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ON CONTROLLABILITY OF A SECOND-ORDER NON-AUTONOMOUS STOCHASTIC DELAY DIFFERENTIAL EQUATION

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Abstract. This paper studies a second-order non-autonomous stochastic delay differential equation in a Hilbert space. The objective is to provide sufficient conditions for approximate and optimal controllability of the stochastic system. To establish these results, we first demonstrate the existence and uniqueness of mild solution. We have used Banach contraction principle, the compact analytic semigroup of bounded linear operators, and stochastic analysis techniques. An example is included as an application to show the effectuality of the result.

1. Introduction

Controllability is a fundamental concept in the theory of control dynamical systems. It takes a significant role in investigating and designing various control dynamics processes. Physical problems, where some randomness appears, can be modeled by stochastic systems. Most researchers have investigated the controllability results for the autonomous and non-autonomous stochastic systems [1, 8, 12, 13, 22, 24, 29]. Controllability for first and second-order non-autonomous systems has been studied by many authors [16, 18, 20, 23, 30, 31].

Controllability theory aims the ability to control a particular system to the desired state. Exact controllability directs the system to an arbitrary final state. However, it is possible to drive the system to an arbitrarily small part of the desired state under approximate controllability. As well as the applications are concerned, the approximate controllability is more relevant to dynamical systems [10, 14, 17, 21, 27, 30].

The work of Albert Einstein and Smoluchowski on the theory of Brownian motions developed a new concept of stochastic differential equations. However, in 1940, a Japanese mathematician Kiyosi Itô established the mathematical theory of stochastic differential equations. A differential equation involving some stochastic parameters is called a stochastic differential equation. These equations are used to model various phenomena in many areas such as epidemiology, biology, mathematical finance, and unstable stock prices. For basic theory of the stochastic differential equations, refer to [2, 3, 11, 25].

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Since 1948, scientists have been attempting to determine what factors influence system's behaviour and how they can be controlled to achieve the desired result. This concept has been rigorously pursued and is now known as optimal control theory. Hestenes presented the first mathematical formulation of an optimal control problem in 1950. It has a wide range of applications in science, engineering, and operations research.

Optimal control theory is concerned with obtaining an optimized objective function for a specific system. It is also known as the extension theory of the calculus of variations. We achieve control policies by optimizing a control system with a cost functional that depends on control and state variables. We minimize the cost function by employing optimal control. For instance, the chemical mixture A can be regarded as a system. It has pH as a state variable x and strength as a control function u, and the goal is to find the final product quality, where the pH value can be controlled by the strength u of some components of A.

The optimal control problem is important in many scientific fields, including engineering, mathematics, and biomedicine. Wei et al. [32] examined the existence of optimal controls for the mixed-type impulsive integro-differential equation. The papers [6, 19, 27] contain some work on optimal controllability. The applicability of delay differential equations with stochastic term leads to the rapid development of differential equation theory, see [4, 5, 7, 9, 28]. These equations provide a new technique in many areas of science and economics. Moreover, this theory and its applications are currently receiving a lot of attention from researchers.

We study the approximate and optimal controllability of the following second-order non-autonomous stochastic delay differential equation:

$$\begin{cases} \dfrac{d}{dt}\big[u'(t) - h_1\big(t, u(t), u_t\big)\big] = A(t)u(t) + Bw(t) + h_2\big(t, u(t), u_t\big) \\ \qquad\qquad\qquad\qquad\qquad + h_3\big(t, u(t), u_t\big)\dfrac{dv(t)}{dt}, \quad t \in J_0 = [0, T_0], \\ u_0 = \phi \in \mathfrak{B}_\theta, \quad u'(0) = \chi_0 \in Z, \end{cases} \tag{1.1}$$

where $\{A(t)\}_{t\in J_0}$ generates a compact analytic semigroup of bounded linear operators in a Hilbert space Z. The domain $D\big(A(t)\big)$ is independent of t and is dense in Z i.e. $\overline{D\big(A(t)\big)} = Z$. The history function $u_t : (-\infty, 0] \to Z$, $u_t(\theta) = u(t+\theta)$ belongs to some abstract phase space $\mathfrak{B}_\theta$. Let W, H be Hilbert spaces such that $B \in L\big(W, Z\big)$ is bounded linear operator. Let Υ_t, $t \in J_0$ be a normal filtration and its complete probability space be $\big(\Omega, \Upsilon, P\big)$. Also, let v be a Q-Weiner process on $\big(\Omega, \Upsilon, P\big)$ having $tr(Q) < \infty$, where Q is a covariance operator. Let $L_2^0 = L_2\big(Q^{1/2}H, Z\big)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}H$ to Z with the norm $\|\psi\|_Q^2 = tr[\psi Q \psi^*]$. Moreover, $w \in L_\Upsilon^p\big(J_0, W\big)$ denotes the control function, u_0, χ_0 are Υ-measurable, Z-valued random variables independent of v. The functions h_1, h_2 and h_3 satisfy some suitable conditions to be specified in the next section.

Let $C\big(J_0, L^p(\Upsilon, Z)\big)$ be the Banach space of continuous maps defined on J_0 into $L^p\big(\Upsilon, Z\big)$ such that

$$\sup_{t\in J_0} E\|u(t)\|^p < \infty,$$

where $L^p\big(\Upsilon, Z\big)$ denotes the Banach space of all Υ-measurable, Z-valued and p-integrable random variables. If we assume that $C_2 = C_p\big(J_0, Z\big)$, then C_2 is a closed subspace of $C\big(J_0, L^p(\Upsilon, Z)\big)$ equipped with the norm

$$\|\psi\|_{C_2} = \left(\sup_{t\in J_0} E\|\psi(t)\|_Z^p\right)^{\frac{1}{p}}.$$

We define the phase space [15] as:

Let $\theta : (-\infty, 0] \to [0, \infty)$ be a continuous function s.t. $\displaystyle\int_{-\infty}^{0} \theta(t)dt < \infty$. For any $\alpha > 0$, define

$\mathfrak{B}_\theta = \Big\{\chi : (-\infty, 0] \to Z$ such that $\big(E\|\chi(\nu)\|^p\big)^{\frac{1}{p}}$ is bounded and measurable on $[-\alpha, 0]$ with $\phi(0) = 0$ and $\displaystyle\int_{-\infty}^{0} \theta(\nu) \sup_{\nu\in[0,s]} \big(E\|\chi(\nu)\|^p\big)^{\frac{1}{p}} d\nu < \infty\Big\}$.

