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SOLVING VOLTERRA-FREDHOLM INTEGRAL EQUATIONS BY LINEAR SPLINE FUNCTION

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ABSTRACT. This study determines the numerical solution of linear mixed Volterra-Fredholm integral equations of the second kind using the linear spline function. The proposed method is based on using the unknown function's linear spline function at an arbitrary point and converting the Volterra-Fredholm integral equation into a system of linear equations with regard to the unknown function using the integration method. By solving the given system, an approximate solution can be easily established. This is done with the help of a computer program that uses the Python code program version 3.9. Furthermore, we demonstrated theoretical results on the method's uniqueness and convergence analyses.

1. Introduction

Many problems of mathematical physics can be stated in the form of integral equations. Some of these will be discussed as examples and treated explicity. To make a list of such applications would be almost impossible. Suffice it to say that there is almost no area of applied mathematics and mathematical physics where integral equations do not play a role, hence, the literature on integral equations and their application is vast.
In recent year, many studies have been carried out and results have been found as the interplay of Fredholm integral equation, Volterra integral equation, mixed Volterra-Fredholm integral equation and numerical part of these three type of integral equation.

In this work, we consider the linear mixed Volterra–Fredholm integral equations (MVFIEs) of the form:

$$u(x) = f(x) + \lambda_1 \int_a^x K(x,t)u(t)dt + \lambda_2 \int_a^b L(x,t)u(t)dt, \qquad (1.1)$$

where the functions $f(x)$, and the kernels $K(x,t)$ and $L(x,t)$ are known L^2 analytic functions and λ_1, λ_2 are arbitrary constants, x is variable and $u(x)$ is the unknown continuous function to be determined. Such equations arise in many applications in areas of physics, fluid dynamics, electrodynamics, and biology. Various formulations of boundary value problems, with Neumann, Dirichlet or mixed boundary

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conditions are reduced to such integral equations. They also provide mathematical models for the development of an epidemic and numerous other physical and biological problems.

It is well-known that the analytical solution of MVFIEs generally does not exist except for special cases, and thus, numerical method was the successful and effective method for solving these problems. Several numerical and approximate methods are used for solving MVFIEs such as Taylor polynomial by Yalcinbacs and Sezer [16]; Yalcinbas [15],least square method and Chebyshev polynomials by Dastjerdi and Ghaini [5], Lagrange collocation method by Wang [14], Series solution, successive approximation method and method of successive substitutions by Saeed and Berdawood [13], Trigonometric Functions and Laguerre Polynomials by Hasan and Sulaiman [7], Touchard Polynomials (T-Ps) method by Al-Miah and Taie [1], Some iterative numerical methods by Micula [12], Taylor polynomial by Didgara and Vahidi [6]. The reader can consult the following references for other information (Jerry [8], Atkinson [2], Bekelman and Gross [3], Lange and Herbert [11], Kaminaka and Wadati [9], Ladopoulos [10]) and the references therein.

This paper wishes to study Equation (1.1) by using linear spline function. The rest of this paper is organized as follows. In Section 2, we introduce our method for solving equation(1.1). In Section 3, we prove the uniqueness and convergence of the prsented method. In Section 4, we investigate several numerical examples, which demonstrate the effectiveness of our technique. In Section 5, some tentative conclusions will be given.

2. Description of the method

In This section, we solve equation(1.1) by using linear spline function (Cheney and Kincaid [4]). The unknown function $u(x)$ in (1.1)approximated by the linear spline function $S(x)$. In the interval $[x_i, x_{i+1}]$ the linear spline function defined by the following formula:

$$S_i(x) = A_i(x)S_i + B_i(x)S_{i+1}, \tag{2.1}$$

where $A_i(x) = \frac{x_{i+1}-x}{h}$ and $B_i(x) = \frac{x-x_i}{h}$ where $h = x_{i+1} - x_i$ for all $i = 0, 1, \cdots, n-1$. Now substituting (2.1) in (1.1) and letting $x = x_i$, we get

