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STUDY OF τ-GENERALISE LAURICELLA FUNCTION WITH PROPERTIES

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Abstract: In this paper we define some new properties on τ-Generalized lauricella functions using the differentiations and integrations properties. Further we discuss some properties related to Euler (beta) transform, Laplace transform, Mellin transform, finally some special cases has been discussed.

1. Introduction and Preliminaries:

The Gauss hypergeometric function and generalized hypergeometric function is defined as follows:

$$ {}_2F_1(a,\ b,\ c;\ z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \text{ where } \{|z| < 1,\ c \neq 0, -1, -2, \ldots\} \quad (1) $$

$$ {}_pF_q(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}, \text{ where } [p = q+1,\ |z| < 1]. \quad (2) $$

and no denominator parameter equal to zero or negative integer.

The Generalisation of hypergeometric function involving double series with two variables given by Appell in (1880). Appell was the first author to treat this matter on a systemic basis, and he defined the four functions F_1, F_2, F_3, F_4 [4] in which F_1 function is defined as

$$ F_1(a;\ b_1,\ b_2;\ c;\ x;\ y) = \sum_{n_1, n_2 = 0}^{\infty} \frac{(a)_{n_1+n_2} (b_1)_{n_1} (b_2)_{n_2}}{(c)_{n_1+n_2}} \frac{x^{n_1}}{n_1!} \cdot \frac{y^{n_2}}{n_2!} $$

where, $[|x| < 1,\ |y| < 1]$ (3)

Further Lauricella proceeded to define and studied the four important functions in multiple series representation [4] in which $F_D^{(n)}$ function is defined as,

$$ F_D^{(n)}(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{(a)_{k_1+\ldots+k_n} (b_1)_{k_1} \cdots (b_n)_{k_n}}{(c)_{k_1+\ldots+k_n}} \frac{(x_1)^{k_1}}{k_1!} \cdots \frac{(x_n)^{k_n}}{k_n!} $$

where, $(|x_1| < 1, |x_2| < 1, |x_3| < 1, \ldots, |x_n| < 1)$, (4)

Further M. Garg and R. Mishra [3] generalized Lauricella function $F_D^{(n)}$ involving τ_i parameter such as,

Keyword. Hypergeometric function, Generalised hypergeometric function, Lauricella Hypergeometric function,Wright type hypergeometric function, Laplace transform, Mellin Transform, Euler Transform, τ-Generalized lauricella function.

$$F_D^{(n)(\tau_1, \ldots, \tau_n)}(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$$
$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1 \ldots k_n = 0}^{\infty} \frac{\Gamma(a + \tau_1 k_1 + \ldots + \tau_n k_n)(b_1)_{k_1} \ldots (b_n)_{k_n}}{\Gamma(c + \tau_1 k_1 + \ldots + \tau_n k_n)} \frac{(x_1)^{k_1}}{k_1!} \ldots \frac{(x_n)^{k_n}}{k_n!}$$

where, $(|x_1| < 1, \ldots, |x_n| < 1)$, $\tau_i > 0$. (5)

The Laplace transform of function $f(t)$ is defined as [6].

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{6}$$

Euler Beta transform is defined as [6]

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}, \tag{7}$$

The pochhammer symbol (or the shifted factorial)[2] denoted by $(\alpha)_n$ and defined by

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1),\ n \geq 1$$

2. Main Result

Theorem 1. If $\tau_1, \tau_2, \ldots \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, b_1, \ldots, b_n, c \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$, $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0$, $\mathrm{Re}(c) > 0$

then

$$c \,.\, F_D^{(n)(\tau_1, \tau_2, \ldots, \tau_n)}\left(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n\right)$$
$$= c \,.\, F_D^{(n)(\tau_1, \tau_2, \ldots, \tau_n)}\left(a; b_1, \ldots, b_n; c+1; x_1, \ldots, x_n\right)$$
$$+ \tau_1 x_1 \frac{\partial}{\partial x_1} F_D^{(n)(\tau_1, \tau_2, \ldots, \tau_n)}\left(a; b_1, \ldots, b_n; c+1; x_1, \ldots, x_n\right)$$
$$+ \ldots + \tau_n x_n \frac{\partial}{\partial x_n} F_D^{(n)(\tau_1, \tau_2, \ldots, \tau_n)}\left(a; b_1, \ldots, b_n; c+1; x_1, \ldots, x_n\right)$$