$\mathfrak{B}_\theta$ is a Banach space with respect to the norm defined by

$$\|\chi\|_{\mathfrak{B}_\theta} = \int_{-\infty}^{0} \theta(\nu) \sup_{\nu\in[0,s]} \big(E\|\chi(\nu)\|^p\big)^{\frac{1}{p}} d\nu.$$

2. Preliminaries and Assumptions

The present section introduces various assumptions, notations, definitions and useful lemmas. Let ψ be a two-parameter evolution operator defined on $J_0 \times J_0$ to *L(Z)*, where *L(Z)* denotes the Banach space of all bounded linear operators on Z. For more details about the evolution operator and semigroup theory, see [17, 26]. We introduce another operator $\xi(t, s)$ associated with the evolution operator $\psi(t, s)$ as

$$\xi(t, s) = -\frac{\partial\psi(t, s)}{\partial s}.$$

Definition 2.1. The set of all possible final states in $[0, T_0]$ defined by

$$R_{T_0}(W) = \big\{u(T_0, u_0, \chi_0, w) : w \in W\big\}$$

is called the reachable set.

Definition 2.2. The control system (1.1) is approximately controllable on $[0, T_0]$, if

$$\overline{R_{T_0}(W)} = Z,$$

where $\overline{R_{T_0}(W)}$ denotes the closure of $R_{T_0}(W)$.

Definition 2.3. (Mild Solution) A stochastic process $u \in C_2$ is a mild solution of (1.1) if for each $w \in L^p_\Upsilon\big(J_0, W\big)$, it satisfies

$$\begin{aligned} u(t) &= \xi(t,0)\phi(0) + \psi(t,0)\big[\chi_0 - h_1\big(0, \phi(0), \phi\big)\big] + \int_0^t \xi(t,s)h_1\big(s, u(s), u_s\big)ds \\ &\quad + \int_0^t \psi(t,s)Bw(s)ds + \int_0^t \psi(t,s)h_2\big(s, u(s), u_s\big)ds \\ &\quad + \int_0^t \psi(t,s)h_3\big(s, u(s), u_s\big)dv(s). \end{aligned} \tag{2.1}$$

Lemma 2.4. [22] *For any $u_{T_0} \in L^p(\Upsilon_{T_0}, Z)$, there exists $X \in L^p_\Upsilon(J_0, L_2^0)$ such that*

$$u_{T_0} = Eu_{T_0} + \int_0^{T_0} X(s)dv(s).$$

Lemma 2.5. [21] *Let $\mu : J_0 \times \Omega \to L_2^0$ be strongly measurable mapping such that $\int_0^{T_0} E\|\mu(s)\|^p_{L_2^0} ds < \infty$. Then*

$$E\bigg\|\int_0^t \mu(s)dv(s)\bigg\|^p \le L_\mu \int_0^t E\|\mu(s)\|^p ds,$$

for all $0 \le t \le T_0$ and $p \ge 2$, where L_μ is the constant involving p and T_0.

Definition 2.6. A controllability map for the system (1.1) on J_0 is the bounded linear map $S^{T_0} : L^2(J_0, W) \to Z$ defined as

$$S^{T_0}w := \int_0^{T_0} \psi(T_0, s)Bw(s)ds,$$

and the controllability Grammian operator for (1.1) is given by

$$F_0^{T_0} := S^{T_0}(S^{T_0})^*,$$

where

$$F_0^{T_0} = \int_0^{T_0} \psi(t,s)BB^*\psi^*(t,s)ds,$$

where $*$ denotes the adjoint. The resolvent of $F_0^{T_0}$ is given by

$$R(\lambda, F_0^{T_0}) = (\lambda I + F_0^{T_0})^{-1}.$$

Consider the following assumptions:

(H1) There exist constants M, M' and M_1, $\hat{N} > 0$ such that

$$\|\psi(t,s)\| \le M, \ \|\xi(t,s)\| \le M', \ \|B\| \le M_1, \ \|u_t\|_{\mathfrak{B}_\theta} \le \hat{N}\|u(t)\|.$$

(H2) There exist constants L_ψ and $L_\xi > 0$ such that

$$\|\psi(t_2, s) - \psi(t_1, s)\| \le L_\psi|t_2 - t_1|,$$

and

$$\|\xi(t_2, s) - \xi(t_1, s)\| \le L_\xi|t_2 - t_1|.$$

(H3) For every $t \in J_0; u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in Z$, there exist constants M_{h_1} and $\tilde{M}_{h_1} > 0$ such that the nonlinear map $h_1 : J_0 \times Z \times \mathfrak{B}_\theta \to Z$ satisfies

(i) $E\|h_1(t, u_1, \tilde{u}_1) - h_1(t, u_2, \tilde{u}_2)\|^p \le M_{h_1}\big[\|u_1 - u_2\|^p + \|\tilde{u}_1 - \tilde{u}_2\|^p_{\mathfrak{B}_\theta}\big]$

(ii) $\|h_1(t,u,\tilde{u})\|^p \leq \tilde{M}_{h_1}\big(1+\|u\|^p+\|\tilde{u}\|^p_{\mathfrak{B}_\theta}\big)$.

(H4) For every $t \in J_0; u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in Z$, there exist constants M_{h_2} and $\tilde{M}_{h_2} > 0$ such that the nonlinear map $h_2 : J_0 \times Z \times \mathfrak{B}_\theta \to Z$ satisfies

(i) $E\|h_2(t,u_1,\tilde{u}_1) - h_2(t,u_2,\tilde{u}_2)\|^p \leq M_{h_2}\big[\|u_1-u_2\|^p + \|\tilde{u}_1-\tilde{u}_2\|^p_{\mathfrak{B}_\theta}\big]$

(ii) $E\|h_2(t,u,\tilde{u})\|^p \leq \tilde{M}_{h_2}\big(1+\|u\|^p+\|\tilde{u}\|^p_{\mathfrak{B}_\theta}\big)$.