$$S_i = f(x_i) + \lambda_1 \sum_{j=0}^{j=i} [\int_{x_j}^{x_{j+1}} K(x_i,t)[A_j S_j + B_j S_{j+1}]dt] + \lambda_2 [\int_a^{x_1} L(x_i,t) S_0(t)dt$$

$$+ \int_{x_1}^{x_2} L(x_i,t)S_1(t)dt + \cdots + \int_{x_{n-1}}^{x_n=b} L(x_i,t)S_{n-1}(t)dt]$$

$$= f(x_i) + \frac{\lambda_1}{h} \sum_{j=0}^{j=i} S_j \int_{x_j}^{x_{j+1}} K(x_i,t)(x_{j+1}-t)dt$$

$$+ \frac{\lambda_1}{h} \sum_{j=0}^{j=i} S_{j+1} \int_{x_j}^{x_{j+1}} K(x_i,t)(t-x_j)dt$$

$$\frac{\lambda 2}{h} \int_a^{x_1} L(x_i,t)[(x_1-t)S_0 + (t-x_0)S_1]dt$$

$$+ \frac{\lambda 2}{h} \int_{x_1}^{x_2} L(x_i,t)[(x_2-t)S_1 + (t-x_1)S_2]dt$$

$$+ \cdots + \frac{\lambda 2}{h} \int_{x_{n-1}}^{x_n=b} L(x_i,t)[(x_n-t)S_{n-1} + (t-x_{n-1})S_n]dt.$$

By computing the integrals in the above equation using trapezoidal rule, we get

$$S_i = f_i + \frac{\lambda_1 h}{2} \underbrace{[\sum_{j=0}^{j=i} S_j K(x_i,x_j) + S_{j+1}K(x_i,x_{j+1})]}_{\text{i=1,2,}\cdots\text{,n-1 and j}\leq i}$$

$$+ \frac{\lambda_2 h}{2} \underbrace{[L(x_i,x_0) + 2\sum_{j=1}^{n-1} L(x_i,x_j)S_j + L(x_i,x_n)]}_{\text{i=1,2,}\cdots\text{,n}}; i = 0,1,...,n. \tag{2.2}$$

In this way, Equation (2.2) construct a system of linear equations with respect to the unknown function S_i. Briefly, this system can be rewritten as follows:

$$CS = F, \tag{2.3}$$

where $S = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_n \end{bmatrix}$, F$=\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$ and $C = \begin{bmatrix} C_0 & C_1 & C_2 & \ldots & C_{n-1} & C_n \end{bmatrix}$ where

$$C_0 = \begin{bmatrix} 1 - \frac{\lambda_2 h}{2} L_{00} \\ -\frac{h}{2}(\lambda_1 K_{10} + \lambda_2 L_{10}) \\ -\frac{h}{2}(\lambda_1 K_{20} + \lambda_2 L_{20}) \\ \vdots \\ -\frac{h}{2}(\lambda_1 K_{(n-1)0} + \lambda_1 L_{(n-1)0}) \\ -\frac{h}{2}(\lambda_1 K_{(n-1)0} + \lambda_1 L_{(n-1)0}) \end{bmatrix}, \quad C_1 = \begin{bmatrix} -\lambda_2 h L_{01} \\ 1 - \frac{h}{2}(\lambda_1 K_{11} + \lambda_2 L_{11}) \\ -h(\lambda_1 K_{21} + \lambda_2 L_{21}) \\ \vdots \\ -h(\lambda_1 K_{(n-1)1} + \lambda_2 L_{(n-1)1}) \\ -h(\lambda_1 K_{n1} + \lambda_2 L_{n1}) \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -\lambda_2 h L_{02} \\ -\lambda_2 h L_{12} \\ 1-\frac{h}{2}(\lambda_1 K_{22}+\lambda_2 L_{22}) \\ \vdots \\ -h(\lambda_1 K_{(n-1)2}+\lambda_2 L_{(n-1)2}) \\ -h(\lambda_1 K_{n2}+\lambda_2 L_{n2}) \end{bmatrix}, \dots,$$

