Global and Stochastic Analysis
Vol. 8 No. 3 (December, 2021)
Special Issue: Modern Stochastic Models and Problems of Actuarial Mathematics

ON THE CLARK-OCONE TYPE FORMULA FOR INTEGRAL TYPE WIENER FUNCTIONAL

BADRI MAMPORIA, EKATERINE NAMGALAURI, AND OMAR PURTUKHIA

Abstract. The integral type Wiener functional is considered and the stochastic integral representation formula of the Clark-Ocone type is established.

1. Introduction and Auxiliary Results

As is well known in modern stochastic analysis special place take the so-called martingale representation theorems, which implies the representation of the adapted functionals in the form of stochastic integrals. In the 80th of the past century, it turned out (see, [1]) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. In particular, using the integrand of the stochastic integral appearing in the integral representation, one can construct hedging strategies in the European options of different type.

The first proof of the martingale representation theorem was implicitly provided by Ito (1951) himself. This theorem states that any square-integrable Wiener functional is equal to a stochastic integral with respect to Wiener process. One of the pioneer work on explicit descriptions of the integrand is certainly the one by Clark ([2]): if F is a $\Im_t^W$-measurable random variable with $EF^2 < \infty$, then there exists the adapted process $\psi(\cdot,\cdot) \in L_2([0,T] \times \Omega)$, such that the integral representation:

$$F = EF + \int_0^T \psi(t,\omega) dW_t \quad (P - a.s.)$$

holds.

However, this result says nothing about explicitly finding the process $\psi(t,\omega)$. A rather general result in this direction is well known Clark-Ocone formula (see [3]), for the formulation of which we recall some definitions from [4] (see, also [5]).

The class of smooth Wiener functionals S is the class of a random variables which has the form

$$F = f(W_{t_1}, ..., W_{t_n}),\ f \in C_p^\infty(R^n),\ t_i \in [0,T],\ n \geq 1,$$

where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth.

2000 *Mathematics Subject Classification.* Primary 60H07, 60H30; Secondary 62P05.

Key words and phrases. Wiener functional, stochastic (Malliavin) derivative, Clark representation, Clark-Ocone representation.

The stochastic (Malliavin) derivative of a smooth random variable $F \in S$ is the stochastic process $D_t F$ given by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, ..., W_{t_n}) I_{[0,t_i]}(t).$$

Denote by $D_{2,1}$ the Hilbert space that is the closure of the class of smooth Wiener functionals with the following Sobolev type norm:

$$||F||_{2,1} = ||F||_{L_2(\Omega)} + |||D.F|||_{L_2(\Omega;L_2([0,T]))}.$$

The Clark-Ocone formula is the following theorem from [3].

Theorem 1.1. *If F is differentiable in Malliavin sense, $F \in D_{2,1}$, then the stochastic integral representation is fulfilled*

$$F = E[F] + \int_0^T E[D_t F|\Im_t]\mathrm{d}W_t \quad (P - a.s.).$$

Shiryaev and Yor ([6]) (see, also Graversen, Shiryaev and Yor ([7]) proposed a method based on Ito's formula to find explicit martingale representations for Wiener functionals which yields in particular the explicit martingale representation of the running maximum of Wiener process. Even though they consider Clark-Ocone formula as a general way to find stochastic integral representations, they raise the question if it is possible to handle it efficiently even in simple cases. Later on, using the Clark-Ocone formula, Renaud and Remillard ([8]) have established explicit martingale representations for path-dependent Wiener functionals.

Application of the Clark-Ocone formula needs as a rule, on the one hand, essential efforts, and, on the other hand, in the cases if the functional F has no stochastic derivative, its application is impossible. Jaoshvili and Purtukhia (see [9]) in the frame of the classical Ito calculus constructed $\psi(t,\omega)$ explicitly, by using both the standard L_2 theory and the theories of weighted Sobolev spaces, for some class of functionals F that do not have a stochastic derivative.

Let $\mathcal{B}(R)$ be a Borel $\sigma-$ algebra on R, λ be a Lebesgue measure, and $\rho(x,T) := \exp\{-\frac{x^2}{2T}\}$.