(H5) For every $t \in J_0; u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in Z$, there exist constants M_{h_3} and $\tilde{M}_{h_3} > 0$ such that the nonlinear map $h_3 : J_0 \times Z \times \mathfrak{B}_\theta \to Z$ satisfies

(i) $E\|h_3(t,u_1,\tilde{u}_1) - h_3(t,u_2,\tilde{u}_2)\|^p \leq M_{h_3}\big[\|u_1-u_2\|^p + \|\tilde{u}_1-\tilde{u}_2\|^p_{\mathfrak{B}_\theta}\big]$

(ii) $E\|h_3(t,u,\tilde{u})\|^p \leq \tilde{M}_{h_3}\big(1+\|u\|^p+\|\tilde{u}\|^p_{\mathfrak{B}_\theta}\big)$.

(H6) (i) The resolvent operator $(\lambda I - A(t))^{-1}$, satisfies the following condition:

$$\big\|(\lambda I - A(t))^{-1}\big\| \leq \frac{C_0}{|\lambda|+1} \text{ for } Re(\lambda) \geq 0.$$

(ii) For each $t \in J_0$, the operator $\lambda\big(\lambda I + F_0^{T_0}\big)^{-1} \to 0$ in the strong operator topology as $\lambda \to 0^+$.

(iii) There exist constants $L_A > 0$ and $0 < \alpha \leq 1$ such that

$$\big\|\big(A(t) - A(\nu)\big)A(s)^{-1}\big\| \leq L_A|t-\nu|^\alpha, \quad \text{for } t, \nu, s \in J_0.$$

For any $\lambda > 0$ and u_{T_0}, we define the control function

$$\begin{aligned} w^\lambda\big(t,u,u_t\big) &= B^*\psi^*(T_0,t)\Big\{\big(\lambda I + F_0^{T_0}\big)^{-1}\Big(Eu_{T_0} - \xi(T_0,0)\phi(0) \\ &\quad -\psi(T_0,0)\big[\chi_0 - h_1\big(0,\phi(0),\phi\big)\big]\Big) + \int_0^{T_0}\big(\lambda I + F_0^{T_0}\big)^{-1}X(s)dv(s)\Big\} \\ &\quad -B^*\psi^*(T_0,t)\int_0^{T_0}\big(\lambda I + F_0^{T_0}\big)^{-1}\xi(T_0,s)h_1\big(s,u(s),u_s\big)ds \\ &\quad -B^*\psi^*(T_0,t)\int_0^{T_0}\big(\lambda I + F_0^{T_0}\big)^{-1}\psi(T_0,s)h_2\big(s,u(s),u_s\big)ds \\ &\quad -B^*\psi^*(T_0,t)\int_0^{T_0}\big(\lambda I + F_0^{T_0}\big)^{-1}\psi(T_0,s)h_3\big(s,u(s),u_s\big)dv(s). \end{aligned}$$

3. Existence and Uniqueness of Mild Solution

Lemma 3.1. *There exist constants $\hat{K}_1, \hat{K}_2 > 0$ such that* $E\big\|w^\lambda(t,u_1,(u_1)_t) - w^\lambda(t,u_2,(u_2)_t))\big\|^p$

$$\leq \frac{\hat{K}_1}{(|\lambda|+1)^p}\int_0^t \Big[E\big\|u_1(s)-u_2(s)\big\|^p + E\big\|(u_1)_s - (u_2)_s\big\|^p_{\mathfrak{B}_\theta}\Big]ds,$$

and

$$E\big\|w^\lambda(t,u,(u)_t)\big\|^p \leq \frac{\hat{K}_2}{(|\lambda|+1)^p}\Big[1 + \int_0^t \Big(E\big\|u(s)\big\|^p + E\big\|u_s\big\|^p_{\mathfrak{B}_\theta}\Big)ds\Big].$$

Proof. $E\big\|w^{\lambda}\big(t,u_1,(u_1)_t\big)-w^{\lambda}\big(t,u_2,(u_2)_t\big)\big\|^p$

$$\begin{aligned}
\leq\ & 3^{p-1}E\bigg\|B^*\psi^*(T_0,t)\int_0^t\big(\lambda I+F_0^{T_0}\big)^{-1}\xi(T_0,s)\big[h_1\big(s,u_1(s),(u_1)_s\big)\\
&\qquad\qquad -h_1\big(s,u_2(s),(u_2)_s\big)\big]ds\bigg\|^p\\
&+3^{p-1}E\bigg\|B^*\psi^*(T_0,t)\int_0^t\big(\lambda I+F_0^{T_0}\big)^{-1}\psi(T_0,s)\big[h_2\big(s,u_1(s),(u_1)_s\big)\\
&\qquad\qquad -h_2\big(s,u_2(s),(u_2)_s\big)\big]ds\bigg\|^p\\
&+3^{p-1}E\bigg\|B^*\psi^*(T_0,t)\int_0^t\big(\lambda I+F_0^{T_0}\big)^{-1}\psi(T_0,s)\big[h_3\big(s,u_1(s),(u_1)_s\big)\\
&\qquad\qquad -h_3\big(s,u_1(s),(u_2)_s\big)\big]dv(s)\bigg\|^p.
\end{aligned}$$

Using (H3)-(H6) and Holder's inequality, we have
$E\big\|w^{\lambda}\big(t,u_1,(u_1)_t\big)-w^{\lambda}\big(t,u_2,(u_2)_t\big)\big\|^p$

$$\begin{aligned}
\leq\ & \frac{3^{p-1}\big(M_1MC_0\big)^p}{(|\lambda|+1)^p}\Big[\big(M'\big)^pT_0^{\frac{p}{q}}M_{h_1}+M^pT_0^{\frac{p}{q}}M_{h_2}+M^pL_{h_3}M_{h_3}\Big]\\
&\times\int_0^t\Big(E\big\|u_1(s)-u_2(s)\big\|^p+E\big\|(u_1)_s-(u_2)_s\big\|^p_{\mathfrak{B}_\theta}\Big)ds\\
=\ & \frac{\hat{K}_1}{(|\lambda|+1)^p}\int_0^t\Big(E\big\|u_1(s)-u_2(s)\big\|^p+E\big\|(u_1)_s-(u_2)_s\big\|^p_{\mathfrak{B}_\theta}\Big)ds,
\end{aligned}$$

where

$$\hat{K_1}=3^{p-1}\big(M_1MC_0\big)^p\Big[\big(M'\big)^pT_0^{\frac{p}{q}}M_{h_1}+M^pT_0^{\frac{p}{q}}M_{h_2}+M^pL_{h_3}M_{h_3}\Big].$$

Similarly, one can prove the second inequality.

