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SOME CLASS OF THE INTERPOLATING MARTINGALE MEASURES ON A COUNTABLE PROBABILITY SPACE

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Abstract. This article deals with such martingale measures that satisfy the noncoincidence barycenter condition (NBC), a condition that makes it possible to interpolate with such martingale measure incomplete market to complete one with respect to an arbitrary interpolating special Haar filtering.

We obtain sufficient conditions on market parameters that ensure the existence of a martingale measure. The obtained results can form the basis for the algorithm and software complex. The program based on the method and the created software complex will allow one to apply Haar interpolations method to the calculations on the arbitrage-free financial markets, which will greatly facilitate the choice of optimal strategies of investors in the financial markets.

1. Introduction

We consider one-step (B, S)-market defined on the set $\{\Omega, \mathbf{F}\}$ where $\Omega = \{\omega_k\}_{k=1}^m$, $3 \le m \le \infty$, $\mathbf{F} = (\mathcal{F}_0, \mathcal{F}_1)$ is a one-step filtration with $\mathcal{F}_0 = \{\Omega, \emptyset\}$, and $\mathcal{F}_1$ is the σ-algebra of all subsets of Ω. Let $N = \{1, 2, \ldots, m\}$, if $m < \infty$, and N is the set of natural numbers if $m = \infty$. We denote by $Z = (Z_k, \mathcal{F}_k)_{k=0}^1$ an $\mathbf{F}$-adapted random process, which we think of as the discounted value of the shares $(Z_0 = a, Z_1(\omega_k) = b_k, k \in N)$. A set of nondegenerate martingale measures (m.m.) P of the (B, S)-market under consideration will be denoted as $\mathcal{P}(Z, \mathbf{F})$. It is clear that

$$\mathcal{P}(Z, \mathbf{F}) = \{P = (p_k)_{k=1}^m, m \in N : \sum_{k=1}^m p_k = 1, \sum_{k=1}^m p_k b_k = a, p_k > 0, k \in N\}.$$

Suppose that $\inf_k b_k < a < \sup_k b_k$. This is a sufficient condition for the fact that $\mathcal{P}(Z, \mathbf{F}) \ne \emptyset$. Thus, our market is arbitrage-free and incomplete.

For the transition from incomplete arbitrage-free markets to the complete one, we use the Haar interpolation method [1], the essence of which is as follows. Considering arbitrage-free but incomplete markets, we expand the initial filtration of

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the financial market's $\mathbf{F}$ in such a way that it turns into *Haar filtering*(H.f.) (*special Haar filtering* (s.H.f.)) $\mathbf{H}$, in which, when passing from the time instant n to the time instant $n+1$, exactly one atom is split into two parts (respectively, only one atom, the one that was obtained as a result of splitting at the previous step, is split into two parts), and the other atoms remain unchanged. Then, using the probabilistic (martingale) solution of the Dirichlet problem for the discounted share price in relation to the H.f (s.H.f.), we obtain a uniquely determined interpolation of the discounted share price for specially selected time intervals. Finally, with the martingale interpolation obtained in this way, we construct a financial market, defined both on the initial and newly introduced intermediate values of the time parameter.

At the initial values of the time parameter, the stock's prices and the prices of the bank account of this market coincide with those originally set up, i.e. we obtain an interpolation of the initial financial market.

We will refer to this interpolation as *Haar interpolation filtering* (H.i.f.) and, respectively, *special Haar interpolation filtering* (s.H.i.f.).

We denote by $|A|$ the number of elements of some set A.

Definition 1.1. We say that a measure $P \in \mathcal{P}(Z, \mathbf{F})$ possesses **the property of universal Haar uniqueness** (PUHU) (**weakened property of universal Haar uniqueness** (WPUHU)), if for any H.i.f. (s.H.i.f.) $\mathbf{H}$ interpolating the filtering $\mathbf{F}$, the corresponding interpolating process Y admits a unique martingale measure (coinciding with the original measure P), that is, the following equality holds $|\mathcal{P}(Y, \mathbf{H})| = 1$.

The weakened property of universal Haar uniqueness (WPUHU) is less restrictive than PUHU since it is not associated with all Haar interpolation of the initially specified filtering but only with their subclass, called "special Haar interpolations".

Such interpolation properties as PUHU and WPUHU are inconvenient from the point of view of their analytical verification for the martingale measure under consideration. Therefore we will pass to the properties equivalent to them, called the noncoincidence barycenter condition (NBC) and the weakened noncoincidence barycenter condition (WNBC), respectively.

