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A DETERIORATING SYSTEM MAINTENANCE MODEL WITH IMPERFECT DELAYED REPAIR UNDER PARTIAL SUM PROCESS

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Abstract. In this paper, the partial sum process is introduced and proved that it is a decreasing monotone process. Considering the maintenance model for a deteriorating system with imperfect delayed repair for which successive operating times follows partial sum process and successive repair times assumes geometric process, an expression for the mean cost in long-run under N-policy is derived explicitly. An optimal policy N^* for minimizing the mean cost in long-run is determined analytically. To illustrate the theoretical results, a numerical example given.

1. Introduction

The study of a maintenance model for a basic repairable system is a fundamental and significant problem in reliability. A common assumption in the initial period of studying maintenance issues is that repair is perfect, a repairable framework after the repair is as good as new. Deteriorating systems have a different problem as the one portrayed above. For instance, in machine maintenance problems, after every repair, the working time of a machine will end up shorter and shorter, so the absolute working time or the existence of the machine must be limited. However, in perspective on the aging and aggregate wear, the repair time will turn out to be longer and tend to increase so that at the end the machine is non-repairable. Therefore, there is need to consider a repair replacement model for deteriorating systems, the progressive survival times are diminishing, while the repair times are expanding. The monotone process model would therefore be the most suitable model for a deteriorating system.

Lam (1988) first presented a Geometric Process Repair model to model a deteriorating system with the above characteristics.

Definition 1.1. The sequence $\{X_n,\ n = 1, 2, 3, ...\}$ of non negative independent random variables is called a geometric process, if the distribution function of X_n is given by $F\left(a^{n-1}x\right)$ for $n = 1, 2, 3, ...$, where $a(> 0)$is a constant

Finkelstein (1993) generalizes the *geometric process* to *particular deteriorating renewal process* in which the distribution function of X_n is $F\left(a_n x\right)$ where a_n are scale parameters. It involves a large number of parameter which can be troublesome in actual applications.

To overcome this, the *partial sum process* is introduced in which the parameters a_n are related by $a_n = a_0 + a_1 + ... + a_{n-1}$ for $n = 1, 2, 3, ...$

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Definition 1.2. Let $\{X_n,\ n = 1, 2, 3, ...\}$ be a sequence of independent non-negative random variables and let $F(x)$ be the distribution function of X_1. Then $\{X_n,\ n = 1, 2, 3, ...\}$ is called a partial sum process, if the distribution function of X_{i+1} is $F(\beta_i x)\quad (i = 1, 2, 3, ...)$ where $\beta_i > 0$ are constants with $\beta_i = \beta_0 + \beta_1 + \beta_2 + \cdots \beta_{i-1}$ and $\beta_0 = \beta > 0$.

Lemma 1.3. *For real $\beta_i\ (i = 1, 2, 3, ...)$, $\beta_i = 2^{i-1}\beta$.*

Proof. When $i = 1,\ \beta_1 = \beta_0 = \beta$. Thus, the result is true for $i = 1$.
Assume that the result is true for $i = n$.

$$\begin{aligned}\beta_{n+1} &= (\beta_0 + \beta_1 + \beta_2 + ... + \beta_{n-1}) + \beta_n \\ &= 2\beta_n \\ &= 2 \times 2^{n-1}\beta \qquad (by\ induction\ assumption) \\ &= 2^{(n+1)-1}\beta\end{aligned}$$

Thus, the result is true for $i = n + 1$ also. □

By Lemma 1.3, the distribution function of X_{i+1} is $F\left(2^{i-1}\beta\right)$ for $i = 1, 2, 3, ...$

Lemma 1.4. *The partial sum process $\{X_n,\ n = 1, 2, 3, ...\}$ with parameter $\beta(> 0)$ is stochastically decreasing and hence it is a monotone process.*

Proof. Note that for any $\alpha \geq 0$,

$$F(\alpha) \leq F(\beta\alpha) \leq F(2\beta\alpha) \leq ... \leq F\left(2^{n-1}\beta\alpha\right)$$
$$\Rightarrow P(X_1 > \alpha) \geq P(X_2 > \alpha) \geq P(X_3 > \alpha) \geq ... \geq P(X_n > \alpha)$$

This implies that $\{X_n,\ n = 1, 2, 3, ...\}$ is stochastically decreasing(Ross(1983)). □

It is easy to verify the following lemma.