□

Theorem 3.2. *Assume that conditions (H1)-(H6) hold. Then the system (1.1) has a unique mild solution on* $[0,T_0]$ *provided that*

$$\begin{aligned}
(4^{p-1})^n(1+\hat{N})^n(T_0)^n\bigg[&(M')^pT_0^{\frac{p}{q}}M_{h_1}+(MM_1)^pT_0^{\frac{p}{q}}\frac{\hat{K}_1}{(|\lambda|+1)^p}+M^pT_0^{\frac{p}{q}}M_{h_2}\\
&+M^pL_{h_3}M_{h_3}\bigg]^n<1.
\end{aligned}$$

Proof. For any $\lambda>0$, define the operator

$$F_\lambda: C\big(J_0,L^p(\Upsilon,Z)\big)\to C\big(J_0,L^p(\Upsilon,Z)\big)\quad\text{by}$$

$$\begin{aligned}
\big(F_\lambda u\big)(t) &= \xi(t,0)\phi(0)+\psi(t,0)\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big] \\
&\quad +\int_0^t \xi(t,s)h_1\big(s,u(s),u_s\big)ds+\int_0^t \psi(t,s)Bw^\lambda\big(s,u(s),u_s\big)ds \\
&\quad +\int_0^t \psi(t,s)h_2\big(s,u(s),u_s\big)ds+\int_0^t \psi(t,s)h_3\big(s,u(s),u_s\big)dv(s).
\end{aligned}$$

Step 1: For any $u\in C\big(J_0,L^p(\Upsilon,Z)\big)$, $F_\lambda u$ is continuous on J_0 in the L^p-sense. Let $t_1,t_2\in[0,T_0]$ such that $t_1<t_2$, we have

$$\begin{aligned}
&E\big\|\big(F_\lambda u\big)(t_2)-\big(F_\lambda u\big)(t_1)\big\|^p \\
&\quad\le 10^{p-1}\Bigg[E\big\|\big[\xi(t_2,0)-\xi(t_1,0)\big]\phi(0)\big\|^p \\
&\qquad +E\big\|\big(\psi(t_2,0)-\psi(t_1,0)\big)\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big]\big\|^p \\
&\qquad +E\left\|\int_0^{t_1}\big[\xi(t_2,s)-\xi(t_1,s)\big]h_1\big(s,u(s),u_s\big)ds\right\|^p \\
&\qquad +E\left\|\int_{t_1}^{t_2}\xi(t_2,s)h_1\big(s,u(s),u_s\big)ds\right\|^p \\
&\qquad +E\left\|\int_0^{t_1}\big[\psi(t_2,s)-\psi(t_1,s)\big]Bw^\lambda\big(s,u(s),u_s\big)ds\right\|^p \\
&\qquad +E\left\|\int_{t_1}^{t_2}\psi(t_2,s)Bw^\lambda\big(s,u(s),u_s\big)ds\right\|^p \\
&\qquad +E\left\|\int_0^{t_1}\big[\psi(t_2,s)-\psi(t_1,s)\big]h_2\big(s,u(s),u_s\big)ds\right\|^p \\
&\qquad +E\left\|\int_{t_1}^{t_2}\psi(t_2,s)h_2\big(s,u(s),u_s\big)ds\right\|^p \\
&\qquad +E\left\|\int_0^{t_1}\big[\psi(t_2,s)-\psi(t_1,s)\big]h_3\big(s,u(s),u_s\big)dv(s)\right\|^p \\
&\qquad +E\left\|\int_{t_1}^{t_2}\psi(t_2,s)h_3\big(s,u(s),u_s\big)dv(s)\right\|^p\Bigg].
\end{aligned}$$

Using assumptions (H1), (H2) and Holder's inequality, we get
$E\big\|(F_\lambda u)(t_2)-(F_\lambda u)(t_1)\big\|^p$

$$\begin{aligned}
&\le 10^{p-1}\Big[E\big\|\big[\xi(t_2,0)-\xi(t_1,0)\big]\phi(0)\big\|^p \\
&\qquad +E\big\|\big[\psi(t_2,0)-\psi(t_1,0)\big]\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big]\big\|^p \\
&\qquad +t_1^{\frac{p}{q}}\int_0^{t_1}E\big\|\big[\xi(t_2,s)-\xi(t_1,s)\big]h_1\big(s,u(s),u_s\big)\big\|^p ds \\
&\qquad +(M')^p(t_2-t_1)^{\frac{p}{q}}\int_{t_1}^{t_2}E\big\|h_1\big(s,u(s),u_s\big)\big\|^p ds
\end{aligned}$$

$$
\begin{aligned}
&+t_1^{\frac{p}{q}}\int_0^{t_1}E\big\|\big[\psi(t_2,s)-\psi(t_1,s)\big]Bw^{\lambda}\big(s,u(s),u_s\big)\big\|^p ds\\
&+M^p(t_2-t_1)^{\frac{p}{q}}\int_{t_1}^{t_2}E\big\|Bw^{\lambda}\big(s,u(s),u_s\big)\big\|^p ds\\
&+t_1^{\frac{p}{q}}\int_0^{t_1}E\big\|\big[\psi(t_2,s)-\psi(t_1,s)\big]h_2\big(s,u(s),u_s\big)\big\|^p ds\\
&+M^p(t_2-t_1)^{\frac{p}{q}}\int_{t_1}^{t_2}E\big\|h_2\big(s,u(s),u_s\big)\big\|^p ds\\
&+L_{h_3}\int_0^{t_1}E\big\|\big[\psi(t_2,s)-\psi(t_1,s)\big]h_3\big(s,u(s),u_s\big)\big\|^p ds\\
&+M^pL_{h_3}\int_{t_1}^{t_2}E\big\|h_3\big(s,u(s),u_s\big)\big\|^p ds\Big].
\end{aligned}
$$

By using strong continuity of $\xi(t,s)$, $\psi(t,s)$ and Lebesgue' s dominated convergence theorem, we conclude that $E\big\|(F_\lambda u)(t_2)-(F_\lambda u)(t_1)\big\|^p \to 0$ as $t_2\to t_1$, which implies that $F_\lambda u$ is continuous on $[0,T_0]$.