$$C_{n-1} = \begin{bmatrix} -\lambda_2 h L_{0(n-1)} \\ -\lambda_2 h L_{1(n-1)} \\ -\lambda_2 h L_{2(n-1)} \\ \vdots \\ 1-\frac{h}{2}(\lambda_1 K_{(n-1)(n-1)}+\lambda_2 L_{(n-1)(n-1)}) \\ -h(\lambda_1 K_{n(n-1)}+\lambda_2 L_{n(n-1)})) \end{bmatrix},$$

and $$C_n = \begin{bmatrix} -\frac{\lambda_2 h}{2} L_{0n} \\ -\frac{\lambda_2 h}{2} L_{1n} \\ -\frac{\lambda_2 h}{2} L_{2n} \\ \vdots \\ -\frac{\lambda_2 h}{2} L_{(n-1)(n-1)} \\ 1-\frac{h}{2}(\lambda_1 K_{nn}+\lambda_2 L_{nn}) \end{bmatrix}.$$

In the sequel, making use of a standard rule to the resulting system yields an approximate solution of Equation (1.1) as $S_i(x)$ given by the Equation (2.1).

3. Uniquness and convergence theorem

In this section, we consider the uniqueness and convergence analysis of the above method by the following theorem:

Theorem 3.1. *Let $u(x)$ be the exact solution of the Equation (1.1) and $S_i(x)$ be the approximation solution of (1.1), where $S_i(x)$ given by (2.1), then, the solution of (1.1) by (2.1) is unique and convergent if* $0 < \alpha < 1$.

Proof. First, we proof the uniqueness. Let $S(x)$ and $\acute{S}(x)$ be two different approximate solutions for Equation (1.1), we will have

$$\begin{aligned}
\left|S(x)-\acute{S}(x)\right| &= \left|f(x)+\lambda_1\int_a^x K(x,t)S(t)dt+\lambda_2\int_a^b L(x,t)S(t)dt\right. \\
&\left.-f(x)-\lambda_1\int_a^x K(x,t)\acute{S}(t)dt-\lambda_2\int_a^b L(x,t)\acute{S}(t)dt\right| \\
&= \left|\lambda_1\int_a^x K(x,t)S(t)dt+\lambda_2\int_a^b L(x,t)S(t)dt\right. \\
&\left.-\lambda_1\int_a^x K(x,t)\acute{S}(t)dt-\lambda_2\int_a^b L(x,t)\acute{S}(t)dt\right| \\
&= \left|\lambda_1\int_a^x K(x,t)(S(t)-\acute{S}(t))dt+\lambda_2\int_a^b L(x,t)(S(t)-\acute{S}(t))dt\right| \\
&\le \left|\lambda_1\int_a^x K(x,t)(S(t)-\acute{S}(t))dt\right|+\left|\lambda_2\int_a^b L(x,t)(S(t)-\acute{S}(t))dt\right| \\
&\le |\lambda_1|\int_a^x |K(x,t)||(S(t)-\acute{S}(t))|dt \\
&+|\lambda_2|\int_a^b |L(x,t)||(S(t)-\acute{S}(t))|dt \\
&\le |\lambda_1|M_1\int_a^x |(S(t)-\acute{S}(t))|dt+|\lambda_2|M_2\int_a^b |(S(t)-\acute{S}(t))|dt,
\end{aligned}$$

where $|K(x,t)| \le M_1$ and $|L(x,t)| \le M_2$. Since $a \le x \le b$, we have

$$\begin{aligned}
|S(x)-\acute{S}(x)| &\le (|\lambda_1|+|\lambda_2|)M\int_a^b |(S(t)-\acute{S}(t))| \\
&\le (|\lambda_1|+|\lambda_2|)M\beta|(S(t)-\acute{S}(t))| \\
&= \alpha|(S(t)-\acute{S}(t))|,
\end{aligned}$$