Definition 1.2. We say that the measure $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies **noncoincidence barycenter condition** (NBC), if for any two (sorted in ascending order) disjoint subsets of the indices I and J (for the numbers b_i), I and $J \subset N$, $|I| \leqslant |J|$, the following inequality holds:

$$\frac{\sum\limits_I b_i p_i}{\sum\limits_I p_i} \neq \frac{\sum\limits_J b_j p_j}{\sum\limits_J p_j}. \tag{1.1}$$

Definition 1.3. We say that the measure $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies **weakened noncoincidence barycenter condition (WNBC)**, if $\forall i \in N$ and for any set of indices $J \subset N \diagdown \{i\}$ with a finite complement $\overline{J} = N \diagdown J$, the following inequality holds (if $m < \infty$ the set $\overline{J}$ is always finite):

$$b_i \neq \frac{\sum_J b_j p_j}{\sum_J b_j p_j}.$$

The set m.m. of the process Z satisfying NBC (WNBC) is denoted by NBC(Z) (WNBC(Z)).

In the case of finite σ-algebra $\mathcal{F}_1$ the set NBC(Z) has been studied in some detail (see e.g. [1]. In particular, it has been found that NBC(Z) $\neq \oslash$ if $\mathcal{P}(Z, \mathbf{F}) \neq \oslash$ and $a \neq b_i, \forall i$. In the case of countable σ-algebra, it is also obvious that if there is a probability measure $P \in \mathcal{P}(Z, \mathbf{F})$ satisfying NBC, then the numbers a, b_1, b_2, ... are different. If NBC(Z) $\neq \oslash$, then filtration $\mathbf{F}$ is the natural filtration of process Z. It is also shown that the fact that the filtration $\mathbf{F}$ is a natural filtering of the process Z, does not imply the existence of the measure $P \in \mathcal{P}(Z, \mathbf{F})$, which satisfies the NBC. The corresponding sufficient conditions could not be obtained. The classes of sets of m.m. satisfying WNBC, are also studied in [3]-[6].

The aim of this paper is to prove the existence of m.m. that satisfy the NBC in the case when $m = \infty$ ([7]). Note that at the moment no examples of an m.m. satisfying the NBC, have been found.

2. Sufficient conditions for the nonemptiness NBC(Z)

So, let $m = \infty$. The following results are new and give sufficient conditions for the existence of m.m. satisfying the NBC.

Lemma 2.1. *Let $b_1 < b_2 < b_3 < \ldots$, and $b_i - b_{i-1} \geqslant b_{i-1}$, $\forall i \geqslant 2$. If*

$$b_{i-1} \min_{2 \leqslant j \leqslant i-1} p_j > \sum_{j=i+1}^{\infty} b_j p_j, \quad \forall i \geqslant 2, \tag{2.1}$$

then the measure $P \in NBC(Z)$.

Proof. Let us check that inequalities (1.1) are satisfied.

Let I and J be two (ordered in ascending order) disjoint finite or infinite subsets of the indices of the set $\{1, 2, ...\}$: $I = \{i_1, i_2, \ldots\}$, $J = \{j_1, j_2, \ldots\}$. We can consider $i_1 > j_1$. Represent the set I as$I_1 \cup I_2$ and J as $J_1 \cup J_2$, where $I_1 = \{i_1\}$, $I_2 \subset \{i_1 + 1, i_1 + 2, \ldots\}$, $J_1 \subset \{1, 2, 3, \ldots, i_1 - 1\}$, $J_2 \subset \{i_1 + 1, i_1 + 2, \ldots\}$. Then inequalities (1.1) are equivalent to the following ones:

$$\frac{\sum_I b_i p_i}{\sum_I p_i} \neq \frac{\sum_J b_j p_j}{\sum_J p_j} \Longleftrightarrow \frac{b_{i_1} p_{i_1} + \sum_{I_2} b_i p_i}{p_{i_1} + \sum_{I_2} p_i} \neq \frac{\sum_{J_1} b_j p_j + \sum_{J_2} b_j p_j}{\sum_{J_1} p_j + \sum_{J_2} p_j} \Longleftrightarrow$$

$$\Leftrightarrow (b_{i_1} p_{i_1} + \sum_{I_2} b_i p_i) \left(\sum_{J_1} p_j + \sum_{J_2} p_j \right) \neq \left(\sum_{J_1} b_j p_j + \sum_{J_2} b_j p_j \right) (p_{i_1} + \sum_{I_2} p_i) \Leftrightarrow$$

$$\Leftrightarrow b_{i_1} p_{i_1} \sum_{J_1} p_j - p_{i_1} \sum_{J_1} b_j p_j + \sum_{I_2} b_i p_i \sum_{J_1} p_j - \sum_{I_2} p_i \sum_{J_1} b_j p_j \neq$$

$$\neq p_{i_1}\sum_{J_2}b_jp_j+\sum_{I_2}p_i\sum_{J_2}b_jp_j-\sum_{I_2}b_ip_i\sum_{J_2}p_j-b_{i_1}p_{i_1}\sum_{J_2}p_j.$$