Step 2: We show that $F_\lambda(C_2)\subset C_2$.

$$
\begin{aligned}
&E\big\|\big(F_\lambda u\big)(t)\big\|^p_{C_2}\\
\leq\;& 6^{p-1}\Big[\sup_{t\in J_0}E\big\|\xi(t,0)\phi(0)\big\|^p+\sup_{t\in J_0}E\big\|\psi(t,0)\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big]\big\|^p\\
&+\sup_{t\in J_0}\int_0^t E\big\|\xi(t,s)h_1\big(s,u(s),u_s\big)\big\|^p ds\\
&+\sup_{t\in J_0}\int_0^t E\big\|\psi(t,s)Bw^{\lambda}\big(s,u(s),u_s\big)\big\|^p ds\\
&+\sup_{t\in J_0}\int_0^t E\big\|\psi(t,s)h_2\big(s,u(s),u_s\big)\big\|^p ds\\
&+L_{h_3}\sup_{t\in J_0}\int_0^t E\big\|\psi(t,s)h_3\big(s,u(s),u_s\big)\big\|^p ds\Big]\\
\leq\;& 6^{p-1}\Big[(M')^pE\|\phi(0)\|^p+M^pE\big\|\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big]\big\|^p\\
&+(M')^p\tilde{M}_{h_1}T_0\Big(1+\|u\|^p_{C_2}+\|u_t\|^p_{\mathfrak{B}_\theta}\Big)\\
&+\frac{M^pM_1^p\hat{K}_2T_0}{(|\lambda|+1)^p}\Big(1+\|u\|^p_{C_2}+\|u_t\|^p_{\mathfrak{B}_\theta}\Big)\\
&+M^p\tilde{M}_{h_2}T_0\Big(1+\|u\|^p_{C_2}+\|u_t\|^p_{\mathfrak{B}_\theta}\Big)\\
&+L_{h_3}M^p\tilde{M}_{h_3}T_0\Big(1+\|u\|^p_{C_2}+\|u_t\|^p_{\mathfrak{B}_\theta}\Big)\Big].
\end{aligned}
$$

Above inequality implies that $\|F_\lambda u\|^p_{C_2}<\infty$. Since $F_\lambda u$ is continuous $[0,T_0]$, we have $F_\lambda(C_2)\subset C_2$.

Step 3: We show that for each fixed λ, there exists $n \in \mathbb{N}$ such that F_λ^n is a contraction on C_2. To prove this, let $u_1, u_2 \in C_2$ and $t \in [0, T_0]$, we have

$$\begin{aligned}
& E\big\|\big(F_\lambda u_1\big)(t) - \big(F_\lambda u_2\big)(t)\big\|^p \\
&\leq 4^{p-1}\bigg[E\bigg\|\int_0^t \xi(t,s)\big[h_1\big(s,u_1(s),(u_1)_s\big) - h_1\big(s,u_2(s),(u_2)_s\big)\big]ds\bigg\|^p \\
&\quad + E\bigg\|\int_0^t \psi(t,s)B\big[w^\lambda\big(s,u_1(s),(u_1)_s\big) - w^\lambda\big(s,u_2(s),(u_2)_s\big)\big]ds\bigg\|^p \\
&\quad + E\bigg\|\int_0^t \psi(t,s)\big[h_2\big(s,u_1(s),(u_1)_s\big) - h_2\big(s,u_2(s),(u_2)_s\big)\big]ds\bigg\|^p \\
&\quad + E\bigg\|\int_0^t \psi(t,s)\big[h_3\big(s,u_1(s),(u_1)_s\big) - h_3\big(s,u_2(s),(u_2)_s\big)\big]dv(s)\bigg\|^p\bigg].
\end{aligned}$$

Using (H1)-(H5) and Lemma 3.1, we get

$$\begin{aligned}
& E\big\|\big(F_\lambda u_1\big)(t) - \big(F_\lambda u_2\big)(t)\big\|^p \\
&\leq 4^{p-1}\bigg[(M')^p T_0^{\frac{p}{q}} M_{h_1} + (MM_1)^p T_0^{\frac{p}{q}} \frac{\hat{K}_1}{(|\lambda|+1)^p} + M^p T_0^{\frac{p}{q}} M_{h_2} + M^p L_{h_3} M_{h_3}\bigg] \\
&\quad \times \int_0^t \Big(E\big\|u_1(s) - u_2(s)\big\|^p + E\big\|(u_1)_s - (u_2)_s\big\|^p_{\mathfrak{B}_\theta}\Big)ds \\
&\leq 4^{p-1}\bigg[(M')^p T_0^{\frac{p}{q}} M_{h_1} + (MM_1)^p T_0^{\frac{p}{q}} \frac{\hat{K}_1}{(|\lambda|+1)^p} + M^p T_0^{\frac{p}{q}} M_{h_2} + M^p L_{h_3} M_{h_3}\bigg] \\
&\quad \times \int_0^t (1+\hat{N})E\big\|u_1(s) - u_2(s)\big\|^p ds \\
&\leq 4^{p-1}(1+\hat{N})T_0\bigg[(M')^p T_0^{\frac{p}{q}} M_{h_1} + (MM_1)^p T_0^{\frac{p}{q}} \frac{\hat{K}_1}{(|\lambda|+1)^p} \\
&\qquad + M^p T_0^{\frac{p}{q}} M_{h_2} + M^p L_{h_3} M_{h_3}\bigg]\big\|u_1 - u_2\big\|^p_{C_2}.
\end{aligned}$$