Proof: Consider the term

$$\tau_1 x_1 \frac{\partial}{\partial x_1} F_D^{(n)(\tau_1, \tau_2, \ldots, \tau_n)}\left(a; b_1, \ldots, b_n; c+1; x_1, \ldots, x_n\right) + \ldots$$
$$+ \tau_n x_n \frac{\partial}{\partial x_n} F_D^{(n)(\tau_1, \tau_2, \ldots, \tau_n)}\left(a; b_1, \ldots, b_n; c+1; x_1, \ldots, x_n\right)$$
$$= \frac{\tau_1 x_1 \Gamma(c+1)}{\Gamma a} \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{\Gamma\left(a + \tau_1 k_1 + \ldots + \tau_n k_n\right)\left(b_1\right)_{k_1} \ldots \left(b_n\right)_{k_n}}{\Gamma\left(c + 1 + \tau_1 k_1 + \ldots + \tau_n k_n\right)} \frac{k_1 x_1^{k_1 - 1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \ldots \frac{x_n^{k_n}}{k_n!} + \ldots$$
$$+ \frac{\tau_n x_n \Gamma(c+1)}{\Gamma a} \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{\Gamma\left(a + \tau_1 k_1 + \ldots + \tau_n k_n\right)\left(b_1\right)_{k_1} \ldots \left(b_n\right)_{k_n}}{\Gamma\left(c + 1 + \tau_1 k_1 + \ldots + \tau_n k_n\right)} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \ldots \frac{k_n x_n^{k_n - 1}}{k_n!}$$

$$= \frac{\Gamma(c+1)}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+1+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \ldots \frac{x_n^{k_n}}{k_n!}\left[\tau_1 k_1+\ldots+\tau_n k_n\right]$$

$$= \frac{c\Gamma(c)}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)\left(b_1\right)_{k_1}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \ldots \frac{x_n^{k_n}}{k_n!}$$

$$-\frac{c\Gamma(c+1)}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)\left(b_1\right)_{k_1}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c+1+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \ldots \frac{x_n^{k_n}}{k_n!}$$

$$= c\mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c;x_1,\ldots,x_n\right) - c\mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c+1;x_1,\ldots,x_n\right)$$

Thus, we get the result

Theorem 2. If $\tau_1, \tau_2, \ldots \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, b_1, \ldots, b_n, c, \delta \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$, $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0$, $\mathrm{Re}(c) > 0$,

then

$$\frac{\Gamma(c+\delta)}{\Gamma\delta} \int_0^1 u^{c-1}(1-u)^{\delta-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c;x_1 u^{\tau_1},\ldots,x_n u^{\tau_n}\right) du$$

$$= \Gamma c\left(\mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c+\delta;x_1,\ldots,x_n\right)\right)$$

Proof: consider left hand side

$$\frac{\Gamma(c+\delta)}{\Gamma\delta} \int_0^1 u^{c-1}(1-u)^{\delta-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c;x_1 u^{\tau_1},\ldots,x_n u^{\tau_n}\right) du,$$

Using (5) in above result and changing the order of summation and integration, we have

$$= \frac{\Gamma(c+\delta)}{\Gamma\delta} \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{(x_1)^{k_1}}{k_1!}$$

$$\ldots \frac{(x_n)^{k_n}}{k_n!} \int_0^1 u^{c+\tau_1 k_1+\ldots+\tau_n k_n-1}(1-u)^{\delta-1} du$$

Usingbeta function we get the desired result.

Theorem 3. If $\tau_1, \tau_2, \ldots \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, b_1, \ldots, b_n, c \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$, $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0$, $\mathrm{Re}(c) > 0$, $w_i > 0$,

then

$$\int_0^z t^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c;w_1 t^{\tau_1},\ldots,w_n t^{\tau_n}\right) dt$$

$$= \frac{z^c}{c} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c+1;w_1 z^{\tau_1},\ldots,w_n z^{\tau_n}\right)$$

Proof: We starts with left side of the above equation

$$\int_0^z t^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c;w_1 t^{\tau_1},\ldots,w_n t^{\tau_n}\right)dt$$

$$= \int_0^z t^{c-1} \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(w_1 t^{\tau_1}\right)^{k_1}}{k_1!}\ldots\frac{\left(w_n t^{\tau_n}\right)^{k_n}}{k_n!}dt$$