Now let us use the machinery from [4]. Since

$$b_{i_1}p_{i_1}\sum_{J_1}p_j-p_{i_1}\sum_{J_1}b_jp_j+\sum_{I_2}b_ip_i\sum_{J_1}p_j-\sum_{I_2}p_i\sum_{J_1}b_jp_j=$$

$$=\left(b_{i_1}\sum_{J_1}p_j-\sum_{J_1}b_jp_j\right)p_{i_1}+\sum_{I_2}b_ip_i\sum_{J_1}p_j-\sum_{I_2}p_i\sum_{J_1}b_jp_j>$$

$$>\left(b_{i_1}\sum_{J_1}p_j-b_{i_1-1}\sum_{J_1}p_j\right)p_{i_1}+\sum_{I_2}b_ip_i\sum_{J_1}p_j-b_{i_1-1}\sum_{I_2}p_i\sum_{J_1}p_j=$$

$$=(b_{i_1}-b_{i_1-1})p_{i_1}\sum_{J_1}p_j+\left(\sum_{I_2}b_ip_i-b_{i_1-1}\sum_{I_2}p_i\right)\sum_{J_1}p_j\geqslant$$

$$\geqslant b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_jp_{i_1}+\left(\sum_{I_2}b_ip_i-b_{i_1-1}\sum_{I_2}p_i\right)\min_{2\leqslant j\leqslant i_1-1}p_j\geqslant$$

$$\geqslant b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_jp_{i_1}+\left(b_{i_1+1}\sum_{I_2}p_i-b_{i_1-1}\sum_{I_2}p_i\right)\min_{2\leqslant j\leqslant i_1-1}p_j\geqslant$$

$$\geqslant b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_jp_{i_1}+(b_{i_1+1}-b_{i_1-1})\min_{2\leqslant j\leqslant i_1-1}p_j\sum_{I_2}p_i=$$

$$=b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_jp_{i_1}+(b_{i_1+1}-b_{i_1}+b_{i_1}-b_{i_1-1})\min_{2\leqslant j\leqslant i_1-1}p_j\sum_{I_2}p_i\geqslant$$

$$\geqslant b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_jp_{i_1}+b_{i_1}\min_{2\leqslant j\leqslant i_1-1}p_j\sum_{I_2}p_i+b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_j\sum_{I_2}p_i\geqslant$$

$$\geqslant p_{i_1}\sum_{j=i_1+1}^{\infty}b_jp_j+b_{i_1}\min_{2\leqslant j\leqslant i_1-1}p_j\sum_{I_2}p_i+b_{i_1-1}\min_{2\leqslant j\leqslant i_1-1}p_j\sum_{I_2}p_i\geqslant$$

$$\geqslant p_{i_1}\sum_{j=i_1+1}^{\infty}b_jp_j+\sum_{I_2}p_i\sum_{j=i_1+1}^{\infty}b_jp_j\geqslant p_{i_1}\sum_{J_2}b_jp_j+\sum_{I_2}p_i\sum_{J_2}b_jp_j\geqslant$$

$$\geqslant p_{i_1}\sum_{J_2}b_jp_j+\sum_{I_2}p_i\sum_{J_2}b_jp_j-\sum_{I_2}b_ip_i\sum_{J_2}p_j-b_{i_1}p_{i_1}\sum_{J_2}p_j,$$

inequalities (1.1) are fulfilled and the measure P satisfies the NBC. Q.E.D. □

Remark 2.2. If the sequence p_1, p_2, ... decreases monotonically, inequalities (2.1) coincide with the inequalities

$$b_{i-1}p_{i-1}>\sum_{j=i+1}^{\infty}b_jp_j,\quad\forall i\geqslant 2.\tag{2.2}$$

Example 2.3. Let $b_1 = 1 < a = \frac{5}{3} < b_2 = 2 < b_3 = 4 < b_4 = 8 < \cdots < b_i = 2^{i-1} < \dots,\ b_i - b_{i-1} = 2^{i-1} - 2^{i-2} = 2^{i-2} = b_{i-1}$. Then the measure $P = \left(\frac{2}{3}; \frac{1}{2^2}; \frac{1}{2^4}; \dots; \frac{1}{2^{2i-2}}; \dots\right) \in \mathcal{P}(Z, \mathbf{F})$ satisfies condition (2.2) and, consequently, possesses NBC.

