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A HYBRID METHOD FOR SOLVING THE MINIMUM TIME CONTROL PROBLEM

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Abstract. In this paper, a new hybrid method is presented to solve a minimum time optimal control problem. For this, we consider an auxiliary problem with free terminal time which is transformed into another one with a fixed terminal time. The proposed method combines the support method for optimal control problems with a fixed terminal time and an indirect finishing procedure, similar to the shooting method.
This approach by the auxiliary problem can be considered as a two-phase optimal control method, where we try to make the norm tending to zero while minimizing the time.

1. Introduction

Studying a dynamic system is often synonymous with improving its behavior [8]. Optimal control offers an ideal framework for evaluating the performance of a dynamic system. This is done by optimizing its objective functional whose value is determined by the behavior of the system on which we act by means of a control [31, 22].

For some optimal control problems the terminal time is not fixed and it will be considered as an unknown variable to be found. These problems known as optimal control problems with free terminal time have been extensively studied in the literature. The authors in [1], solve an optimal control problem with free final time by a method based on a transformation and a modified quasilinearization technique. In [23], the authors deal with a time delay optimal control problem in which the terminal time is a free parameter and they used an optimization method based on the gradient. Second order sufficient conditions for control problems with control-state constraints and free final time are presented in [25], using a transformation of the free terminal time problem into a fixed terminal time problem based on a Riccati's approach. Also, an optimal control problem with free terminal time is treated in [34] by using the constraint transcription and local smoothing technique to approximate the state inequality constraints to conventional ones, and the problem is solved by means of the filled function method. Another variant of these problems is studied in [2], where the problem treated has the particularity of having controls and constrained target sets varying over time.

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We deal with a minimum time optimal control problem when it is essentially about driving a system to a desired state in minimum time. So this is a particular case of the optimal control problems for a special form of the functional to be optimized. Since the publication of the first works on this subject in the sixties for linear systems [21], several researchers have studied the model [27, 10, 12, 29, 26, 11]. Soon after, the support method [15, 6, 32, 7] is at the origin of several works dealing with the numerical resolution of optimal control problems [14, 35]. The theoretical and practical interests have made that the minimum time optimal control problem remains one of the most attractive subjects in optimal control. Based on the concept of support, a numerical method was described in [18] using differential equations for the optimal support and the optimal value of the cost function. In [20], by taking the initial state values depending on a parameter, the authors have studied the problem of the solution structure identification for small parameter perturbations.

Methods based on other concepts are also proposed for solving the time optimal control problem. The authors in [30] have considered the problem of structural stability for the minimum time problem in the case when it has a bang-bang strongly local optimal control which exhibits a double switch. In [24], the authors have obtained the optimal time by solving a sequence of norm optimal control problems that is equivalent to solve a nonlinear equation by an iteratively reweighted least square method. Other instances of the time optimal control problem have been treated such as multivariable and discrete dynamic systems [28, 9, 36].

In this paper, we present a new hybrid method to solve the time optimal control problem. This method combines the support method of a nonlinear-quadratic problem of optimal control [16, 17] and an indirect finishing procedure [15, 5, 3, 4, 19]. For this, we proceed in two steps, in the first one we transform an auxiliary optimal control problem with a free terminal time into a bilinear-quadratic problem with a fixed terminal time. We solve the latter by means of the support method and this gives an idea about the optimal structure of the commutations, that allows us to calculate an approximate optimal control and an approximate minimal time. In the second step, we use a finishing procedure similar to the shooting method [33] in order to compute the minimum time more precisely. This procedure consists of solving a system of equations using the Newton method, where the initial iteration is deduced from the approximations obtained in the first step.

2. Problem statement and definitions

Consider the following time optimal control problem:

$$\min J(u_1, t_*) = t_*, \tag{2.1}$$

$$\dot{x}(t) = Ax(t) + bu_1(t),\ x(0) = x^0, \tag{2.2}$$

$$x(t_*) = 0, \tag{2.3}$$

$$u_1(t) \in U = [-L, L], t \geq 0, \tag{2.4}$$

where $\dot{x}(t) = \dfrac{dx}{dt}$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is an n-vector of state at instant t; $x(0) = x^0 \neq 0$ is an initial state; $u_1(t)$ is a piecewise continuous scalar function

called a control that may take any value from the set U; A is an $n \times n$ matrix which characterizes the system; b is an n-vector and L a positive constant.
The functional $J(u_1, t_*)$ is called the quality criterion, where the terminal time t_* is not fixed but rather considered as an unknown variable to be found. The symbol $(^T)$ denotes the transposition operation. Furthermore, we suppose that the dynamic system (2.2) is controllable. This means that we will assume that the state vector x can be brought from $x(0) = x^0$ to $x(t_*) = 0$ in a finite time using an admissible control $u \in U$ defined as follows :

Definition 2.1. The control $u_1(t)$, $t \in T(t_*) = [0, t_*]$, is called an admissible control of the problem (2.1)-(2.4) if:

(i) it is a piecewise continuous function on T, continuous on the right in its break points:

$$\lim_{t \to t_j, t > t_j} u_1(t) = u_1(t_j + 0) = u_1(t_j),\ j = 1, \dots, s;$$

(ii) $-L \leq u_1(t) \leq L$, $t \in T$;

(iii) the corresponding trajectory satisfies: $x(t_*) = 0$.