Lemma 1.5. *Let $E(X_1) = \gamma$. Then for $i = 1, 2, 3, ...$*

$$E(X_{i+1}) = \frac{\gamma}{2^{i-1}\beta}$$

In the greater part of the research works for the simple repairable systems, there is a supposition that the framework will be fixed immediately when it fails. This, however, is not often the situation. In the real world, for instance, the system after failure can't be fixed promptly on the grounds that the repairman might be on a vacation. This will cause a deferred fix time. In any case, repairs are not constantly postponed. That is the repair can be done immediately or the repair can be delayed. This type of repair is often referred to as the imperfect delayed repair(see, Zhang(2017)).

In Geometric process, the operating times of a system are uniformly decreasing. But practically, it is not uniform. Thus, we propose the "Partial sum process" to model it, because *partial sum process* has more number of parameters.

In the following sections, a deteriorating system maintenance model with imperfect delayed repair for which successive operating times follows partial sum process and successive repair times assumes geometric process is studied.

In section 2, the model descriptions are given. The mean cost in long-run expression is determined explicitly and an optimal policy N^* for minimizing the mean cost in long-run is determined analytically in section 3. To demonstrate the theoretical results, a numerical example is provided in section 4.

2. Model Descriptions

We consider the deteriorating system maintenance model with the descriptions as follows.

2.1 A new simple repairable system is used at the start. At whatever point the system fails, it might be repaired or replaced by a new and similar one.

2.2 Let X_1 be the operating time before the first failure and let $F(x)$be the distribution function of X_1. Assume that $\gamma_1 = E(X_1) = \gamma > 0$. Let X_{i+1} be the operating time after the i-th repair for $i = 1, 2, 3, ...$. Then the distribution function of $X_{i+1}$is $F\left(2^{i-1}\beta\right)$ where $\beta > 0$ is a constant and $\gamma_{i+1} = E(X_{i+1}) = \dfrac{\gamma}{2^{i-1}\beta}$ for $i = 1, 2, 3, ...$. The successive operating times $\{X_n,\ n = 1, 2, 3, ...\}$ after repair constitute a *partial sum process*.

2.3 After the n^{th} failure, Let Y_n be the repair time and let the distribution function of Y_n be $F\left(a^{n-1}x\right)$, where $0 < a \leq 1$ is a constant. That is the successive repair times $\{Y_n,\ n = 1, 2, 3, ...\}$ constitute a increasing geometric process(Lam(1988)) or a renewal process. Also, assume that $E(Y_1) = \xi \geq 0$, where $\xi = 0$ means that expected repair time is negligible. By Lam(1988), $E(Y_i) = \dfrac{\xi}{a^{i-1}},\ \ i = 1, 2, 3, ...$.

2.4 When the system fails, the repair is postponed with probability p and is immediate with probability $1 - p$.

2.5 After the n-th failure, let $\{D_n,\ n = 1, 2, 3, ...\}$ denote the delayed repair time and let it be a sequence of iid random variables. Further, let $E(D_n) = \nu \geq 0,\ n = 1, 2, 3, ...$, where $\nu = 0$ negligible delayed repair time.

2.6 Let the random variable Z denote the replacement time with mean τ.

2.7 $X_n,\ Y_n, D_n,\ n = 1, 2, 3, ...$ and Z are independent of each other.

2.8 The reward rate is r, the repair cost rate is c and the the basic replacement cost is R. Let c_P be the proportional cost associated with the duration of the replacement time Z.