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(w_1\right)^{k_1}}{k_1!}\ldots\frac{\left(w_n\right)^{k_n}}{k_n!}\int_0^z t^{c-1+\tau_1 k_1+\ldots+\tau_n k_n}dt$$

after solving the integration, we have

$$= z^c \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+1+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(w_1 z^{\tau_1}\right)^{k_1}}{k_1!}\ldots\frac{\left(w_n z^{\tau_n}\right)^{k_n}}{k_n!}$$

Now multiplying and dividing by c, we get the required result.

$$= \frac{z^c}{c} \frac{\Gamma(c+1)}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+1+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(w_1 z^{\tau_1}\right)^{k_1}}{k_1!}\ldots\frac{\left(w_n z^{\tau_n}\right)^{k_n}}{k_n!}$$

$$= \frac{z^c}{c} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c+1;w_1 z^{\tau_1},\ldots,w_n z^{\tau_n}\right)$$

Theorem 4. If $\tau_1, \tau_2, \ldots \tau_n \in \mathbb{R}_+ = (0,\infty)$; $a, b_1, \ldots, b_n, c \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$, $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0, \mathrm{Re}(c) > 0$, $w_i > 0$,
then,

$$\left(\frac{d}{dz}\right)^m \left[z^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c;w_1 z^{\tau_1},\ldots,w_n z^{\tau_n}\right)\right]$$

$$= z^{c-m-1} \frac{\Gamma c}{\Gamma(c-m)} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a;b_1,\ldots,b_n;c-m;w_1 z^{\tau_1},\ldots,w_n z^{\tau_n}\right)$$

Proof: Let's start with

$$\left(\frac{d}{dz}\right)^m \left[z^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a,b_1,\ldots,b_n;c;w_1 z^{\tau_1},\ldots,w_n z^{\tau_n}\right)\right]$$

$$= \left(\frac{d}{dz}\right)^m \left[z^{c-1} \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(w_1 z^{\tau_1}\right)^{k_1}}{k_1!}\ldots\frac{\left(w_n z^{\tau_n}\right)^{k_n}}{k_n!}\right]$$

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(w_1\right)^{k_1}}{k_1!}\ldots$$

$$\frac{\left(w_n\right)^{k_n}}{k_n!}\left(\frac{d}{dz}\right)^m \left[z^{c-1+\tau_1 k_1+\ldots+\tau_n k_n}\right]$$

here, let $c + \tau_1 k_1 + \ldots + \tau_n k_n = r$

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a + \tau_1 k_1 + \ldots + \tau_n k_n\right)(b_1)_{k_1} \ldots (b_n)_{k_n}}{\Gamma\left(c + \tau_1 k_1 + \ldots + \tau_n k_n\right)} \frac{(w_1)^{k_1}}{k_1!} \ldots \frac{(w_n)^{k_n}}{k_n!} \left(\frac{d}{dz}\right)^m \left[z^{r-1}\right]$$

after, differentiating m times, we get

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a + \tau_1 k_1 + \ldots + \tau_n k_n\right)(b_1)_{k_1} \ldots (b_n)_{k_n}}{\Gamma\left(c + \tau_1 k_1 + \ldots + \tau_n k_n\right)} \frac{(w_1)^{k_1}}{k_1!} \ldots$$

$$\frac{(w_n)^{k_n}}{k_n!} \left[(r-1)(r-2)\ldots(r-m) z^{r-m-1}\right]$$

After simplification, we obtain

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a + \tau_1 k_1 + \ldots + \tau_n k_n\right)(b_1)_{k_1} \ldots (b_n)_{k_n}}{\Gamma\left(c + \tau_1 k_1 + \ldots + \tau_n k_n\right)} \frac{(w_1)^{k_1}}{k_1!} \ldots \frac{(w_n)^{k_n}}{k_n!} z^{c+\tau_1 k_1 + \ldots + \tau_n k_n - m - 1}$$

$$\frac{\left(c + \tau_1 k_1 + \ldots + \tau_n k_n - 1\right)!}{\left(c + \tau_1 k_1 + \ldots + \tau_n k_n - m - 1\right)!}$$

$$= z^{c-m-1} \frac{\Gamma c}{\Gamma(c-m)} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a, b_1, \ldots, b_n; c-m; w_1 z^{\tau_1}, \ldots, w_n z^{\tau_n}\right)$$