Definition 2.2. The admissible control $u_1^*(t)$, $t \in [0, \tau_*]$, is said to be optimal if it minimizes the quality criterion:

$$J(u_1^*, \tau_*) = \min_{u_1, t_*} J(u_1, t_*), \tag{2.5}$$

where u_1 is any control taken among the admissible controls and $t_* > 0$. The corresponding trajectory $x^*(t)$, $t \in [0, \tau_*]$, is said to be an optimal trajectory.
Furthermore, we call a suboptimal control (or $\varepsilon-$optimal), any admissible control $u_1^\varepsilon(t)$, $t \in [0, t_*^\varepsilon]$, satisfying the inequality

$$J(u_1^\varepsilon, t_*^\varepsilon) - J(u_1^*, \tau_*) \leq \varepsilon, \tag{2.6}$$

where the couple (u_1^*, τ_*) is optimal in the problem (2.1)-(2.4), and ε is a nonnegative number chosen as an accuracy.
For the classical minimum time problem, we recall the well known maximum principle of Pontryagin.

Theorem 2.3 (Maximum Principle [31])**.** *For the instant τ_* and the admissible control $u_1^*(t)$, $t \in T = [0, \tau_*]$, to be optimal in the problem (2.1)-(2.4), it is necessary and sufficient that the following conditions hold:*

(i) *The Hamiltonian $H(x, \Psi, u_1) = \Psi^T(Ax + bu_1)$ reaches its maximum:*

$$H(x^*(t), \Psi^*(t), u_1^*(t)) = \max_{v \in U} H(x^*(t), \Psi^*(t), v),\ \ t \in T = [0, \tau_*],$$

(ii) *and must be zero at the terminal time*

$$H(x^*(\tau_*), \Psi^*(\tau_*), u_1^*(\tau_*)) = 0,$$

where $x^*(t)$ is the trajectory of the direct system (2.2) corresponding to $u_1^*(t)$; $\Psi^*(t)$ being a nonzero solution of the conjugate system:

$$\dot{\Psi} = -A^T \Psi = -\frac{\partial H}{\partial x}.$$

To solve numerically this problem, several methods have been proposed, using the maximum principle, dynamic programming, as well as other approaches [21, 27, 10, 12, 29, 26, 11, 18]. In the following, we present a new constructive approach consisting of two steps. To begin, we consider an auxiliary problem to approximate the minimum time and the zero terminal state. After that, a finishing procedure, similar to the shooting method, is used to calculate more precisely the optimal solution.

3. Auxiliary Problem with Free Terminal Time

Consider the following auxiliary optimal control problem:

$$\min J_a(u_1, t_*) = \frac{1}{2} \parallel x(t_*) \parallel^2 + t_*, \tag{3.1}$$

$$\dot{x}(t) = Ax(t) + bu_1(t),\ x(0) = x^0, \tag{3.2}$$

$$u_1(t) \in U = [-L, L], t \in T = [0, t_*]. \tag{3.3}$$

An admissible control of the problem (3.1)-(3.3) verifies the conditions (i) and (ii) of Definition 2.1. Here the problem of optimization consists in finding a time $\hat{t}_* > 0$, and an admissible control $\hat{u}_1(t)$, $t \in [0, \hat{t}_*]$, realizing the minimum of the functional (3.1). So we have the following lemma:

Lemma 3.1. *Let $\hat{t}_*$ and $\hat{u}_1(t)$, $t \in [0, \hat{t}_*]$, be an optimal couple in the problem (3.1)-(3.3). Then two cases may occur:*

(a) *If $\hat{x}(\hat{t}_*) = 0$, then $\hat{t}_*$ is the minimum time of the problem (2.1)-(2.4).*

(b) *If $\hat{x}(\hat{t}_*) \neq 0$, then the couple $(\hat{u}_1, \hat{t}_*)$ is not admissible in the problem (2.1)-(2.4) and we have:*

$$\tau_* \geq \frac{1}{2} \parallel \hat{x}(\hat{t}_*) \parallel^2 + \hat{t}_*, \tag{3.4}$$

where τ_* is the minimum time of the problem (2.1)-(2.4).

Proof.

(a) Suppose that $\hat{t}_*$ is not the minimum time in the problem (2.1)-(2.4). So there exist a time instant $\bar{t}_* > 0$ and a control $\bar{u}_1$ such that $\bar{x}(\bar{t}_*) = 0$ and $\bar{t}_* < \hat{t}_*$. Hence,

$$J_a(\bar{u}_1, \bar{t}_*) = \frac{1}{2} \parallel \bar{x}(\bar{t}_*) \parallel^2 + \bar{t}_* < \frac{1}{2} \parallel \hat{x}(\hat{t}_*) \parallel^2 + \hat{t}_*,$$

which contradicts the optimality of $(\hat{u}_1, \hat{t}_*)$ in the problem (3.1)-(3.3). Therefore, $\hat{t}_*$ is the minimum time of the problem (2.1)-(2.4).

(b) Let (u_1^*, τ_*) be a couple which is the solution to the problem (2.1)-(2.4). In virtue of the optimality of the couple $(\hat{u}_1, \hat{t}_*)$ in the problem (3.1)-(3.3), then we will have:

$$J_a(u_1^*, \tau_*) = \frac{1}{2} \parallel x^*(\tau_*) \parallel^2 + \tau_* \geq \frac{1}{2} \parallel \hat{x}(\hat{t}_*) \parallel^2 + \hat{t}_*.$$

Since $x^*(\tau_*) = 0$, thus we obtain

$$\tau_* \geq \frac{1}{2} \parallel \hat{x}(\hat{t}_*) \parallel^2 + \hat{t}_*.$$