2.9 The system can not incur costs or generate income when waiting for repairs.

2.10 The n-th cycle of the system $(n = 1, 2, 3, ...)$ is the time period between the $(n - 1)$-st repair completion and the n-th repair completion. In the n-th cycle, let A_n denote the event that the repair is postponed and let $\overline{A_n}$ denote the event that the repair is undelayed.

3. The replacement policy N

Definition 3.1. The replacement policy N is a policy that replaces the system after the N-th failure since the last replacement.

The period between the system initialization and the first replacement or a period of two successive replacements is called a cycle. The subsequent cycles will establish a renewal process. Then, according to renewal reward theorem, Ross(1983), the mean cost per unit time in long-run under N-policy is given by

$$C(N) = \frac{\text{the mean of incurred cost in a cycle}}{\text{the mean duration of a cycle}}$$

$$= \frac{cE\left(\sum_{i=1}^{N-1} Y_i\right) + R + c_p E(Z) - rE\left(\sum_{i=1}^{N} X_i\right)}{E\left(\sum_{i=1}^{N} X_i\right) + E\left[\sum_{i=1}^{N-1} (D_i + Y_i)\,\chi_{A_i}\right] + E\left(\sum_{i=1}^{N-1} Y_i \chi_{\overline{A_i}}\right) + E(Z)}$$

where $\chi_A(.)$ denotes the indicator function. Then

$$C(N) = \frac{cE\left(\sum_{i=1}^{N-1} Y_i\right) + R + c_p E(Z) - rE\left(\sum_{i=1}^{N} X_i\right)}{E\left(\sum_{i=1}^{N} X_i\right) + E\left[\sum_{i=1}^{N-1} (D_i + Y_i)\right] p + E\left(\sum_{i=1}^{N-1} Y_i\right)(1-p) + E(Z)}$$

$$= \frac{c\sum_{i=1}^{N-1} \xi_i + R + c_p\tau - r\sum_{i=1}^{N} \gamma_i}{\sum_{i=1}^{N} \gamma_i + p(N-1)\nu + \sum_{i=1}^{N-1} \xi_i + \tau}$$

$$= \frac{(c+r)\sum_{i=1}^{N-1} \xi_i + R + \left(c_p + r\right)\tau + rp(N-1)\nu}{\sum_{i=1}^{N} \gamma_i + p(N-1)\nu + \sum_{i=1}^{N-1} \xi_i + \tau} - r \tag{3.1}$$

$$= \frac{(c+r)\,\xi\sum_{i=1}^{N-1} \frac{1}{a^{i-1}} + R + \left(c_p + r\right)\tau + rp(N-1)\nu}{\gamma\left(1 + \frac{1}{\beta}\sum_{i=2}^{N} \frac{1}{2^{i-2}}\right) + p(N-1)\nu + \xi\sum_{i=1}^{N-1} \frac{1}{a^{i-1}} + \tau} - r \tag{3.2}$$

For minimizing $C(N)$, we shall evaluate the difference $C(N+1) - C(N)$.
From equation (1),

$$C(N+1) - C(N)$$

$$= \frac{\left(\begin{array}{l} (c+r)\left(\xi_N\left[\sum_{i=1}^{N} \gamma_i + p(N-1)\nu + \tau\right] - \sum_{i=1}^{N-1} \xi_i\left[\gamma_{N+1} + p\nu\right]\right) \\ -\left(R + \left(c_p + r\right)\tau + rpN\nu\right)\left[\gamma_{N+1} + p\nu + \xi_N\right] \\ +rp\nu\left[\sum_{i=1}^{N+1} \gamma_i + pN\nu + \sum_{i=1}^{N} \xi_i + \tau\right] \end{array}\right)}{\left(\sum_{i=1}^{N+1} \gamma_i + pN\nu + \sum_{i=1}^{N} \xi_i + \tau\right)\left(\sum_{i=1}^{N} \gamma_i + p(N-1)\nu + \sum_{i=1}^{N-1} \xi_i + \tau\right)}$$