Theorem 5. If $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, b_1, \ldots, b_n, c \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$ $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0$, $\mathrm{Re}(c) > 0$,

then

$$\mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a; b_1, \ldots, b_n; c - (\tau_1 + \ldots + \tau_n); x_1, \ldots, x_n\right)$$

$$-\mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a-1; b_1, \ldots, b_n; c - (\tau_1 + \ldots + \tau_n); x_1, \ldots, x_n\right)$$

$$= \frac{\tau_1 b_1 x_1 \Gamma\left(c - (\tau_1 + \ldots + \tau_n)\right)}{\Gamma a}$$

$$\sum_{m_1,k_2,\ldots k_n=0}^{\infty} \frac{\Gamma\left(a - 1 + \tau_1 + \tau_1 m_1 + \ldots + \tau_n k_n\right)\left(b_1+1\right)_{m_1} (b_2)_{k_2} \ldots \left(b_n\right)_{k_n}}{\Gamma\left(c - (\tau_2 + \ldots + \tau_n) + \tau_1 m_1 + \ldots + \tau_n k_n\right)} \frac{(x_1)^{m_1}}{m_1!} \ldots$$

$$\frac{(x_n)^{k_n}}{k_n!} + \ldots + \frac{\tau_n b_n x_n \Gamma\left(c - (\tau_1 + \ldots + \tau_n)\right)}{\Gamma a}$$

$$\sum_{k_1\ldots k_{n-1},m_n=0}^{\infty} \frac{\Gamma\left(a - 1 + \tau_n + \tau_1 k_1 + \ldots \tau_{n-1} k_{n-1} + \tau_n m_n\right)\left(b_1\right)_{k_1} (b_2)_{k_2} \ldots \left(b_n+1\right)_{m_n}}{\Gamma\left(c - (\tau_1 + \ldots + \tau_{n-1}) + \tau_1 k_1 + \ldots + \tau_n m_n\right)}$$

$$\frac{(x_1)^{k_1}}{k_1!} \ldots \frac{(x_n)^{m_n}}{m_n!}$$

Proof: Consider left hand side

$$= \frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma a}\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{\Gamma\left(a+\tau_1k_1+\ldots+\tau_nk_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}\frac{\left(x_1\right)^{k_1}}{k_1\,!}\ldots$$

$$\frac{\left(x_n\right)^{k_n}}{k_n\,!}-\frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma(a-1)}$$

$$\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{\Gamma\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}\frac{\left(x_1\right)^{k_1}}{k_1\,!}\ldots\frac{\left(x_n\right)^{k_n}}{k_n\,!}$$

Using gamma function, we get

$$= \frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma(a-1)}\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{\Gamma\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}\frac{\left(x_1\right)^{k_1}}{k_1\,!}$$

$$\ldots\frac{\left(x_n\right)^{k_n}}{k_n\,!}\left\{\frac{\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)}{(a-1)}-1\right\}$$

$$= \frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma(a)}\sum_{k_1=1,k_2,\ldots,k_n=0}^{\infty}\frac{\Gamma\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}$$

$$\frac{\left(x_1\right)^{k_1}}{k_1\,!}\ldots\frac{\left(x_n\right)^{k_n}}{k_n\,!}\tau_1k_1+\ldots+\frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma(a)}$$

$$\sum_{k_1,\ldots,k_{n-1}=0,k_n=1}^{\infty}\frac{\Gamma\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}\frac{\left(x_1\right)^{k_1}}{k_1\,!}\ldots\frac{\left(x_n\right)^{k_n}}{k_n\,!}\tau_nk_n$$

using $\left(b_n+1\right)_{k_n-1}=\frac{\left(b_n\right)_{k_n}}{b_n}$, we have

$$= \frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma a}$$

$$\sum_{k_1=1,k_2,\ldots,k_n=0}^{\infty}\frac{\Gamma\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)\,b_1\left(b_1+1\right)_{k_1-1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}\frac{x_1\left(x_1\right)^{k_1-1}}{(k_1-1)!}\ldots$$

$$\frac{\left(x_n\right)^{k_n}}{k_n\,!}\tau_1+\ldots+\frac{\Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma a}$$

$$\sum_{k_1,\ldots,k_{n-1}=0,k_n=1}^{\infty}\frac{\Gamma\left(a-1+\tau_1k_1+\ldots+\tau_nk_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots b_n\left(b_n+1\right)_{k_n-1}}{\Gamma\left(c-(\tau_1+\ldots+\tau_n)+\tau_1k_1+\ldots+\tau_nk_n\right)}\frac{\left(x_1\right)^{k_1}}{k_1\,!}\ldots$$

