbf{N} \setminus \{1\}$ there exists the inverse operator $A^{-m} \in \mathcal{L}(\mathbf{L}^2)$, which is compact due to the compactness of the attachment $\mathrm{dom}A^m$ $(= \mathrm{im}A^{-1}) \hookrightarrow \mathbf{L}^2$. It is easy to show that the eigenvalues $\{|\lambda_k|^{-1}\}$ of the operator A^{-m} have the following asymptotic behavior

$$|\lambda_k|^{-m} \sim k^{-2m/d^2} \text{ under } k \to \infty.$$

This means that if $m/d^2 > 1$, the operator A^{-m} is nuclear. We fix such $m \in \mathbf{N} \setminus \{1\}$.

Further arguments are not fundamentally different from the reasoning in paragraph 2 of this article, and therefore omitted.

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