$$= \frac{\left(\begin{array}{l} (c+r)\left(\xi_N\left[\sum_{i=1}^{N}\gamma_i + p\,(N-1)\,\nu + \tau\right] - \sum_{i=1}^{N-1}\xi_i\,[\gamma_{N+1}+p\nu]\right) \\ -\left(R+\left(c_p+r\right)\tau\right)[\gamma_{N+1}+p\nu+\xi_N] \\ -rpN\nu\,[\gamma_{N+1}+p\nu+\xi_N] + rp\nu\left[\sum_{i=1}^{N+1}\gamma_i + pN\nu + \sum_{i=1}^{N}\xi_i + \tau\right] \end{array}\right)}{\left(\sum_{i=1}^{N+1}\gamma_i + pN\nu + \sum_{i=1}^{N}\xi_i + \tau\right)\left(\sum_{i=1}^{N}\gamma_i + p\,(N-1)\,\nu + \sum_{i=1}^{N-1}\xi_i + \tau\right)}$$

$$= \frac{\left(\begin{array}{l} (c+r)\left[\xi_N\left(\sum_{i=1}^{N}\gamma_i + p\,(N-1)\,\nu + \tau\right) - \sum_{i=1}^{N-1}\xi_i\,(\gamma_{N+1}+p\nu)\right] \\ -\left(R+\left(c_p+r\right)\tau\right)[\gamma_{N+1}+p\nu+\xi_N] \\ +rp\nu\left[\sum_{i=1}^{N+1}\gamma_i + \sum_{i=1}^{N}\xi_i + \tau - N\,(\gamma_{N+1}+\xi_N)\right] \end{array}\right)}{\left(\sum_{i=1}^{N+1}\gamma_i + pN\nu + \sum_{i=1}^{N}\xi_i + \tau\right)\left(\sum_{i=1}^{N}\gamma_i + p\,(N-1)\,\nu + \sum_{i=1}^{N-1}\xi_i + \tau\right)} \quad (3.3)$$

Let

$$A\,(N) = \frac{\left(\begin{array}{l} (c+r)\left[\xi_N\left(\sum_{i=1}^{N}\gamma_i + p\,(N-1)\,\nu + \tau\right) - \sum_{i=1}^{N-1}\xi_i\,(\gamma_{N+1}+p\nu)\right] \\ +rp\nu\left[\sum_{i=1}^{N+1}\gamma_i + \sum_{i=1}^{N}\xi_i + \tau - N\,(\gamma_{N+1}+\xi_N)\right] \end{array}\right)}{\left(R+\left(c_p+r\right)\tau\right)(\gamma_{N+1}+p\nu+\xi_N)} \quad (3.4)$$

$$= \frac{1}{\left(R+\left(c_p+r\right)\tau\right)}\left[(c+r)\,A_1\,(N) + rp\nu A_2\,(N)\right] \quad (3.5)$$

where

$$A_1\,(N) = \frac{\xi_N\left(\sum_{i=1}^{N}\gamma_i + p\,(N-1)\,\nu + \tau\right) - \sum_{i=1}^{N-1}\xi_i\,(\gamma_{N+1}+p\nu)}{\gamma_{N+1}+p\nu+\xi_N} \quad (3.6)$$

and

$$A_2\,(N) = \frac{\sum_{i=1}^{N+1}\gamma_i + \sum_{i=1}^{N}\xi_i + \tau - N\,(\gamma_{N+1}+\xi_N)}{\gamma_{N+1}+p\nu+\xi_N} \quad (3.7)$$

It is clear that the denominator term in equation (3.3) is positive. Thus from equations (3.3) and (3.4), we have

Lemma 3.2. *For $C\,(N)$ given by equation (3.1) and $A\,(N)$ by the equation (3.4), we have $C\,(N)$ is increasing(decreasing) if and only if $A\,(N) > 1$($A\,(N) < 1$).*