3.1. Maximum principle in the auxiliary problem with free terminal time. Let $\hat{t}_*$ and $\hat{u}_1(t)$, $t \in [0, \hat{t}_*]$, be an optimal solution of the auxiliary problem (3.1)-(3.3). For a such fixed time $\hat{t}_*$, the control $\hat{u}_1(t)$, $t \in [0, \hat{t}_*]$, is an optimal solution of the following problem:

$$\min J_a(u_1) = \frac{1}{2} \parallel x(\hat{t}_*) \parallel^2 + \hat{t}_*, \tag{3.5}$$

$$\dot{x}(t) = Ax(t) + bu_1(t), \; x(0) = x^0, \tag{3.6}$$

$$|u_1(t)| \leq L, \; t \in T = [0, \hat{t}_*]. \tag{3.7}$$

Indeed, if it is assumed that there is another admissible control $\widetilde{u}_1(t)$, $t \in T$, such that $J_a(\widetilde{u}_1) < J_a(\hat{u}_1)$, then this latter inequality would contradict the optimality of the couple $(\hat{u}_1, \hat{t}_*)$ in the problem (3.1)-(3.3). Hence, for the control $\hat{u}_1$, the following theorem holds for the fixed terminal time $\hat{t}_*$:

Theorem 3.2 (Maximum Principle [13])**.** *The control $\hat{u}_1(t)$, $t \in T = [0, \hat{t}_*]$, is optimal in the problem (3.5)-(3.7) if and only if, along $\hat{u}_1(t)$ and the corresponding trajectories $\hat{x}(t)$ (3.6) and $\hat{\Psi}(t)$ of the conjugate system:*

$$\dot{\Psi} = -A^T \Psi, \; \Psi(\hat{t}_*) = -\hat{x}(\hat{t}_*), \tag{3.8}$$

the hamiltonian $H(x, \Psi, u_1) = \Psi^T(Ax + bu_1)$ reaches its maximum:

$$H(\hat{x}(t), \hat{\Psi}(t), \hat{u}_1(t)) = \max_{v \in U} H(\hat{x}(t), \hat{\Psi}(t), v), \; t \in T = [0, \hat{t}_*]. \tag{3.9}$$

The following theorem provides the necessary and sufficient conditions that the optimal time $\hat{t}_*$ in the problem (3.1)-(3.3) must satisfy:

Theorem 3.3. *For the time $\hat{t}_*$ and the admissible control $\hat{u}_1(t)$, $t \in T = [0, \hat{t}_*]$, to be optimal in the problem (3.1)-(3.3), it is necessary and sufficient that the following conditions hold:*

(i) $H(\hat{x}(t), \hat{\Psi}(t), \hat{u}_1(t)) = \max_{v \in U} H(\hat{x}(t), \hat{\Psi}(t), v), t \in T = [0, \hat{t}_*]$;

(ii) $H(\hat{x}(\hat{t}_*), \hat{\Psi}(\hat{t}_*), \hat{u}_1(\hat{t}_*)) = 1$,

where $\hat{x}(t)$ is the trajectory of the direct system (3.2) corresponding to $\hat{u}_1(t)$ and $\hat{\Psi}(t)$ being the solution of the conjugate system (3.8).

Proof. For (i) see Theorem 3.2. For the condition (ii), let us compare the optimal process generated by the optimal control $\hat{u}_1(t)$, $t \in T = [0, \hat{t}_*]$, with two others processes for which the duration of one is greater than the optimal time $\hat{t}_*$, and the duration of the other less than $\hat{t}_*$. Indeed, let's first consider the control:

$$\widetilde{u}_1(t) = \begin{cases} \hat{u}_1(t), t \in [0, \hat{t}_*], \\ \hat{u}_1(\hat{t}_*), t \in [\hat{t}_*, \hat{t}_* + \varepsilon], \varepsilon > 0. \end{cases} \tag{3.10}$$

From the equation (3.6), we obtain the trajectory corresponding to the control $\widetilde{u}_1(t)$:

$$\begin{aligned}\widetilde{x}(t) &= \hat{x}(t),\ t \in [0, \hat{t}_*] \\ \widetilde{x}(\hat{t}_* + \varepsilon) &= \widetilde{x}(\hat{t}_*) + \varepsilon \frac{d\widetilde{x}(\hat{t}_*)}{dt} + o(\varepsilon) \\ &= \hat{x}(\hat{t}_*) + \varepsilon[A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)] + o(\varepsilon).\end{aligned}$$

Therefore, we will have:

$$\begin{aligned}0 \leq \Delta J_a(\hat{u}_1, \hat{t}_*) &= J_a(\widetilde{u}_1, \hat{t}_* + \varepsilon) - J_a(\hat{u}_1, \hat{t}_*) \\ &= \varepsilon \frac{\partial \varphi}{\partial x}(\hat{x}(\hat{t}_*), \hat{t}_*)[A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)] + \varepsilon \frac{\partial \varphi}{\partial t}(\hat{x}(\hat{t}_*), \hat{t}_*) + o(\varepsilon),\end{aligned}$$

where $\varphi(x,t) = \frac{1}{2} \parallel x \parallel^2 + t$, with $\frac{\partial \varphi}{\partial x} = x$ and $\frac{\partial \varphi}{\partial t} = 1$.
So we obtain

$$\begin{aligned}\Delta J_a(\hat{u}_1, \hat{t}_*) &= J_a(\widetilde{u}_1, \hat{t}_* + \varepsilon) - J_a(\hat{u}_1, \hat{t}_*) \\ &= \varepsilon \hat{x}^T(\hat{t}_*)[A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)] + \varepsilon + o(\varepsilon) \\ &= \varepsilon[\hat{x}^T(\hat{t}_*)(A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)) + 1 + \frac{o(\varepsilon)}{\varepsilon}] \geq 0.\end{aligned}$$