Theorem 1.2. [1] *Let the function $f \in L_{2,T/\alpha}, 0 < \alpha < 1$, and it has the generalized derivative of the first order $\partial f/\partial x$, such that $\partial f/\partial x \in L_{2,T/\beta}$, $0 < \beta < 1/2$, then the following integral representation holds*

$$f(W_T) = E[f(W_T)] + \int_0^T E\left[\frac{\partial f}{\partial x}(W_T)|\Im_t^W\right]\mathrm{d}W_t \quad (P - a.s.),$$

where $L_{2,T}$ denotes the set of measurable functions $u : R \to R$, such that $u(\cdot)\rho(\cdot,T)$ $\in L_2 := L_2(R, \mathcal{B}(R), \lambda)$.

It turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation[2]. Glonti and Purtukhia (see [10]) considered Wiener functionals which are

[1] see [9]

[2] It is well-known, that if random variable is stochastically differentiable in Malliavin sense, then its conditional mathematical expectation is differentiable too. On the other hand, it is

not stochastically differentiable and generalized the Clark-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method of finding of integrand.

Theorem 1.3. [3] *Suppose that $g_t := E[F|\Im_t^W]$ is Malliavin differentiable ($g_t \in D_{2,1}^W$) for almost all $t \in [0,T)$. Then we have the stochastic integral representation*

$$g_T = F = E[F] + \int_0^T \nu_s dW_s \quad (P-a.s.),$$

where

$$\nu_s = \lim_{t \uparrow T} E[D_s^W g_t|\Im_s^W] \ \text{ in the } \ L_2([0,T] \times \Omega).$$

Example 1.4. For any real $x \in R$ the nonsmooth functional[4] $F = I_{\{W_T \le x\}}$ have the representation

$$I_{\{W_T \le x\}} = \Phi(\frac{x}{\sqrt{T}}) - \int_0^T \frac{1}{\sqrt{T-s}} \varphi(\frac{x - W_s}{\sqrt{T-s}}) \mathrm{d}W_s,$$

where φ is standard normal distribution density function.

Example 1.5. For any real $x \in R$ the nonsmooth functional $F = I_{\{W_T^+ \le x\}}$ (where $W_T^+ = \max\{0, W_T\}$) admits the representation

$$I_{\{W_T^+ \le x\}} = \sqrt{\frac{T}{2\pi}} + \int_0^T \Phi(\frac{W_s}{\sqrt{T-s}}) \mathrm{d}W_s.$$

It is clear that there are also such functionals which don't satisfy even the weakened conditions, i.e. the nonsmooth functionals whose conditional mathematical expectations is not stochastically differentiable too. In particular, to such functional belongs the integral type functional $\int_0^T u_s(\omega)ds$ with nonsmooth integrand $u_s(\omega)$.

It is well-known that if $u_s(\omega) \in D_{2,1}$ for all s, then $\int_0^T u_s(\omega)ds \in D_{2,1}$ and $D_t\{\int_0^T u_s(\omega)ds\} = \int_0^T D_t u_s(\omega)ds$. But if $u_s(\omega)$ is not differentiable in Malliavin sense, then the Lebesgue average (with respect to ds) also is not differentiable in Malliavin sense (see, for example, [11]). Indeed, in this case the conditional mathematical expectation is not stochastically smooth, because we have:

$$E[\int_0^T u_s(\omega)ds|\Im_t^W] = \int_0^t u_s(\omega)ds + \int_t^T E[u_s(\omega)|\Im_t^W]ds,$$

possible that conditional expectation can be smooth even if random variable is not stochastically smooth. For example, it is well-known that $I_{\{W_T \le x\}} \notin D_{2,1}$, but for all $t \in [0,T)$:

$$E[I_{\{W_T \le x\}}|\Im_t^W] = \Phi\left(\frac{x - W_t}{\sqrt{T-t}}\right) \in D_{2,1},$$

where Φ is standard normal distribution function.

[3] see [10]

[4] The indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one.

where the first summand (integral) is analogous that the initial integral and therefore it is not Malliavin differentiable, but the second summand is differentiable in Malliavin sense when u_s satisfied our weakened condition. It should be noted that such type integral functionals have been considered in our previous works (Glonti, Purtukhia, [11]) and (Glonti, Jaoshvili and Purtukhia, [12]).

Here we consider some stochastically smooth integral type (path-dependent) Wiener functionals and obtain their stochastic integral representation formula of the Clark-Ocone type with explicit expressions for the integrands.