Theorem 2.4. *Let the measure $P \in \mathcal{P}(Z, \boldsymbol{F})$, $b_1 < a < b_2 < b_3 < b_4 < b_5 < \dots$ and*

$$b_i - b_{i-1} \geqslant b_{i-1}, \forall i \geqslant 2. \tag{2.3}$$

Then NBC(Z) is nonempty and strictly imbedded in $\mathcal{P}(Z, \boldsymbol{F})$.

Proof. Without loss of generality we can assume that $b_1 \geq 1$. First we show that the set of m.m., for which the condition (2.1) is satisfied, is nonempty [2]. We describe the set $\mathcal{P}(Z, \mathbf{F})$ as follows:

$$\begin{cases} p_1 + p_2 + \sum\limits_{j=3}^{\infty} p_j = 1 \\ b_1 p_1 + b_2 p_2 + \sum\limits_{j=3}^{\infty} b_j p_j = a \\ p_i > 0,\ i \in \mathbb{N}. \end{cases} \tag{2.4}$$

We will construct a solution of system (2.4) such that (2.1) and (2.3) are satisfied for it . Take $p_j = \frac{1}{b_j 2^{nj}}$, $\forall j \geq 3$. We set the number n so large that the following inequalities hold: $\dfrac{a-\varepsilon}{b_1} > 1$, $\frac{a-\varepsilon}{b_2} < 1$, $1 - \delta > \frac{a-\varepsilon}{b_2}$, $\frac{a-b_1}{b_2-b_1} + \frac{b_1\delta-\varepsilon}{b_2-b_1} > \max\{\frac{\varepsilon}{b_2}, \frac{1}{b_3 2^{3n}}\}$ $\varepsilon = \sum\limits_{j=3}^{\infty} \frac{1}{2^{nj}} = \frac{1}{2^{2n}(2^n-1)}$, nd $\delta = \sum\limits_{j=3}^{\infty} \frac{1}{b_j 2^{nj}} < \varepsilon$. Then system (2.4) takes the form:

$$\begin{cases} p_1 + p_2 = 1 - \delta \\ b_1 p_1 + b_2 p_2 = a - \varepsilon \\ p_i > 0,\ i = 1, 2 \end{cases} \tag{2.5}$$

It is easy to see that system (2.5) has the only solution:

$$\begin{cases} p_1 = \frac{b_2-a}{b_2-b_1} + \frac{\varepsilon - b_2\delta}{b_2-b_1} \\ p_2 = \frac{a-b_1}{b_2-b_1} + \frac{b_1\delta-\varepsilon}{b_2-b_1} \end{cases} \tag{2.6}$$

Thus, we have obtained a nondegenerate m.m. Now let us show that it satisfies the NBC.

From the inequality $\frac{a-b_1}{b_2-b_1} + \frac{b_1\delta-\varepsilon}{b_2-b_1} > \max\{\frac{\varepsilon}{b_2}, \frac{1}{b-3 2^{3n}}\}$, first, it follows that the sequence $p_2, p_3, \dots$ decreases monotonically, and, second, that (2.2) is satisfied for $i = 3$. The fulfillment of inequality (2.2) is trivial for $i \geq 4$. From lemma (2.1) it follows that the constructed m.m. satisfies the NBC.

Show that NBC(Z) is strictly imbedded in $\mathcal{P}(Z, \mathbf{F})$. Consider system (2.4) together with the following equation that contradicts the NBC:

$$b_3 = \frac{b_2 p_2 + \sum\limits_{j=4}^{\infty} b_j p_j}{p_2 + \sum\limits_{j=4}^{\infty} p_j} \Longleftrightarrow \sum_{j=4}^{\infty} b_j p_j = b_3(1 - p_1 - p_3) - b_2 p_2.$$

Substituting this into the first equation of system (2.4), we obtain: $p_1 = \frac{b_3 - a}{b_3 - b_1}$. The second equation of system (2.4) now has the form: $\sum_{j=2}^{\infty} p_j = \frac{a - b_1}{b_3 - b_1}$. From here we easily find the solutions of the posed problem, the m.m., which does not satisfy the NBC. Q.E.D. □

Conclusion

The obtained conditions for the existence of martingale measures that satisfy the NBC, allow one to construct models of arbitrage-free incomplete financial markets with an infinite number of states that, with the help of Haar interpolations, are transformed into complete arbitrage-free ones, which is very important in calculating the prices of various financial obligations and constructing hedging portfolios [8]-[9].

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