For a small enough $\varepsilon > 0$, we deduce that

$$\hat{x}^T(\hat{t}_*)[A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)] + 1 \geq 0.$$

Since $\hat{\Psi}(\hat{t}_*) = -\hat{x}(\hat{t}_*)$, then we can write:

$$H(\hat{x}(\hat{t}_*), \hat{\Psi}(\hat{t}_*), \hat{u}_1(\hat{t}_*)) \leq 1. \tag{3.11}$$

On the other hand, we consider the control $\bar{u}_1(t) = \hat{u}_1(t)$, $t \in [0, \hat{t}_* - \varepsilon]$, $\varepsilon > 0$. Then the corresponding trajectory verifies $\bar{x}(t) = \hat{x}(t), t \in [0, \hat{t}_* - \varepsilon]$. In particular, for a small enough $\varepsilon > 0$ we have:

$$\bar{x}(\hat{t}_* - \varepsilon) = \hat{x}(\hat{t}_* - \varepsilon) = \hat{x}(\hat{t}_*) - \varepsilon \frac{d\hat{x}(\hat{t}_*)}{dt} + o(\varepsilon). \tag{3.12}$$

Therefore, we obtain

$$\begin{aligned}\Delta J_a(\hat{u}_1, \hat{t}_*) &= J_a(\bar{u}_1, \hat{t}_* - \varepsilon) - J_a(\hat{u}_1, \hat{t}_*) \\ &= -\varepsilon \hat{x}^T(\hat{t}_*)[A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)] - \varepsilon + o(\varepsilon) \\ &= \varepsilon[-\hat{x}^T(\hat{t}_*)(A\hat{x}(\hat{t}_*) + b\hat{u}_1(\hat{t}_*)) - 1 + \frac{o(\varepsilon)}{\varepsilon}] \geq 0,\end{aligned}$$

consequently, we deduce:

$$H(\hat{x}(\hat{t}_*), \hat{\Psi}(\hat{t}_*), \hat{u}_1(\hat{t}_*)) \geq 1. \tag{3.13}$$

According to the inequalities (3.11) and (3.13), we conclude that the optimal time $\hat{t}_*$ in the problem (3.1)-(3.3) must verify the following equality:

$$H(\hat{x}(\hat{t}_*), \hat{\Psi}(\hat{t}_*), \hat{u}_1(\hat{t}_*)) = 1. \tag{3.14}$$

□

Remark 3.4. The case (a) of Lemma 3.1 can not occur, because the terminal condition (3.8) would give $H(\hat{x}(\hat{t}_*), \hat{\Psi}(\hat{t}_*), \hat{u}_1(\hat{t}_*)) = 0$, which would contradict the relation (3.14).

3.2. Solving the Auxiliary Problem. To use the optimality conditions of the auxiliary problem (3.1)-(3.3), we begin by transforming the latter into an equivalent control problem on the fixed time interval $[0,1]$. We will treat the unknown variable $t_* > 0$ as an additional control u_2, then the knowledge of u_1 and u_2 gives the possibility to reconstruct the state $x(t)$, $\forall t \in T = [0, t_*]$. So these observations will guide our approach:

(1) $s = \dfrac{t}{t_*}$ is a quantity included between 0 and 1, and this, whatever the value of t_*;

(2) $u_2 = t_*$ is an arbitrary positive constant.

This leads us to formulate everything in terms of the new variable $s \in [0,1]$.
The relation $t = s \times t_*$ allows us to set:

$$\begin{cases} \widetilde{x}(s) = x(s \times t_*), \\ \widetilde{u}_1(s) = u_1(s \times t_*), \qquad s \in [0,1]. \\ \widetilde{u}_2(s) = t_* \equiv const. \end{cases} \tag{3.15}$$

After this change of variable, $\widetilde{x}(\cdot)$ and $\widetilde{u}_1(\cdot)$ have respectively the same dimensions as $x(\cdot)$ and $u_1(\cdot)$. Furthermore, we have

$$\widetilde{x}(0) = x(0) = x^0 \text{ and } \widetilde{x}(1) = x(t_*).$$

Let $\widetilde{x}_{n+1}$ be a new variable such that $\widetilde{x}_{n+1}(s) = s \times t_*$. So this variable verifies

$$\dot{\widetilde{x}}_{n+1}(s) = t_* = \widetilde{u}_2, \;\; s \in [0,1], \;\; \widetilde{x}_{n+1}(0) = 0.$$

To define the new optimal control problem, we consider

$$z(s) = (\widetilde{x}(s), \widetilde{x}_{n+1}(s)) \in \mathbb{R}^n \times \mathbb{R}, \quad \widetilde{u}(s) = (\widetilde{u}_1(s), \widetilde{u}_2) \in U \times \mathbb{R}^+.$$

Thus, by increasing the size of the state vector and that of the control, via the change of variable $t = s \times t_*$, we reformulate the problem (3.1)-(3.3) into an equivalent optimal control problem on the fixed time interval $[0,1]$. Let us start with the quality criterion by defining the following function:

$$\begin{aligned} &\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \\ &z = (\widetilde{x}, \widetilde{x}_{n+1}) \mapsto \varphi(z) = \frac{1}{2}\|\widetilde{x}\|^2 + \widetilde{x}_{n+1}. \end{aligned}$$

So we have

$$\varphi(z(1)) = \frac{1}{2}\|\widetilde{x}(1)\|^2 + \widetilde{x}_{n+1}(1) = \frac{1}{2}\|x(t_*)\|^2 + t_*.$$