2. Main Results

Let W_t, $t \in [0,T]$, be a standard Wiener process on a standard filtered probability space $(\Omega, \Im, \Im_t, P)$ and let $\Im_t = \Im_t^W$ be the augmentation of the filtration generated by W.

Let K be a real number and for simplicity of statement consider the path-dependent Wiener functional

$$G = \left(\frac{1}{T}\int_0^T W_s ds - K\right)^+$$

which, as is easy to see, is a special case of the payoff function of the so-called Asian option in the Bachelier financial market model and derive the stochastic integral representation formula with an explicit form of the integrand.

Proposition 2.1. *For the Wiener functional G the following stochastic integral representation holds*

$$G = EG + \int_0^T \frac{T-t}{T}\left\{1 - \Phi\left(\frac{\sqrt{3}}{\sqrt{(T-t)^3}}\left[KT - \int_0^t (T-s)dW_s\right]\right)\right\} dW_t, \quad (2.1)$$

where

$$EG = \sqrt{T/3}\varphi(K\sqrt{3/T}) - K[1 - \Phi(K\sqrt{3/T})].$$

Proof. It is not difficult to see that the random variable $\int_0^T W_s ds$ has a normal distribution with parameters zero and $T^3/3$. Indeed, due to the stochastic version of integration by parts, we have

$$\int_0^T W_s ds = sW_s|_0^T - \int_0^T s dW_s = \int_0^T (T-s)dW_s, \quad (2.2)$$

and, hence, we easily ascertain that

$$\int_0^T W_s ds \cong N(0, E\int_0^T W_s^2 ds) = N(0, T^3/3) \quad (2.3)$$

and

$$\int_t^T (T-s)dW_s \cong N(0, (T-t)^3/3) := N(0, \sigma^2). \quad (2.4)$$

On the one hand, using (2.3) and the standard integration technique, we obtain

$$EG = \frac{1}{\sqrt{2\pi T^3/3}}\int_{-\infty}^{+\infty} (\frac{x}{T} - K)^+ \exp\{-\frac{x^2}{2T^3/3}\}dx =$$

$$= \frac{1}{\sqrt{2\pi T^3/3}} \int_{KT}^{+\infty} (\frac{x}{T} - K) \exp\{-\frac{x^2}{2T^3/3}\} dx =$$

$$= -\frac{1}{T} \frac{1}{\sqrt{2\pi T^3/3}} \frac{T^3}{3} \int_{KT}^{+\infty} d\big(\exp\{-\frac{x^2}{2T^3/3}\}\big) -$$

$$-K \frac{1}{\sqrt{2\pi}} \int_{K\sqrt{3}/\sqrt{T}}^{+\infty} \exp\{-\frac{x^2}{2}\} dx =$$

$$= \sqrt{\frac{T}{3}} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{3K^2}{2T}\} - K[1 - \Phi(K\sqrt{3/T})] =$$

$$= \sqrt{T/3}\varphi(K\sqrt{3/T}) - K[1 - \Phi(K\sqrt{3/T})]. \tag{2.5}$$

On the other hand, according to the relation $\partial x^+/\partial x = I_{x>0}$, based on the rule of stochastic differentiation of a composite function (see, Proposition 1.2.4 [5]), using the well-known properties of the Malliavin derivative, we can write

$$D_t G = I_{\{\frac{1}{T}\int_0^T W_s ds - K > 0\}} \frac{1}{T} \int_0^T D_t W_s ds =$$

$$= I_{\{\int_0^T W_s ds > KT\}} \frac{1}{T} \int_0^T I_{[0,s]}(t) ds =$$

$$= \frac{T-t}{T} I_{\{\int_0^T W_s ds > KT\}}.$$

Further, thanks to relations (2.2) – (2.4), due to the well-known properties of conditional mathematical expectation, we have

$$E[D_t G|\Im_t] = \frac{T-t}{T} E[I_{\{\int_0^T W_s ds > KT\}}|\Im_t] =$$

$$= \frac{T-t}{T} E[I_{\{\int_0^t (T-s)dW_s + \int_t^T (T-s)dW_s > KT\}}|\Im_t] =$$

$$= \frac{T-t}{T} E[I_{\{x + \int_t^T (T-s)dW_s > KT\}}]\big|_{x=\int_0^t (T-s)dW_s} =$$

$$= \frac{T-t}{\sqrt{2\pi}\sigma T} \int_{KT-x}^{+\infty} \exp\{-\frac{y^2}{2\sigma^2}\} dy\big|_{x=\int_0^t (T-s)dW_s} =$$

$$= \frac{T-t}{T} \int_{(KT-x)/\sigma}^{+\infty} \exp\{-\frac{y^2}{2}\} dy\big|_{x=\int_0^t (T-s)dW_s}. \tag{2.6}$$