where $M = max\,\{M_1,\, M_2\}$, $\alpha = (|\lambda_1|+|\lambda_2|)M\beta$, and $\beta = (b-a)$.
Then

$$\left|(S(t)-\acute{S}(t))\right| \le \alpha\left|(S(t)-\acute{S}(t))\right|,$$

from which we get

$$(1-\alpha)\left|(S(t)-\acute{S}(t))\right| \le 0.$$

Since $0 < \alpha < 1$, then $\left|(S(t)-\acute{S}(t))\right| = 0$, and this is implies that $S(t) = \acute{S}(t)$. Hence the uniqueness proof is complete.

Now, we proof the convergence. From definition of the norms, we have

$$\begin{aligned}
\|u(x)-S_i(x)\| &= \max_{\substack{x\in[a,b]\\ t\in[a,c_1]}} |u(x)-S_i(x)| \\
&= \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \Bigg| f(x)+\lambda_1\int_a^x K(x,t)u(t)dt+\lambda_2\int_a^b L(x,t)u(t)dt \\
&-f(x)-\lambda_1\int_a^x K(x,t)S_i(t)dt-\lambda_2\int_a^b L(x,t)S_i(t)dt\Bigg| \\
&= \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \Bigg|\lambda_1\int_a^x K(x,t)(u(t)-S_i(t)))dt \\
&+\lambda_2\int_a^b L(x,t)(u(t)-S_i(t)))dt\Bigg| \\
&\le \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \left|\lambda_1\int_a^x K(x,t)(u(t)-S_i(t))dt\right| \\
&+\max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \left|\lambda_2\int_a^b K(x,t)(u(t)-S_i(t))dt\right| \\
&\le |\lambda_1| \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \int_a^x |K(x,t)|\,|(u(t)-S_i(t))|\,dt \\
&+|\lambda_2| \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \int_a^b |L(x,t)|\,|(u(t)-S_i(t))|\,dt \\
&\le |\lambda_1|\,M_1 \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \int_a^x |(u(t)-S_i(t))|\,dt \\
&+|\lambda_2|\,M_2 \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \int_a^b |(u(t)-S_i(t))|\,dt \\
&\le (|\lambda_1|\,M_1+|\lambda_2|\,M_2) \max_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \int_a^b |(u(t)-S_i(t))| \\
&\text{since}\quad a\le x\le b \\
&\le (|\lambda_1|\,M_1+|\lambda_2|\,M_2)\beta\,\|u(x)-S_i(x)\|_\infty,
\end{aligned}$$

where $M_1=\max_{\substack{x\in[a,b]\\ t\in[0,c_1]}}|K(x,t)|$, $M_2=\max_{\substack{x\in[a,b]\\ t\in[0,c_1]}}|L(x,t)|$ and $\beta=(b-a)$.
Hence

$$(1-\alpha)\,\|u(x)-S_i(x)\|_\infty \le 0 \quad \text{where} \quad \alpha=(|\lambda_1|\,M_1+|\lambda_2|\,M_2)\beta.$$

Then if $0 < \alpha < 1$ and $n \to \infty$, we have

$$\lim_{n\to\infty} \|u(x) - S_i(x)\| = 0$$

This completes the convergence proof.

4. Numerical examples

In this section, we present three examples to illustrate the effciency and the accuracy of the proposed method. The computed errors e_i are defined by $e_i = |u_i - S_i|$, where u_i is the exact solution of Equation(1.1) and S_i is an approximate solution of the same equation. Also we compute Least square error(LSE)= $\sum_{i=0}^{n}(u_i - S_i)^2$ and all computations are performed using Python program.

Example 4.1. Consider Mixed Volterra-Fredholm integral equation

$$u(x) = -\frac{x^2}{2} - \frac{7x}{2} + 2 + \int_0^x u(t)dt + \int_0^1 xu(t)dt.$$

The exact solution of this equation is given by $u(x) = x + 2$.
Table (1) demonstrates LSE obtained from applying our method on Example (1) for $n = 5$.