Using successive iterations, we get

$$\begin{aligned}
& E\big\|\big(F_\lambda^n u_1\big)(t) - \big(F_\lambda^n u_2\big)(t)\big\|^p \\
&\leq (4^{p-1})^n(1+\hat{N})^n(T_0)^n\bigg[(M')^p T_0^{\frac{p}{q}} M_{h_1} + (MM_1)^p T_0^{\frac{p}{q}} \frac{\hat{K}_1}{(|\lambda|+1)^p} \\
&\qquad + M^p T_0^{\frac{p}{q}} M_{h_2} + M^p L_{h_3} M_{h_3}\bigg]^n \big\|u_1 - u_2\big\|^p_{C_2}.
\end{aligned}$$

Taking supremum over $[0, T_0]$, we get

$$\begin{aligned}
&\big\|\big(F_\lambda^n u_1\big) - \big(F_\lambda^n u_2\big)\big\|_{C_2}^p \\
&\leq \quad (4^{p-1})^n (1+\hat{N})^n (T_0)^n \bigg[(M')^p T_0^{\frac{p}{q}} M_{h_1} + (MM_1)^p T_0^{\frac{p}{q}} \frac{\hat{K}_1}{(|\lambda|+1)^p} \\
&\qquad + M^p T_0^{\frac{p}{q}} M_{h_2} + M^p L_{h_3} M_{h_3}\bigg]^n \big\|u_1 - u_2\big\|_{C_2}^p,
\end{aligned}$$

where n is sufficiently large such that

$$\begin{aligned}
&(4^{p-1})^n (1+\hat{N})^n (T_0)^n \bigg[(M')^p T_0^{\frac{p}{q}} M_{h_1} + (MM_1)^p T_0^{\frac{p}{q}} \frac{\hat{K}_1}{(|\lambda|+1)^p} \\
&\qquad + M^p T_0^{\frac{p}{q}} M_{h_2} + M^p L_{h_3} M_{h_3}\bigg]^n < 1.
\end{aligned}$$

Thus, F_λ^n is a contraction mapping. Therefore, by Banach contraction principle, F_λ has a unique fixed point $u_\lambda \in C_2$ which is a mild solution of (1.1). □

4. Approximate Controllability

Theorem 4.1. *Let the assumptions (H1)-(H6) hold and the functions $h_i : J_0 \times Z \times \mathfrak{B}_\theta \to Z$, where $i = 1, 2, 3$ be uniformly bounded. Then the system (1.1) is approximately controllable on $[0, T_0]$.*

Proof. From Theorem 3.2, $F_\lambda u$ has a fixed point u_λ in C_2 which is a mild solution for the control function:

$$w^\lambda\big(t, u_\lambda\big) = B^* \psi^*(T_0, t)\big(\lambda I + F_0^{T_0}\big)^{-1} p(u_\lambda),$$

where

$$\begin{aligned}
p(u_\lambda) \quad = \quad & E u_{T_0} - \xi(t,0)\phi(0) - \psi(t,0)\big[\chi_0 - h_1\big(0, \phi(0), \phi\big)\big] + \int_0^t X(s)dv(s) \\
& - \int_0^t \xi(T_0, s) h_1\big(s, u_\lambda(s), (u_\lambda)_s\big)ds - \int_0^t \psi(T_0, s) h_2\big(s, u_\lambda(s), (u_\lambda)_s\big)ds \\
& - \int_0^t \psi(T_0, s) h_3\big(s, u_\lambda(s), (u_\lambda)_s\big)dv(s).
\end{aligned}$$

Further, we have

$$\begin{aligned}
u_\lambda(T_0) &= \xi(T_0,0)\phi(0)+\psi(T_0,0)\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big]\\
&\quad+\int_0^{T_0}\xi(T_0,s)h_1\big(s,u_\lambda(s),(u_\lambda)_s\big)ds\\
&\quad+\int_0^{T_0}\psi(T_0,s)Bw^\lambda\big(s,u_\lambda(s),(u_\lambda)_s\big)ds\\
&\quad+\int_0^{T_0}\psi(T_0,s)h_2\big(s,u_\lambda(s),(u_\lambda)_s\big)ds\\
&\quad+\int_0^{T_0}\psi(T_0,s)h_3\big(s,u_\lambda(s),(u_\lambda)_s\big)dv(s)\\
&= Eu_{T_0}+\int_0^{T_0}X(s)dv(s)-p(u_\lambda)+{F_0}^{T_0}\big(\lambda I+F_0^{T_0}\big)^{-1}p(u_\lambda)\\
&= Eu_{T_0}+\int_0^{T_0}X(s)dv(s)-\lambda R\big(\lambda,{F_0}^{T_0}\big)p(u_\lambda).
\end{aligned}$$

Since the functions $h_i:J_0\times Z\times\mathfrak{B}_\theta\to Z$ where $i=1,2,3$ are uniformly bounded. It follows that $h_i\big(s,u(s),u_s\big)$ are bounded in $L^2(J_0,Z)$. Thus, there exist subsequences $h_i\big(s,u_\lambda(s),(u_\lambda)_s\big)$ converges to $h_i(s)$.
We define

$$\begin{aligned}
\alpha &= Eu_{T_0}+\int_0^{T_0}X(s)dv(s)-\xi(T_0,0)\phi(0)-\psi(T_0,0)\big[\chi_0-h_1\big(0,\phi(0),\phi\big)\big]\\
&\quad-\int_0^{T_0}\xi(T_0,s)h_1(s)ds-\int_0^{T_0}\psi(T_0,s)h_2(s)ds-\int_0^{T_0}\psi(T_0,s)h_3(s)dv(s).
\end{aligned}$$