$$\frac{x_n\left(x_n\right)^{k_n-1}}{(k_n-1)!}\tau_n$$

putting, $k_1 = m_1 + 1$, ..., and $k_n = m_n + 1$, then we get

$$= \frac{\tau_1 b_1 x_1 \Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma a}$$

$$\sum_{m_1, k_2, \ldots k_n=0}^{\infty} \frac{\Gamma\left(a-1+\tau_1+\tau_1 m_1+\ldots+\tau_n k_n\right)\left(b_1+1\right)_{m_1}(b_2)_{k_2}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c-(\tau_2+\ldots+\tau_n)+\tau_1 m_1+\ldots+\tau_n k_n\right)} \frac{\left(x_1\right)^{m_1}}{m_1!}\ldots$$

$$\frac{\left(x_n\right)^{k_n}}{k_n!}+\ldots+\frac{\tau_n b_n x_n \Gamma\left(c-(\tau_1+\ldots+\tau_n)\right)}{\Gamma a}$$

$$\sum_{k_1 \ldots k_{n-1}, m_n=0}^{\infty} \frac{\Gamma\left(a-1+\tau_n+\tau_1 k_1+\ldots\tau_{n-1} k_{n-1}+\tau_n m_n\right)\left(b_1\right)_{k_1}(b_2)_{k_2}\ldots\left(b_n+1\right)_{m_n}}{\Gamma\left(c-(\tau_1+\ldots+\tau_{n-1})+\tau_1 k_1+\ldots+\tau_n m_n\right)}$$

$$\frac{\left(x_1\right)^{k_1}}{k_1!}\ldots\frac{\left(x_n\right)^{m_n}}{m_n!}$$

Theorem 6. (Laplace Transform). If $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, \gamma, b_1, \ldots, b_n, c \in \mathbb{C}$; $\operatorname{Re}(a) > 0$, $\operatorname{Re}(b_1) > 0$, $\operatorname{Re}(b_2) > 0$, ..., $\operatorname{Re}(b_n) > 0$, $\operatorname{Re}(c) > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(\gamma) > 0$ and $\left|\frac{x_1}{s^{\tau_1}}\right|, \ldots, \left|\frac{x_n}{s^{\tau_n}}\right| < 1$,

then

$$\int_0^{\infty} e^{-sz} z^{\gamma-1} \mathrm{F}_{\mathrm{D}}^{\tau_1, \tau_2, \ldots, \tau_n}\left(a, b_1, \ldots, b_n; c; x_1 z^{\tau_1}, \ldots, x_n z^{\tau_n}\right) dz$$

$$= \frac{\gamma}{s^{\gamma}} \mathrm{F}_{\mathrm{D}}^{\tau_1, \tau_2, \ldots, \tau_n}\left(a, \gamma, b_1, \ldots, b_n; c; \frac{x_1}{s^{\tau_1}}, \ldots, \frac{x_n}{s^{\tau_n}}\right)$$

Proof: consider left hand side and using equation (5) and apply property of Laplace transform, we have

$$= \int_0^{\infty} e^{-sz} z^{\gamma-1} \frac{\Gamma c}{\Gamma a} \sum_{k_1, \ldots, k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)\left(b_1\right)_{k_1}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(x_1 z^{\tau_1}\right)^{k_1}}{k_1!}\ldots\frac{\left(x_n z^{\tau_n}\right)^{k_n}}{k_n!} dz$$

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1, \ldots, k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)\left(b_1\right)_{k_1}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(x_1\right)^{k_1}}{k_1!}\ldots$$

$$\frac{\left(x_n\right)^{k_n}}{k_n!} \int_0^{\infty} e^{-sz} z^{\gamma+\tau_1 k_1+\ldots+\tau_n k_n-1} dz$$

Now using Laplace Transform property, $\mathrm{L}(t^{n-1}) = \frac{\Gamma n}{s^n}$, we get

$$= \frac{1}{s^{\gamma}} \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)\left(b_1\right)_{k_1}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{1}{k_1!}\left(\frac{x_1}{s^{\tau_1}}\right)^{k_1} \cdots \frac{1}{k_n!}\left(\frac{x_n}{s^{\tau_n}}\right)^{k_n}$$