Table 1. The Numerical Results for Example (1) with $n = 5$.

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	2	2	0	0
0.2	2.2	2.2	4.440892×10^{-16}	1.972152×10^{-31}
0.4	2.4	2.4	4.440892×10^{-16}	1.972152×10^{-31}
0.6	2.6	2.6	1.332267×10^{-15}	1.774937×10^{30}
0.8	2.8	2.8	4.440892×10^{-16}	1.972152×10^{-31}
1	3	3	8.881784×10^{-16}	7.888609×10^{-31}
LSE				3.155443×10^{-30}

Example 4.2. Consider Mixed Volterra-Fredholm integral equation

$$u(x) = 2cos(x) - 1 + \int_0^x (x - t)u(t)dt + \int_0^\pi u(t)dt.$$

The exact solution of this equation is given by $u(x) = cos(x)$.
Table (2) demonstrates LSE obtained from applying our method on Example (2) for $n = 5$.

Example 4.3. Consider Mixed Volterra-Fredholm integral equation

$$u(x) = -\frac{x^5}{10} + 2x^3 - \frac{x^2}{2} - \frac{3x}{2} + \frac{1}{10} + \int_0^x (x + t)u(t)dt + \int_0^1 (x - t)u(t)dt.$$

The exact solution of this equation is given by $u(x) = 2x^3 + 1$.
Table (3) demonstrates LSE obtained from applying our method on Example (3) for $n = 5$.

Table 2. The Numerical Results for Example (2) with $n = 5$.

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.98481596	0.01518404	$2.30555188 \times 10^{-4}$
$\frac{\pi}{5}$	0.809016	0.797244	0.011772	1.385840×10^{-4}
$\frac{2\pi}{5}$	0.309016	0.306379	0.002637	6.957214×10^{-6}
$\frac{3\pi}{5}$	-0.309016	-0.299600	0.009416	8.867220×10^{-5}
$\frac{4\pi}{5}$	-0.809016	-0.787789	0.021227	4.505994×10^{-4}
π	-1	-0.968951	0.031048	9.639885×10^{-4}
LSE				1.879356×10^{-3}

Table 3. The Numerical Results for Example (3) with $n = 5$.

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.933137	0.066862	0.004470
0.2	1.016	0.921386	0.094613	0.008951
0.4	1.128	1.001530	0.126469	0.015994
0.6	1.432	1.267976	0.164023	0.026903
0.8	2.024	1.8149013	0.2090981	0.043722
1	3	2.73618219	0.26381781	0.06959984
LSE				0.16964242

Table 4. LSE for different values of n for Examples (1)-(3).

	LSE for		
n	**Example 1**	**Example 2**	**Example 3**
10	5.581190×10^{-29}	1.963980×10^{-4}	1.991238×10^{-2}
20	1.577721×10^{-29}	2.211122×10^{-5}	2.365762×10^{-3}
30	9.269115×10^{-29}	6.311298×10^{-6}	6.871413×10^{-4}
40	2.839899×10^{-29}	2.612048×10^{-6}	2.868519×10^{-4}
50	4.358456×10^{-29}	1.321872×10^{-6}	1.459168×10^{-4}

5. Conclusion

In this work, we use linear spline function for solving Volterra-Fredholm integral equations, and this method is powerful numerical method. The numerical results given in the previous section shows the suggested method that can successfully treat the problem of Volterra-Fredholm type. From Table (4), we found that the suggested method has very satisfactory stability properties as n increases, the error reduces initially and then finally stabilizes. Also, we conclude that we get good accuracy when exact solution is linear function. Accuracy remains the same regardless where n increased. The method can be easily extended to systems of mixed Volterra-Fredholm integral equations and systems of Volterra-Fredholm integro-differential equations.

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