Let us write the dynamic system according to the new variable $s \in [0,1]$:

$$\begin{cases} \dot{\widetilde{x}}(s) = \dfrac{d}{ds}x(s \times t_*) = \dot{x}(s \times t_*)t_* = [A\widetilde{x}(s) + b\widetilde{u}_1(s)]\widetilde{u}_2, \;\; \widetilde{x}(0) = x^0, \\ \dot{\widetilde{x}}_{n+1}(s) = \widetilde{u}_2, \;\; \widetilde{x}_{n+1}(0) = 0. \end{cases} \tag{3.16}$$

Hence, this gives the following equivalent control problem with the fixed terminal time $s_* = 1$:

$$\min \widetilde{J}_a(\widetilde{u}) = \frac{1}{2} z^T(s_*)\widetilde{D}z(s_*) + \widetilde{c}^T z(s_*), \tag{3.17}$$

$$\dot{z}(s) = \begin{pmatrix} [A\widetilde{x}(s) + b\widetilde{u}_1(s)]\widetilde{u}_2 \\ \widetilde{u}_2 \end{pmatrix},\ z(0) = \begin{pmatrix} x^0 \\ 0 \end{pmatrix}, \tag{3.18}$$

$$\widetilde{u}(s) = (\widetilde{u}_1(s), \widetilde{u}_2) \in U \times \mathbb{R}^+,\ s \in [0, s_*], \tag{3.19}$$

where $\widetilde{D} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ is an $(n+1)\times(n+1)$ matrix and $\widetilde{c} = (0, 1)$ an $(n+1)$-vector whose the first n components are equal to zero. The vector $z(s) \in \mathbb{R}^{n+1}$ represents the state vector at instant s, $z(0) = z^0 = (x^0, 0)$ being the initial state of the system; $\widetilde{u}(s) = (\widetilde{u}_1(s), \widetilde{u}_2)$, $s \in [0, s_*]$, is the control of the system, where $\widetilde{u}_1(s)$ is a piecewise continuous scalar function with values in $U = [-L, L]$ and $\widetilde{u}_2$ is a parameter of $\mathbb{R}^+$. Let $\widetilde{u}^*(s) = (\widetilde{u}_1^*(s), \widetilde{u}_2^* = t_*^*)$ be the optimal control obtained by solving the equivalent auxiliary problem (3.17)-(3.19) by the method [16], and $\widetilde{x}^*(s)$, $s \in [0, 1]$ be the corresponding optimal trajectory.

4. Finishing Procedure

The Finishing Procedure of the support method [15, 19], similar to the shooting method, constitutes the second phase of the elaborated approach. It consists in calculating the minimum time of the problem (2.1)-(2.4) with a good accuracy.

Here, we consider that the problem (2.1)-(2.4) is simple, so the optimal control $\widetilde{u}_1^*(s)$ has $(n-1)$ switching points such that $0 < s_1^0 < s_2^0 < \dots < s_{n-1}^0 < 1$. Returning to the variable t, we set:

$$\tau_*^0 = t_* = \frac{1}{2} \parallel \widetilde{x}^*(1) \parallel^2 + t_*^*,\ \tau_i^0 = s_i^0 t_*,\ i = 1, \dots, n-1,\ T_s^0 = (\tau_1^0, \dots, \tau_{n-1}^0)^T.$$

We form the following $n \times (n-1)$ matrix supposed to be of complete rank:

$$Q_0 = (q(t_*, \tau_i^0),\ i = 1, \dots, n-1), \tag{4.1}$$

with $q(t_*, t) = e^{A(t_*-t)}b,\ t \in [0, t_*]$.

We calculate the vector of the potentials $\lambda^0 = (y^0, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that $Q_0^T \lambda^0 = 0$, and the cocontrol $E^0(t) = (\Psi^0)^T(t)b$, where $\Psi^0(t)$, $t \in T = [0, t_*]$, is the solution of the conjugate system

$$\dot{\Psi} = -A^T\Psi,\ \Psi(t_*) = -\lambda^0.$$

Hence

$$E^0(t) = -(\lambda^0)^T e^{A(t_*-t)}b = -(\lambda^0)^T q(t_*, t) = -((y^0)^T, 1)q(t_*, t),\ t \in [0, t_*].$$

Furthermore, we calculate the quasicontrol

$$\omega^0(t) = L\ signE^0(t),\ t \in T = [0, t_*],$$

and the trajectory $\kappa^0(t)$, $t \in T = [0, t_*]$, of the system (2.2), corresponding to $\omega^0(t)$, $t \in T$. Since the problem is simple, we assume that $\dot{E}^0(\tau_i^0) \neq 0,\ i = 1, \dots, n-1$.

So the finishing procedure consists in finding the solution $\upsilon = (y, \tau_*, T_s) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}$, $\tau_* \in \mathbb{R}^+, T_s = (\tau_1, \tau_2, \dots, \tau_{n-1})^T$, of $(2n-1)$ equations:

$$\begin{aligned} F_1(\upsilon) &= F_1(y, \tau_*, T_s) = \kappa(\tau_*, T_s) = (\kappa_j(\tau_*, T_s) = 0,\ j = 1, \dots, n), \\ F_2(\upsilon) &= F_2(y, \tau_*, T_s) = E(y, \tau_*, T_s) = (E_i(y, \tau_*, \tau_i) = E_i = 0,\ i = 1, \dots, n-1), \end{aligned} \tag{4.2}$$

where $E_i(y, \tau_*, t) = (y^T, 1)e^{A(\tau_* - t)}b$, $t \in [0, \tau_*]$, $i = 1, \dots, n-1$, and $\kappa(t, T_s)$, $t \in [0, \tau_*]$ is the trajectory (2.2) corresponding to the quasicontrol $\omega(t, T_s),\ t \in [0, \tau_*]$:

$$\omega(t, T_s) = \begin{cases} -L\ \mathrm{sign}\dot{E}^0(\tau_1^0), & \text{if } t \in [0, \tau_1[; \\ -L\ \mathrm{sign}\dot{E}^0(\tau_i^0), & \text{if } t \in [\tau_{i-1}, \tau_i[,\ \ i = 2, \dots, n-1; \\ \ \ L\ \mathrm{sign}\dot{E}^0(\tau_{n-1}^0), & \text{if } t \in [\tau_{n-1}, \tau_*]. \end{cases} \tag{4.3}$$