Theorem 7. (Euler (beta) transform). If $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, \alpha, b_1, \ldots, b_n, c \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$, $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(\alpha) > 0$, then

$$\int_0^1 z^{\alpha-1}(1-z)^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a; b_1,\ldots,b_n; c; x_1(1-z)^{\tau_1},\ldots,x_n(1-z)^{\tau_n}\right) dz$$

$$= \frac{\Gamma\alpha\Gamma c}{\Gamma(\alpha+c)} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a; b_1,\ldots,b_n; \alpha+c; x_1,\ldots,x_n\right)$$

Proof: using equation(5) in left hand side and apply Beta function property(7) we get

$$= \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma\left(a+\tau_1 k_1+\ldots+\tau_n k_n\right)\left(b_1\right)_{k_1}\ldots\left(b_n\right)_{k_n}}{\Gamma\left(c+\tau_1 k_1+\ldots+\tau_n k_n\right)} \frac{\left(x_1\right)^{k_1}}{k_1!} \cdots$$

$$\frac{\left(x_n\right)^{k_n}}{k_n!} \int_0^1 z^{\alpha-1}(1-z)^{c+\tau_1 k_1+\ldots+\tau_n k_n-1} dz$$

and finally we apply beta function then we get required result.

Theorem 8. If $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}_+ = (0, \infty)$; $a, b_1, \ldots, b_n, c, \mu, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b_1) > 0$, $\mathrm{Re}(b_2) > 0, \ldots, \mathrm{Re}(b_n) > 0$, $\mathrm{Re}(c) > 0$, $\mathrm{Re}(\mu) > 0$
then

$$\frac{\Gamma(c+\mu)}{\Gamma\mu} \int_t^x (x-s)^{\mu-1}(s-t)^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a, b_1,\ldots,b_n; c; \lambda_1 (s-t)^{\tau_1},\ldots,\lambda_n (s-t)^{\tau_n}\right) ds$$

$$= \Gamma c\,(x-t)^{c+\mu-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}\left(a, b_1,\ldots,b_n; c+\mu; \lambda_1 (x-t)^{\tau_1},\ldots,\lambda_n (x-t)^{\tau_n}\right)$$

Proof: Consider

$$\frac{\Gamma(c+\mu)}{\Gamma\mu} \frac{1}{(x-t)^{\mu-1}} \int_t^x (x-s)^{\mu-1}(s-t)^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}$$

$$\left(a, b_1,\ldots,b_n; c; \lambda_1 (s-t)^{\tau_1},\ldots,\lambda_n (s-t)^{\tau_n}\right) ds$$

$$= \frac{\Gamma(c+\mu)}{\Gamma\mu} \int_t^x \left(\frac{(x-t)-(s-t)}{(x-t)}\right)^{\mu-1} (s-t)^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}$$

$$\left(a, b_1,\ldots,b_n; c; \lambda_1 (s-t)^{\tau_1},\ldots,\lambda_n (s-t)^{\tau_n}\right) ds$$

Put, $u = \frac{(s-t)}{(x-t)} \quad \Rightarrow \quad du = \frac{1}{(x-t)} ds$

$$= \frac{\Gamma(c+\mu)}{\Gamma\mu} \int_t^x (1-u)^{\mu-1} (u)^{c-1} (x-t)^{c-1} \mathrm{F}_{\mathrm{D}}^{(n)(\tau_1,\tau_2,\ldots,\tau_n)}$$

$$\left(a, b_1, \ldots, b_n; c; \lambda_1 u^{\tau_1} (x-t)^{\tau_1}, \ldots, \lambda_n u^{\tau_n} (x-t)^{\tau_n}\right) ds$$

Using equation (5) and simplify, we get

$$= \frac{\Gamma(c+\mu)}{\Gamma\mu} \frac{\Gamma c}{\Gamma a} \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\Gamma(a+\tau_1 k_1 + \ldots + \tau_n k_n)(b_1)_{k_1} \ldots (b_n)_{k_n}}{\Gamma(c+\tau_1 k_1 + \ldots + \tau_n k_n)} \frac{(\lambda_1)^{k_1}}{k_1 !} \ldots$$

$$\frac{(\lambda_n)^{k_n}}{k_n !} (x-t)^{c+\tau_1 k_1 + \ldots + \tau_n k_n} \int_0^1 (1-u)^{\mu-1} (u)^{c+\tau_1 k_1 + \ldots + \tau_n k_n - 1} du.$$

Again using beta function, we obtain the desired result.

Special case: If we put $\tau_2, \tau_3, \ldots, \tau_n = 0$, then the above theorems gives the results due to S.B. Rao et. al., [5].

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