We have,

$$\begin{aligned}
&E\big\|p(u_\lambda)-\alpha\big\|^p\\
&\leq 3^{p-1}\int_0^{T_0}E\left\|\xi(T_0,s)\big[h_1\big(s,u_\lambda(s),(u_\lambda)_s\big)-h_1(s)\big]\right\|^p ds\\
&\quad+3^{p-1}\int_0^{T_0}E\left\|\psi(T_0,s)\big[h_2\big(s,u_\lambda(s),(u_\lambda)_s\big)-h_2(s)\big]\right\|^p ds\\
&\quad+3^{p-1}\int_0^{T_0}E\left\|\psi(T_0,s)\big[h_3\big(s,u_\lambda(s),(u_\lambda)_s\big)-h_3(s)\big]\right\|^p dv(s)\\
&\leq 3^{p-1}M'^p\int_0^{T_0}E\left\|h_1\big(s,u_\lambda(s),(u_\lambda)_s\big)-h_1(s)\right\|^p ds\\
&\quad+3^{p-1}M^p\int_0^{T_0}E\left\|h_2\big(s,u_\lambda(s),(u_\lambda)_s\big)-h_2(s)\right\|^p ds\\
&\quad+3^{p-1}M^p\int_0^{T_0}E\left\|h_3\big(s,u_\lambda(s),(u_\lambda)_s\big)-h_3(s)\right\|^p dv(s)\to 0\\
&\qquad\qquad as\quad \lambda\to 0^+.
\end{aligned}$$

Again,

$$\begin{aligned} E\left\|u_\lambda - Eu_{T_0} - \int_0^{T_0} X(s)dv(s)\right\|^p &= E\left\|\lambda R\big(\lambda, F_0^{T_0}\big)p(u_\lambda)\right\|^p \\ &\le E\left\|\lambda R\big(\lambda, F_0^{T_0}\big)(\alpha)\right\|^p \\ &\quad + E\left\|\lambda R\big(\lambda, F_0^{T_0}\big)\left[p(u_\lambda) - \alpha\right]\right\|^p \to 0 \\ &\quad as \quad \lambda \to 0^+. \end{aligned}$$

This completes the proof.

□

5. Optimal Controllability

In order to discuss the optimal controllability, we define the performance index

$$\tilde{I}(w) = E\left\{\int_0^{T_0} \tilde{G}\big(t, u(t), u_t, w(t)\big)dt\right\}, \tag{5.1}$$

where $\tilde{G}$ is a functional defined on $J_0 \times Z \times Z \times W_{ad}$, where W_{ad} denotes the set of all admissible control and consequently is closed and convex in $L^2(J_0, W)$.

Theorem 5.1. *If all the conditions of Theorem 3.2 hold, then there exists an optimal control of the problem (1.1) provided that*

$$4^{p-1}\big[M'^p M_{h_1}^p + M^p\big(M_{h_2}^p + L_{h_3}M_{h_3}^p\big)\big]T_0(1+\hat{N}) < 1.$$

Proof. It is sufficient to prove that there exists $w^0 \in W_{ad}$ which minimize $\tilde{I}(w)$.

If $\inf\limits_{w\in W_{ad}} \tilde{I}(w) = \infty$, then result is trivially true.

If $\inf\limits_{w\in W_{ad}} \tilde{I}(w) = \epsilon_0 < \infty$, then we can find a sequence $\{w^n\}$ in W_{ad} such that $\tilde{I}(w^n) \to \epsilon_0$. As W_{ad} is a closed and convex subset of $L^2(J_0, W)$, the sequence $\{w^n\}$ has a weakly convergent subsequence $\{w^m\}$ converging to $w^0 \in W_{ad}$. Using Theorem 3.2, for each $w^m \in W_{ad}$, there exists a mild solution u^m of (1.1) such that:

$$\begin{aligned} u^m(t) &= \xi(t,0)\phi(0) + \psi(t,0)\big[\chi_0 - h_1\big(0,\phi(0),\phi\big)\big] \\ &\quad + \int_0^t \xi(t,s)h_1\big(s, u^m(s), u_s^m\big)ds + \int_0^t \psi(t,s)Bw^m(s)ds \\ &\quad + \int_0^t \psi(t,s)h_2\big(s, u^m(s), u_s^m\big)ds + \int_0^t \psi(t,s)h_3\big(s, u^m(s), u_s^m\big)dv(s). \end{aligned}$$

Similarly, corresponding to w^0, there exists a mild solution u^0 of (1.1) such that:

$$\begin{aligned} u^0(t) &= \xi(t,0)\phi(0) + \psi(t,0)\big[\chi_0 - h_1\big(0,\phi(0),\phi\big)\big] + \int_0^t \xi(t,s)h_1\big(s, u^0(s), u_s^0\big)ds \\ &\quad + \int_0^t \psi(t,s)Bw^0(s)ds + \int_0^t \psi(t,s)h_2\big(s, u^0(s), u_s^0\big)ds \\ &\quad + \int_0^t \psi(t,s)h_3\big(s, u^0(s), u_s^0\big)dv(s). \end{aligned}$$

We have

$$
\begin{aligned}
&E\big\|u^m(t)-u^0(t)\big\|^p\\
&\le \quad 4^{p-1}E\bigg\|\int_0^t \xi(t,s)\big[h_1\big(s,u^m(s),u_s^m\big)-h_1\big(s,u^0(s),u_s^0\big)\big]ds\bigg\|^p\\
&\quad +4^{p-1}E\bigg\|\int_0^t \psi(t,s)\big[Bw^m(s)-Bw^0(s)\big]ds\bigg\|^p\\
&\quad +4^{p-1}E\bigg\|\int_0^t \psi(t,s)\big[h_2\big(s,u^m(s),u_s^m\big)-h_2\big(s,u^0(s),u_s^0\big)\big]ds\bigg\|^p\\
&\quad +4^{p-1}E\bigg\|\int_0^t \psi(t,s)\big[h_3\big(s,u^m(s),u_s^m\big)-h_3\big(s,u^0(s),u_s^0\big)\big]dv(s)\bigg\|^p.
\end{aligned}
$$