From (4.3) and the Cauchy Formula, we get:

$$\kappa(\tau_*, T_s) \quad = \quad e^{A\tau_*}x^0 + \int_0^{\tau_*} e^{A(\tau_* - t)}b\omega(t, T_s)dt.$$

By virtue of Lemma 3.1, so we can write:

$$\begin{aligned} \kappa(\tau_*, T_s) &= e^{A(\tau_* - t_* + t_*)}x^0 + \int_0^{t_*} e^{A(\tau_* - t_* + t_* - t)}b\omega(t, T_s)dt + \int_{t_*}^{\tau_*} e^{A(\tau_* - t_* + t_* - t)}b\omega(t, T_s)dt \\ &= e^{A(\tau_* - t_*)}[e^{At_*}x^0 + \int_0^{t_*} e^{A(t_* - t)}b\omega(t, T_s)dt + \int_{t_*}^{\tau_*} e^{A(t_* - t)}b\omega(t, T_s)dt] \\ &= e^{A(\tau_* - t_*)}X(\tau_*, T_s). \end{aligned}$$

Since $e^{A(\tau_* - t_*)}$ is always invertible, then after applying (4.3), the n first equations (4.2) are equivalent to:

$$X(\tau_*, T_s) = \kappa^0(t_*) + L\int_{t_*}^{\tau_*} q(t_*, t)sign\dot{E}^0(\tau_{n-1}^0)dt - 2L\sum_{k=1}^{n-1} sign\dot{E}^0(\tau_k^0)\int_{\tau_k^0}^{\tau_k} q(t_*, t)dt = 0.$$

Hence, this gives the following simplified system: $F(\upsilon) = (F_1(\upsilon), F_2(\upsilon))^T = 0$, i.e,

$$\begin{cases} F_1(\upsilon) &= F_1(y, \tau_*, T_s) = X(\tau_*, T_s) = (X_j(\tau_*, T_s) = 0,\ j = 1, \dots, n), \\ F_2(\upsilon) &= F_2(y, \tau_*, T_s) = E(y, \tau_*, T_s) = (E_i(y, \tau_*, \tau_i) = E_i = 0,\ i = 1, \dots, n-1). \end{cases} \tag{4.4}$$

In particular, for $\tau_* = t_*$ we get:

$$X(t_*, T_s) = \kappa^0(t_*) - 2L\sum_{k=1}^{n-1} \mathrm{sign}\dot{E}^0(\tau_k^0)\int_{\tau_k^0}^{\tau_k} q(t_*, t)dt, \tag{4.5}$$

and

$$\kappa(t_*, T_s) \quad = \quad e^{A(t_* - t_*)}X(t_*, T_s) = X(t_*, T_s) = 0,$$

and we obtain the standard formula (4.5) for the fixed terminal time $\tau_* = t_*$ [15, 16, 19].

We solve the system (4.4) via the Newton method, starting with the initial approximation $\upsilon^0 = (y^0, \tau_*^0, T_s^0)$. The $(k+1)^{th}$ approximation υ^{k+1} is equal to:

$$\upsilon^{(k+1)} = \upsilon^{(k)} - J^{-1}(\upsilon^{(k)})F(\upsilon^{(k)}), \tag{4.6}$$

where

$$J(v)=\begin{pmatrix} \dfrac{\partial X(\tau_*,T_s)}{\partial y} & \dfrac{\partial X(\tau_*,T_s)}{\partial \tau_*} & \dfrac{\partial X(\tau_*,T_s)}{\partial T_s} \\ \dfrac{\partial E(y,\tau_*,T_s)}{\partial y} & \dfrac{\partial E(y,\tau_*,T_s)}{\partial \tau_*} & \dfrac{\partial E(y,\tau_*,T_s)}{\partial T_s} \end{pmatrix}$$

is the Jacobian matrix of the system (4.4).

5. Numerical Example

For illustration, we consider the following minimum time optimal control problem [18]:

$$\min J(u,t_*)=t_* \tag{5.1}$$

$$\begin{aligned} \dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= u_1,\ |u_1|\le 1,\ t\in T(t_*)=[0,t_*], \end{aligned} \tag{5.2}$$

$$x=(x_1,x_2,x_3)^T,\ x^0=x(0)=(16,0,0)^T,\ x(t_*)=0, \tag{5.3}$$

with

$$A=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix},\ b=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

With the variable $s\in[0,1]$, we obtain the following auxiliary bilinear-quadratic problem of optimal control with a fixed terminal time:

$$\min \widetilde{J}_a(\widetilde{u})=\frac{1}{2}z^T(1)\widetilde{D}z(1)+\widetilde{c}^T z(1), \tag{5.4}$$

$$\begin{aligned} \dot{z}_1 &= \dot{\widetilde{x}}_1=\widetilde{x}_2\widetilde{u}_2,\\ \dot{z}_2 &= \dot{\widetilde{x}}_2=\widetilde{x}_3\widetilde{u}_2,\\ \dot{z}_3 &= \dot{\widetilde{x}}_3=\widetilde{u}_1\widetilde{u}_2,\\ \dot{z}_4 &= \dot{\widetilde{x}}_4=\widetilde{u}_2, \end{aligned} \tag{5.5}$$

$$z=(\widetilde{x},\widetilde{x}_4)=(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3,\widetilde{x}_4)^T,\ z^0=z(0)=(16,0,0,0)^T, \tag{5.6}$$

$$\widetilde{u}(s)=(\widetilde{u}_1(s),\widetilde{u}_2)\in U\times\mathbb{R}^+,\ s\in[0,1],\ U=[-1,1], \tag{5.7}$$

where $\widetilde{D}=\begin{pmatrix} I_3 & 0\\ 0 & 0\end{pmatrix}$ and $\widetilde{c}=(0,0,0,1)^T$.