Next, we shall prove $A(N)$ is non-decreasing in N. From equations (3.6) and (3.7), we have

$$A_1\,(N+1) - A_1\,(N)$$

$$= \frac{\left(\sum_{i=1}^{N+1} \gamma_i + \sum_{i=1}^{N} \xi_i + pN\nu + \tau\right)[(\gamma_{N+1}\xi_{N+1} - \gamma_{N+2}\xi_N) + p\nu\,(\xi_{N+1} - \xi_N)]}{(\gamma_{N+2} + p\nu + \xi_{N+1})\,(\gamma_{N+1} + p\nu + \xi_N)} \tag{3.8}$$

$$A_2\,(N+1) - A_2\,(N)$$

$$= \frac{\left(\sum_{i=1}^{N+1} \gamma_i + \sum_{i=1}^{N} \xi_i + \tau + Np\nu\right)(\gamma_{N+1} - \gamma_{N+2} + \xi_N - \xi_{N+1})}{(\gamma_{N+2} + p\nu + \xi_{N+1})\,(\gamma_{N+1} + p\nu + \xi_N)} \tag{3.9}$$

Lemma 3.3. *$A(N)$ given in equation (3.5) is non-decreasing.*

Proof. Using equations (3.8) and (3.9), we have

$$A\,(N+1) - A\,(N)$$

$$= \frac{1}{\left(R + \left(c_p + r\right)\tau\right)}\begin{bmatrix} (c+r)\,(A_1\,(N+1) - A_1\,(N)) \\ +rp\nu\,(A_2\,(N+1) - A_2\,(N)) \end{bmatrix}$$

$$= \frac{\left(\begin{array}{l} \left(\sum_{i=1}^{N+1} \gamma_i + \sum_{i=1}^{N} \xi_i + pN\nu + \tau\right) \\ \qquad \times \begin{bmatrix} (c+r)\,((\gamma_{N+1}\xi_{N+1} - \gamma_{N+2}\xi_N) + p\nu\,(\xi_{N+1} - \xi_N)) \\ +rp\nu\,(\gamma_{N+1} - \gamma_{N+2} + \xi_N - \xi_{N+1}) \end{bmatrix} \end{array}\right)}{\left(R + \left(c_p + r\right)\tau\right)(\gamma_{N+2} + p\nu + \xi_{N+1})\,(\gamma_{N+1} + p\nu + \xi_N)}$$

$$= \frac{\left(\begin{array}{l} \left(\sum_{i=1}^{N+1} \gamma_i + \sum_{i=1}^{N} \xi_i + pN\nu + \tau\right) \\ \qquad \times \left(\begin{array}{l} c\left[\ (\gamma_{N+1}\xi_{N+1} - \gamma_{N+2}\xi_N) + p\nu\,(\xi_{N+1} - \xi_N)\ \right] \\ +r\,(\gamma_{N+1}\xi_{N+1} - \gamma_{N+2}\xi_N) + rp\nu\,(\gamma_{N+1} - \gamma_{N+2}) \end{array}\right) \end{array}\right)}{\left(R + \left(c_p + r\right)\tau\right)(\gamma_{N+2} + p\nu + \xi_{N+1})\,(\gamma_{N+1} + p\nu + \xi_N)}$$

This implies that $A(N)$ is non-decreasing, because γ_n is non-increasing and ξ_n is non-decreasing. ☐

Using Lemma (3.2) and Lemma (3.3), we have the following theorem.

Theorem 3.4. *For $A\,(N)$ given in equation (4),*

$$N^* = \min\{N|\ A\,(N) \geq 1\} \tag{3.10}$$

is the optimal replacement policy. Moreover, N^* is unique iff $A\,(N^*) > 1$.

4. Numerical Example

Let the parameter values be $c = 10$, $r = 40$, $c_p = 15$, $\gamma = 40$, $\xi = 10$, $\beta = 1.25$, $a = 0.95$, $R = 4000$, $\nu = 0.2, p = 0.1$ and $\tau = 10$.