Now, based on the Clark-Ocone formula, using relations (2.5) and (2.6), we easily complete the proof of the proposition and obtain representation (2.1). □

Corollary 2.2. *For $G = (\int_0^1 W_s ds - K)^+$ the following stochastic integral representation holds*

$$G = EG + \int_0^T (1-t) \left\{ 1 - \Phi\left(\sqrt{3(1-t)^{-3}} \left[K - \int_0^t (1-s) dW_s \right] \right) \right\} dW_t,$$

where

$$EG = \varphi(K\sqrt{3})/\sqrt{3} - K[1 - \Phi(K\sqrt{3})].$$

Let us introduce the following notation:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r x^{1+2r}}{(1+2r)r!};$$

$$\mu = \sqrt{T^3/3}, \quad \sigma = \sqrt{(T-t)^3/3};$$

$$\alpha(2i-1,x) = 0, \quad \alpha(2i,x) = (2i-1)!!\sqrt{\frac{\pi}{2}}\big[erf\big(\frac{x}{\sqrt{2}}\big)+1\big];$$

$$\beta(2i-1) = 0, \quad \beta(2i) = 1;$$

$$\gamma(2i-1,x) = 0, \quad \gamma(2i,x) = (2i-1)!!\sqrt{\frac{\pi}{2}}\big[1-erf\big(\frac{x}{\sqrt{2}}\big)\big];$$

$$\delta(i,x) = e^{-x^2/2}\cdot\sum_{r=1}^{[i/2]-\beta(i)+1}\frac{(i-1)!!}{(i-2r+1)!!}x^{i-(2r-1+\beta(i))},$$

where $[i/2]$ denotes the integer part of $i/2$.

For the following calculations, we need some auxiliary results.

Lemma 2.3. *For any natural number $n \geq 1$ and real number y, the following relations hold:*

$$\int_{-\infty}^{y} x^{2n-1}\exp\{-\frac{x^2}{2}\}dx = -\exp\{-\frac{y^2}{2}\}\sum_{r=1}^{n}\frac{(2n-2)!!}{(2n-2r)!!}y^{2n-2r}; \tag{2.7}$$

$$\int_{y}^{\infty} x^{2n-1}\exp\{-\frac{x^2}{2}\}dx = \exp\{-\frac{y^2}{2}\}\sum_{r=1}^{n}\frac{(2n-2)!!}{(2n-2r)!!}y^{2n-2r}. \tag{2.8}$$

Proof. To prove this, we use the method of mathematical induction. Indeed, for $n = 1$, according to the standard integration technique, we have

$$\int_{-\infty}^{y} x\exp\{-\frac{x^2}{2}\}dx = -\int_{-\infty}^{y} d\big(\exp\{-\frac{x^2}{2}\}\big) = -\exp\{-\frac{y^2}{2}\}.$$

Suppose now that (2.7) is valid for n, and we will show its validity for $n+1$. Using the partial integration formula, we easily obtain

$$\int_{-\infty}^{y} x^{2n+1}\exp\{-\frac{x^2}{2}\}dx = -\int_{-\infty}^{y} x^{2n}d\big(\exp\{-\frac{x^2}{2}\}\big) =$$

$$= -x^{2n}\exp\{-\frac{x^2}{2}\}|_{-\infty}^{y} + 2n\int_{-\infty}^{y} x^{2n-1}\exp\{-\frac{x^2}{2}\}dx =$$

$$= -y^{2n}\exp\{-\frac{y^2}{2}\} - \exp\{-\frac{y^2}{2}\}\sum_{r=1}^{n}\frac{2n(2n-2)!!}{(2n-2r)!!}y^{2n-2r} =$$

$$= -\exp\{-\frac{y^2}{2}\}\sum_{r=1}^{n+1}\frac{(2(n+1)-2)!!}{(2(n+1)-2r)!!}y^{2(n+1)-2r}.$$

Relation (2.8) is verified in a similar way. □

Analogously one can verify the validity of the following result.