Using the support method of optimal control [16], we obtain the optimal control $\widetilde{u}^*(s)=(\widetilde{u}_1^*(s),\widetilde{u}_2^*)$ such as:

$$\widetilde{u}_1^*(s)=\begin{cases} -1, & t\in[0,s_1^0[,\\ 1, & t\in[s_1^0,s_2^0[, \quad \widetilde{u}_2^*=7.1771,\\ -1, & t\in[s_2^0,1],\end{cases}$$

with $s_1^0 = 0.2772$, $s_2^0 = 0.8177$ and $\widetilde{J}(\widetilde{u}^*) = \dfrac{1}{2} \parallel \widetilde{x}^*(1) \parallel^2 + \widetilde{x}_4^*(1) = 7.5155$.
Returning to the variable t, we get:

$$\begin{aligned}
\tau_*^0 &= t_* = \widetilde{J}(\widetilde{u}^*) = 7.5155 \\
\tau_1^0 &= s_1^0 \times t_* = 2.0833 \\
\tau_2^0 &= s_2^0 \times t_* = 6.1454.
\end{aligned}$$

We calculate:

$$e^{At_*} = I_3 + t_* A + \frac{t_*^2}{2} A^2 = \begin{pmatrix} 1 & t_* & \dfrac{t_*^2}{2} \\ 0 & 1 & t_* \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7.5155 & 28.2414 \\ 0 & 1 & 7.5155 \\ 0 & 0 & 1 \end{pmatrix},$$

$$q(t_*, t) = \begin{pmatrix} 1 & t_* - t & \dfrac{1}{2}(t_* - t)^2 \\ 0 & 1 & t_* - t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2}(t_* - t)^2 \\ t_* - t \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(7.5155 - t)^2 \\ 7.5155 - t \\ 1 \end{pmatrix},$$

$$q(t_*, \tau_1^0) = \begin{pmatrix} 14.7544 \\ 5.4322 \\ 1 \end{pmatrix}, \; q(t_*, \tau_2^0) = \begin{pmatrix} 0.9386 \\ 1.3701 \\ 1 \end{pmatrix}, \; Q_0 = \begin{pmatrix} 14.7544 & 0.9386 \\ 5.4322 & 1.3701 \\ 1 & 1 \end{pmatrix},$$

and we solve the system

$$Q_0^T \lambda^0 = \begin{pmatrix} 14.7544 & 5.4322 & 1 \\ 0.9386 & 1.3701 & 1 \end{pmatrix} \begin{pmatrix} y_1^0 \\ y_2^0 \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{cases} 14.7544\, y_1^0 + 5.4322\, y_2^0 + 1 = 0 \\ 0.9386\, y_1^0 + 1.3701\, y_2^0 + 1 = 0. \end{cases}$$

Hence we find $\lambda^0 = (y_1^0, y_2^0, 1)^T = (0.2687, -0.9140, 1)^T$.
We determine the cocontrol $E^0(t)$ and the quasicontrol $\omega^0(t)$, $t \in [0, t_*]$:

$$\begin{aligned}
E^0(t) &= -(y^{0'}, 1) q(t_*, t) \\
&= -(0.2687, -0.9140, 1) \begin{pmatrix} \dfrac{1}{2}(7.5155 - t)^2 \\ 7.5155 - t \\ 1 \end{pmatrix} \\
&= -0.1344 t^2 + 1.1056 t - 1.7202, \\
\omega^0(t) &= \begin{cases} -1, & t \in [0, 2.0833[, \\ +1, & t \in [2.0833, 6.1454[, \\ -1, & t \in [6.1454, 7.5155]. \end{cases}
\end{aligned}$$

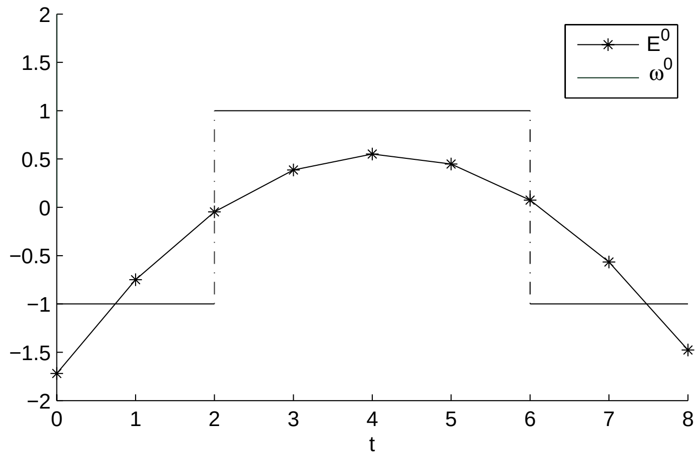

FIGURE 1. The cocontrol $E^0(t)$ and the quasicontrol $\omega^0(t)$, $t \in [0, t_*]$.