From equations(3.2) and (3.4), we have

$$C\,(N) = \frac{500 \times \sum_{i=1}^{N-1} \frac{1}{0.95^{i-1}} + 4550 + 0.8 \times (N-1)}{40\left(1 + \frac{1}{1.25}\sum_{i=2}^{N} \frac{1}{2^{i-2}}\right) + 0.02 \times (N-1) + 10 \sum_{i=1}^{N-1} \frac{1}{0.95^{i-1}} + 10} \tag{4.1}$$

$$A\,(N) = \frac{\left[\begin{array}{l} 50\left(\begin{array}{r} \frac{10}{0.95^{N-1}}\left(40 + \frac{40}{1.25}\sum_{i=2}^{N}\frac{1}{2^{i-2}} + 0.02\,(N-1) + 10\right) \\ -\left(\frac{40}{1.252^{N-1}} + 0.02\right)\sum_{i=1}^{N-1}\frac{10}{0.95^{i-1}} \end{array}\right) \\ +0.8\left(\begin{array}{r} 40 + \frac{40}{1.25}\sum_{i=2}^{N+1}\frac{1}{2^{i-2}} + \sum_{i=1}^{N}\frac{10}{0.95^{i-1}} + 10 \\ -N\left(\frac{40}{1.252^{N-1}} + \frac{10}{0.95^{N-1}}\right) \end{array}\right) \end{array}\right]}{4550\left(\frac{40}{1.252^{N-1}} + 0.02 + \frac{10}{0.95^{N-1}}\right)} \tag{4.2}$$

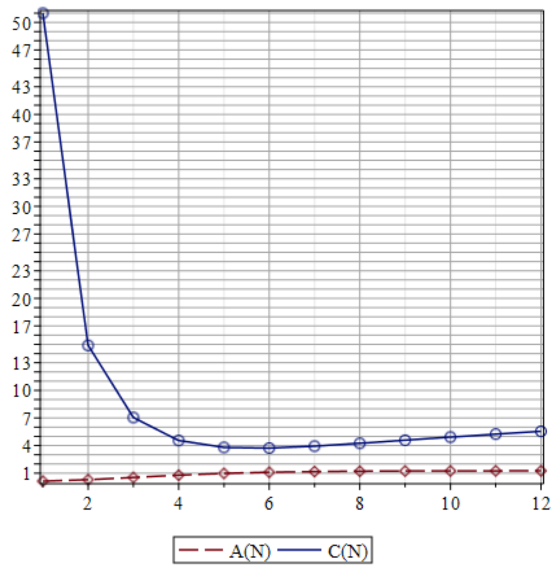
The results of (4.1) and (4.2) are tabulated in Table 1 and plotted in Figure 1.

TABLE 1. The results obtained using equations (4.1) and (4.2)

N	$C\,(N)$	$A\,(N)$	N	$C\,(N)$	$A\,(N)$
1	51.0000	0.1308	7	3.9358	1.1720
2	14.8881	0.2919	8	4.2447	1.2114
3	7.0447	0.5315	9	4.5836	1.2322
4	4.5477	0.7794	10	4.9205	1.2431
5	3.7998	0.9736	11	5.2422	1.2487
6	**3.7309**	**1.0993**	12	5.5441	1.2517

Obviously, At $N = 6$, $C\,(N)$ attains its minimum. Also $N = 6$ is the first value of N for which $A\,(N) \geq 1$. Further A (6)=1.0993 > 1.
Thus, the unique optimum replacement policy N^* is 6 and therefore at the sixth failure, system ought to be replaced.

FIGURE 1. The plots of $C(N)$ and $A(N)$ against N



5. Conclusion

The *partial sum process* is introduced for the first time and studied its application to deteriorating system maintenance model under imperfect delayed repair. An expression for the mean cost in long-run under N-policy is derived explicitly. An optimal policy N^* for minimizing the mean cost in long-run is determined analytically. The numerical example is also provided. In future, we can use *partial sum process* as monotone decreasing process for many maintenance models of deteriorating systems.

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