Lemma 2.4. *For any natural number $n \geq 1$ and real number y, the following relations hold:*

$$\int_{-\infty}^{y} x^{2n} \exp\{-\frac{x^2}{2}\}dx = (2n-1)!!\sqrt{\frac{\pi}{2}}\big(erf\big(\frac{y}{\sqrt{2}}\big)+1\big)-$$

$$-y\exp\{-\frac{y^2}{2}\}\sum_{r=1}^{n}\frac{(2n-1)!!}{(2n-2r+1)!!}y^{2n-2r}; \tag{2.9}$$

$$\int_{y}^{\infty} x^{2n} \exp\{-\frac{x^2}{2}\}dx = (2n-1)!!\sqrt{\frac{\pi}{2}}\big(1-erf\big(\frac{y}{\sqrt{2}}\big)\big)+$$

$$+y\exp\{-\frac{y^2}{2}\}\sum_{r=1}^{n}\frac{(2n-1)!!}{(2n-2r+1)!!}y^{2n-2r}. \tag{2.10}$$

Proposition 2.5. *Combining relations (2.7) and (2.9) and relations (2.8) and (2.10) from the previous lemmas, respectively, we conclude that*

$$\int_{-\infty}^{y} x^{n} \exp\{-\frac{x^2}{2}\}dx = \alpha(n,y) - y^{\beta(n)}\exp\{-\frac{y^2}{2}\}\times$$

$$\sum_{r=1}^{[n/2]-\beta(n)+1}\frac{(n-1)!!}{(n-2r+1)!!}y^{n-(2r-1+\beta(n))}; \tag{2.11}$$

$$\int_{y}^{\infty} x^{n} \exp\{-\frac{x^2}{2}\}dx = \gamma(n,y) + y^{\beta(n)}\exp\{-\frac{y^2}{2}\}\times$$

$$\sum_{r=1}^{[n/2]-\beta(n)+1}\frac{(n-1)!!}{(n-2r+1)!!}y^{n-(2r-1+\beta(n))}. \tag{2.12}$$

Theorem 2.6. *For any even natural number n the functional $G = \big[\big(\int_0^T W_s ds\big)^n - K\big]^+$ admits the following stochastic integral representation*

$$G = E[G] + \frac{n}{\sqrt{2\pi}}\sum_{i=0}^{n-1} C_{n-1}^{i}\int_0^T \sigma^i (T-t)[\theta_1(i,t)+\theta_2(i,t)]dW_t,$$

where

$$E[G] = -2K[1-\Phi(K^{1/n}/\mu)]+$$

$$+2\mu^n\frac{1}{\sqrt{2\pi}}[(n-1)!!\sqrt{\frac{\pi}{2}}\big(1-erf\big(\frac{K^{1/n}}{\sqrt{2}\mu}\big)\big)+\frac{K^{1/n}}{\mu}\cdot\delta(n,\frac{K^{1/n}}{\mu})],$$

$$\theta_1(i,t) = x^{n-1-i}\{\gamma\big(i,\frac{K^{1/n}-x}{\sigma}\big)+\big(\frac{K^{1/n}-x}{\sigma}\big)^{\beta(i)}\delta\big(i,\frac{K^{1/n}-x}{\sigma}\big)\}\big|_{x=\int_0^t (T-s)dW_s}$$

and

$$\theta_2(i,t) = x^{n-1-i}\times$$

$$\{\alpha\big(i,\frac{-K^{1/n}-x}{\sigma}\big)-\big(\frac{-K^{1/n}-x}{\sigma}\big)^{\beta(i)}\delta\big(i,\frac{-K^{1/n}-x}{\sigma}\big)\}\big|_{x=\int_0^t (T-s)dW_s}.$$