Using (H1), (H3)-(H5), Lemma 2.5 and Holder's inequality, we obtain

$$
\begin{aligned}
&E\big\|u^m(t)-u^0(t)\big\|^p\\
&\le \quad 4^{p-1}M'^pM_{h_1}^p\int_0^t\Big[E\big\|u^m(s)-u^0(s)\big\|^p+E\big\|u_s^m(s)-u_s^0(s)\big\|_{\mathfrak{B}_\theta}^p\Big]ds\\
&\quad +4^{p-1}M^pM_1^p\int_0^t E\big\|w^m(s)-w^0(s)\big\|^p ds\\
&\quad +4^{p-1}M^pM_{h_2}^p\int_0^t\Big[E\big\|u^m(s)-u^0(s)\big\|^p+E\big\|u_s^m(s)-u_s^0(s)\big\|_{\mathfrak{B}_\theta}^p\Big]ds\\
&\quad +4^{p-1}M^pM_{h_3}^pL_{h_3}\int_0^t\Big[E\big\|u^m(s)-u^0(s)\big\|^p+E\big\|u_s^m(s)-u_s^0(s)\big\|_{\mathfrak{B}_\theta}^p\Big]ds\\
&\le \quad 4^{p-1}\big[M'^pM_{h_1}^p+M^p\big(M_{h_2}^p+L_{h_3}M_{h_3}^p\big)\big]T_0(1+\hat{N})E\big\|u^m(s)-u^0(s)\big\|^p\\
&\quad +4^{p-1}M^pM_1^pT_0E\big\|w^m(s)-w^0(s)\big\|^p.
\end{aligned}
$$

Since $4^{p-1}\big[M'^pM_{h_1}^p+M^p\big(M_{h_2}^p+L_{h_3}M_{h_3}^p\big)\big]T_0(1+\hat{N})<1$ and $E\big\|w^m(t)-w^0(t)\big\|^p\to 0$, we conclude that $u^m\to u^0$.

Applying Balder's theorem [6] , we get

$$
\begin{aligned}
\epsilon_0 &= \lim_{m\to\infty}E\bigg\{\int_0^{T_0}\tilde{G}\big(t,u^m(t),u_t^m,w^m(t)\big)dt\bigg\}.\\
&\le E\bigg\{\int_0^{T_0}\tilde{G}\big(t,u^0(t),u_t^0,w^0(t)\big)dt\bigg\}\\
&= \tilde{I}(w^0)\ge\epsilon_0.
\end{aligned}
$$

This shows that $\tilde{I}(w^0)=\epsilon_0$, i.e. $\tilde{I}$ attains its minimum value at $w^0\in W_{ad}$. □

6. Application

Consider the following example:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial z(x,t)}{\partial t}\right) = -c(x,t)\dfrac{\partial^2}{\partial x^2}z(x,t) + \dfrac{\partial}{\partial t}h_1\big(t, z(x,t), z(x,t-\delta)\big) \\ \qquad\qquad\qquad +Bw(x,t) + h_2\big(t, z(x,t), z(x,t-\delta)\big) \\ \qquad\qquad\qquad +h_3\big(t, z(x,t), z(x,t-\delta)\big)\dfrac{dv(t)}{dt}, \\ \qquad\qquad\qquad\qquad\qquad x \in [0,1], \quad t \in [0, T_0], \\ z(x,t-\delta) = \phi(x,t-\delta), \quad \delta > 0, \\ z'(x,0) = \chi_0, \end{cases} \tag{6.1}$$

where $c(x,t)$ is uniformly Hölder continuous i.e. there exist $K > 0$ and $\bar{\alpha} \in (0,1)$ such that

$$\big\|c(x,t_1) - c(x,t_2)\big\| \leq K|t_1 - t_2|^{\bar{\alpha}},$$

$\phi(x,t-\delta) \in \mathfrak{B}_\theta$, and $\chi_0 \in Z$.
Define the functions

$$h_1\big(t, z(x,t), z(x,t-\delta)\big) = 3t^2 \cos\big(2 + |z(x,t)| + |z(x,t-\delta)|\big),$$
$$h_2\big(t, z(x,t), z(x,t-\delta)\big) = \sin\big(\pi t + |z(x,t)| + |z(x,t-\delta)|\big),$$

and

$$h_3\big(t, z(x,t), z(x,t-\delta)\big) = \frac{2e^t}{1+e^t}\sin\big(1 + |z(x,t)| + |z(x,t-\delta)|\big).$$

Functions h_1, h_2 and h_3 satisfy the assumptions (H3), (H4) and (H5), respectively. $v(t)$ is defined on a filtered probability space $\big(\Omega, \Upsilon, Q\big)$. To write system (6.1) into abstract form, let $Z = L^2[0,1], H = \mathbb{R}$ and define the operator $A(t)$ by

$$A(t)z(x,t) = -c(x,t)\frac{\partial^2}{\partial x^2}z(x,t)$$

with

$$D\big(A(t)\big) = \big\{z \in Z | z, z' \text{ are absolutely continuous } z'' \in Z \text{ and } z(0) = z(1) = 0\big\},$$

which is independent of t.

$A(t)$ generates an analytic compact semigroup defined by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}\langle z, y_n\rangle y_n,$$

where

$$y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$$

are eigen functions corresponding to the eigenvalues $\lambda_n = -n^2$, where $n \in \mathbb{N}$.

Define an infinite dimensional space

$$W = \left\{w : w = \sum_{n=2}^{\infty} w_n y_n(x) \Big| \sum_{n=2}^{\infty} w_n^2 < \infty\right\} \text{ with the norm } \|w\| = \left(\sum_{n=2}^{\infty} w_n^2\right)^{1/2}.$$

Define the operator $B : W \to Z$ by $Bw(t) = 2w_2(t)y_1(x) + \sum_{n=2}^{\infty} w_n(t)y_n(x)$, where $B \in L(W,Z)$.

Clearly, the problem (6.1) satisfies all the conditions of Theorem 4.1. Therefore, the system (6.1) is approximately controllable on $[0, T_0]$.

7. Conclusion

The main focus of this paper is to establish some sufficient conditions for the controllability of the second-order non-autonomous stochastic delay differential equation. Initially, we studied the existence and uniqueness of the mild solution of (1.1) and then, we examined the approximate and optimal controllability of the system. We used the semigroup theory, stochastic analysis techniques, and Banach contraction principle to obtain the results. An example is also included to show the efficacy of the result. In future, we will study fractional order semilinear stochastic differential equation having several delays in control.

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