Hence, the finishing procedure is to find the solution $\upsilon = (y, \tau_*, T_s) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ of the following 5 equations:

$$X(\tau_*, T_s) = \begin{pmatrix} \frac{1}{6}(7.5155-\tau_*)^3 + \frac{1}{3}(7.5155-\tau_1)^3 - \frac{1}{3}(7.5155-\tau_2)^3 - 54.7493 \\ \frac{1}{2}\tau_*^2 - 7.5155\tau_* + \tau_1^2 - 15.0310\tau_1 - \tau_2^2 + 15.0310\tau_2 \\ -\tau_* - 2\tau_1 + 2\tau_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E(y, \tau_*, T_s) = \begin{pmatrix} E_1(y, \tau_*, \tau_1) \\ E_2(y, \tau_*, \tau_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y_1(\tau_* - \tau_1)^2 + y_2(\tau_* - \tau_1) + 1 \\ \frac{1}{2}y_1(\tau_* - \tau_2)^2 + y_2(\tau_* - \tau_2) + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For this, we form the Jacobian matrix:

$$J(\upsilon) = \begin{pmatrix} (\frac{\partial X(\tau_*,T_s)}{\partial y})_{3\times 2} & (\frac{\partial X(\tau_*,T_s)}{\partial \tau_*})_{3\times 1} & (\frac{\partial X(\tau_*,T_s)}{\partial T_s})_{3\times 2} \\ (\frac{\partial E(y,\tau_*,T_s)}{\partial y})_{2\times 2} & (\frac{\partial E(y,\tau_*,T_s)}{\partial \tau_*})_{2\times 1} & (\frac{\partial E(y,\tau_*,T_s)}{\partial T_s})_{2\times 2} \end{pmatrix}, \quad T_s = (\tau_1, \tau_2).$$

So

$$J(\upsilon) = \begin{pmatrix} 0 & 0 & -\frac{1}{2}(7.5155-\tau_*)^2 & -(7.5155-\tau_1)^2 & (7.5155-\tau_2)^2 \\ 0 & 0 & \tau_* - 7.5155 & -2(7.5155-\tau_1) & 2(7.5155-\tau_2) \\ 0 & 0 & -1 & -2 & 2 \\ \frac{1}{2}(7.5155-\tau_1)^2 & 7.5155-\tau_1 & y_1(t_*-\tau_1)+y_2 & -(y_1(t_*-\tau_1)+y_2) & 0 \\ \frac{1}{2}(7.5155-\tau_2)^2 & 7.5155-\tau_2 & y_1(t_*-\tau_2)+y_2 & 0 & -(y_1(t_*-\tau_2)+y_2) \end{pmatrix}.$$

We solve the system via the Newton method, starting with the initial approximation $\upsilon^0 = (y^0, \tau_*^0 = t*, T_s^0) = (0.2687, -0.9140, 7.5155, 2.0833, 6.1454)^T$, and the following results for $\varepsilon = 10^{-4}$ are found:

k	$\upsilon^{(k)}$	$J(\upsilon^{(k)})^{-1}F(\upsilon^{(k)})$	$\upsilon^{(k+1)}$	$\parallel \upsilon^{(k+1)} - \upsilon^{(k)}) \parallel$
0	$\begin{pmatrix} 0.2687 \\ -0.9140 \\ 7.5155 \\ 2.0833 \\ 6.1454 \end{pmatrix}$	$\begin{pmatrix} 0.1651 \\ -0.3820 \\ -0.5818 \\ 0.0796 \\ 0.0931 \end{pmatrix}$	$\begin{pmatrix} 0.1036 \\ -0.5320 \\ 8.0973 \\ 2.0037 \\ 6.0523 \end{pmatrix}$	0.7257
1	$\begin{pmatrix} 0.1036 \\ -0.5320 \\ 8.0973 \\ 2.0037 \\ 6.0523 \end{pmatrix}$	$\begin{pmatrix} -0.0610 \\ 0.1323 \\ 0.0951 \\ 0.0034 \\ 0.0509 \end{pmatrix}$	$\begin{pmatrix} 0.1647 \\ -0.6643 \\ 8.0022 \\ 2.0003 \\ 6.0014 \end{pmatrix}$	0.1813
2	$\begin{pmatrix} 0.1647 \\ -0.6643 \\ 8.0022 \\ 2.0003 \\ 6.0014 \end{pmatrix}$	$\begin{pmatrix} -0.0020 \\ 0.0024 \\ 0.0022 \\ 0.0003 \\ 0.0014 \end{pmatrix}$	$\begin{pmatrix} 0.1667 \\ -0.6667 \\ 8.0000 \\ 2.0000 \\ 6.0000 \end{pmatrix}$	0.0041
3	$\begin{pmatrix} 0.1667 \\ -0.6667 \\ 8.0000 \\ 2.0000 \\ 6.0000 \end{pmatrix}$	$10^{-5} \begin{pmatrix} -0.1224 \\ 0.0656 \\ 0.0905 \\ 0.0160 \\ 0.0613 \end{pmatrix}$	$\begin{pmatrix} 0.1667 \\ -0.6667 \\ 8.0000 \\ 2.0000 \\ 6.0000 \end{pmatrix}$	1.7747×10^{-6}

For this provided example, the minimum time in the problem (2.1)-(2.4) is $\tau_* = 8$, with the instants of commutation $\tau_1 = 2$ and $\tau_2 = 6$, corresponding to the exact optimal control of the problem [18], where

$$u^0(t) = \begin{cases} -1, & t \in [0, 2[, \\ +1, & t \in [2, 6[, \\ -1, & t \in [6, 8]. \end{cases}$$

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