Proof. Using the relation (2.3) and Lemma 2.3, we can write

$$E[G] = \frac{1}{\sqrt{2\pi}\mu}\int_{-\infty}^{\infty}(x^n - K)^+ \exp\{-\frac{x^2}{2\mu^2}\}dx =$$
$$= \frac{1}{\sqrt{2\pi}\mu}\int_{-\infty}^{\infty}(x^n - K)I_{\{|x|>K^{1/n}\}}\exp\{-\frac{x^2}{2\mu^2}\}dx =$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mu^n x^n - K)I_{\{|x|>\mu^{-1}K^{1/n}\}}\exp\{-\frac{x^2}{2}\}dx =$$
$$= 2\mu^n\frac{1}{\sqrt{2\pi}}\int_{K^{1/n}/\mu}^{\infty}x^n\exp\{-\frac{x^2}{2}\}dx - 2K[1-\Phi(K^{1/n}/\mu)] =$$
$$= 2\mu^n\frac{1}{\sqrt{2\pi}}[(n-1)!!\sqrt{\frac{\pi}{2}}\big(1-erf\big(\frac{K^{1/n}}{\sqrt{2}\mu}\big)\big) + \frac{K^{1/n}}{\mu}\delta(n,\frac{K^{1/n}}{\mu})]-$$
$$-2K[1-\Phi(K^{1/n}/\mu)].$$

By virtue of the rule of stochastic differentiation of a composite function (see Proposition 1.2.4 [5]), using the well-known properties of the Malliavin derivativea and the relation (2.2), we have

$$D_tG = I_{\{\left(\int_0^T W_s ds\right)^n>K\}}n\Big(\int_0^T W_s ds\Big)^{n-1}\int_0^T I_{[0,s]}(t)ds =$$
$$= n(T-t)\Big(\int_0^T W_s ds\Big)^{n-1}I_{\{\left|\int_0^T(T-s)dW_s\right|>K^{1/n}\}}.$$

Next, according to the well-known properties of conditional mathematical expectation, due to the relation (2.4), we can write that

$$E[D_tG|\Im_t] = n(T-t)\times$$
$$E\big[\big(x+\int_t^T(T-s)dW_s\big)^{n-1}I_{\{\left|x+\int_t^T(T-s)dW_s\right|>K^{1/n}\}}\big]\big|_{x=\int_0^t(T-s)dW_s} =$$
$$= n(T-t)\frac{1}{\sqrt{2\pi}\sigma}\big[\int_{-\infty}^{\infty}(x+y)^{n-1}I_{\{|x+y|>K^{1/n}\}}\exp\{-\frac{y^2}{2\sigma^2}\}dy\big]\big|_{x=\int_0^t(T-s)dW_s} =$$
$$= n(T-t)\sum_{i=0}^{n-1}C_{n-1}^i x^{n-1-i}\frac{1}{\sqrt{2\pi}\sigma}\big[\int_{K^{1/n}-x}^{\infty}y^i\exp\{-\frac{y^2}{2\sigma^2}\}dy\big]\big|_{x=\int_0^t(T-s)dW_s}+$$
$$+n(T-t)\sum_{i=0}^{n-1}C_{n-1}^i x^{n-1-i}\frac{1}{\sqrt{2\pi}\sigma}\big[\int_{-\infty}^{-K^{1/n}-x}y^i\exp\{-\frac{y^2}{2\sigma^2}\}dy\big]\big|_{x=\int_0^t(T-s)dW_s} =$$
$$= n(T-t)\sum_{i=0}^{n-1}C_{n-1}^i x^{n-1-i}\frac{\sigma^i}{\sqrt{2\pi}}\big[\int_{(K^{1/n}-x)/\sigma}^{\infty}y^i\exp\{-\frac{y^2}{2}\}dy\big]\big|_{x=\int_0^t(T-s)dW_s}+$$
$$+n(T-t)\sum_{i=0}^{n-1}C_{n-1}^i x^{n-1-i}\frac{\sigma^i}{\sqrt{2\pi}}\big[\int_{-\infty}^{(-K^{1/n}-x)/\sigma}y^i\exp\{-\frac{y^2}{2\sigma^2}\}dy\big]\big|_{x=\int_0^t(T-s)dW_s}.$$

Based on the above obtained relations, according to the Clark-Ocone formula, using relations (2.11) and (2.12), the proof of the theorem is easily completed. □

Acknowledgment. The paper is dedicated to the memory of prof. Omar Glonti.

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Badri Mamporia: Georgian Technical University, Niko Muskhelishvili Institute of Computational Mathematics, Tbilisi 0179, Georgia
Email address: badrimamporia@yahoo.com

Ekaterine Namgalauri: Department of Mathematics, Ivane Javakhishvili Tbilisi State University, Tbilisi 0186, Georgia
Email address: ekanamgalauri96@gmail.com

Omar Purtukhia: Department of Mathematics and A. Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University, Tbilisi 0186, Georgia
Email address: o.purtukhia@gmail.com