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# HANDLING SYSTEMS OF FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS USING MODIFIED FRACTIONAL EULER METHOD

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Abstract. This paper aims to propose a numerical approach to deal with systems of fractional stochastic differential equations. This approach, which is regarded as a modification of the fractional Euler method (FEM), is called the modified fractional Euler method (MFEM). A susceptible infected recovered model (SIR model) is taken into consideration as an example of a system of fractional stochastic differential equations. Such a model is solved with the use of the MFEM in its deterministic and stochastic cases for the purpose of comparing the fulfilled results.

## 1. Introduction

Stochastic Differential Equations (SDEs) have proven to be a valuable tool for modeling dynamic systems subject to both deterministic and stochastic influences. In recent years, interest has grown in extending these modeling capabilities to include fractional calculus, enabling the analysis of systems with long-term memory and non-local interactions. The fusion of fractional calculus with stochastic dynamics has led to the development of Fractional Stochastic Differential Equations (FSDEs), which offer a promising framework to represent complex real-world phenomena [1, 2]. In this research, we focus on addressing systems of FSDEs and propose an innovative approach to their numerical solution. The Modified Fractional Euler Method (MFEM) stands as the centerpiece of our investigation, offering a robust technique to efficiently tackle FSDEs. The MFEM builds upon the classical Euler method while incorporating fractional calculus concepts, providing accurate approximations of solutions even for highly non-linear and multi-dimensional systems [3].

Fractional calculus generalizes the notion of derivatives and integrals to non-integer orders, enabling the modeling of systems with memory and non-local interactions. We investigate FSDEs, which encompass fractional derivatives in combination with stochastic elements. These equations offer a versatile approach to describe complex systems that exhibit both deterministic trends and random fluctuations see [4, 5]. The primary motivation behind exploring the MFEM lies in the

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growing demand for accurate and computationally feasible methods to study FSDEs. These equations frequently arise in diverse fields, including finance, physics, signal processing, and biology, where fractional dynamics and stochasticity jointly influence the system's behavior. For more details, see [6, 7, 8].

The Euler method has long been an essential tool for numerically approximating solutions to ordinary differential equations. Building upon this classical approach, we introduce the MFEM, a novel algorithm designed to handle FSDEs. The MFEM incorporates fractional calculus techniques, empowering it to efficiently capture the long-term memory effects present in fractional dynamics while preserving the accuracy required for stochastic systems [9]. The numerical solution of FSDEs is often challenging due to the interplay between fractional derivatives and stochastic processes. Through the MFEM, we aim to strike a balance between computational efficiency and solution accuracy. Our method enables researchers and practitioners to simulate FSDEs efficiently without compromising the precision of results, making it an attractive choice for real-world applications. The proposed MFEM demonstrates its effectiveness in tackling FSDEs of high dimensionality and nonlinearity. As many real-world systems exhibit complex interactions and multi-dimensional behavior, our method's ability to handle such scenarios is a significant advantage. For deep knowledge about these methods, the reader may refer to the references [10, 11, 12, 13, 14, 15, 16, 17].

In this research, we present the MFEM as a powerful tool for effectively solving systems of FSDEs. The combination of fractional calculus and stochastic dynamics enables the modeling of intricate systems influenced by both deterministic trends and random fluctuations. By offering a computationally efficient and accurate approach to tackle FSDEs, our research contributes to advancing the understanding of complex real-world phenomena across diverse scientific domains.

## 2. Preliminaries

In this section, we recall some preliminaries and basic results related to fractional calculus. For more about stochastic differential equations and the fractional definite integral, see [18].

**Definition 2.1.** Let $\alpha$ be a real nonnegative number. Then the Riemann-Liouville fractional-order integrator $J_a^\alpha$ is defined by:

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \ a \le x \le b. \tag{2.1}$$

**Definition 2.2.** Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$ such that $m-1 < \alpha \le m$. Then the Caputo fractional-order differentiator of order $\alpha$ is given by:

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \ x > a. \tag{2.2}$$

**Theorem 2.3.** [18] *(Generalized Taylor's Theorem) Suppose that* $D_*^{k\alpha} f(x) \in C(0,b]$ *for* $k = 0, 1, \cdots, n+1$*, where* $0 < \alpha \le 1$*. Then the function* $f$ *can be* *expanded about $x = x_0$ as:*

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D_*^{i\alpha} f(x_0) + \frac{x^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D_*^{(n+1)\alpha} f(\xi), \tag{2.3}$$

*with $0 < \xi < x$, $\forall x \in (0, b]$.*

Now, by using the first three terms of the generalized Taylor theorem and for $\xi \in (a, b)$, $t_i \in [a, b]$ in which the interval is divided as $a = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b$ with $h = \frac{b-a}{n}$ for $i = 1, 2, \cdots, n$, we can expand $y(t)$ about $t = t_i$ to develop a new further modification for the FEM, called MFEM. This formula has the form [3]:

$$\begin{aligned} y_{t_{i+1}} = y(t_i) &+ \frac{h^\alpha}{\Gamma(\alpha+1)} f\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, w_i + \frac{h^\alpha}{2\Gamma(\alpha+1)} f(t_i, y(t_i))\right) \\ &+ \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha}(\xi), \end{aligned} \tag{2.4}$$

**Theorem 2.4.** [19] *Let $d\xi(t) = adt + bdw(t)$, and let $f(x,t)$ be a continuous function in $(x,t) \in \mathbb{R}^1 \times [0, \infty)$ with partial derivatives $f_x, f_{xx}, f_t$. Then the process $f(\xi(t), t)$ has a stochastic differential form:*

$$\begin{aligned} df(\xi(t), t) = &\left[ f_t(\xi(t), t) + f_x((\xi(t), t)a(t)) + \frac{1}{2} f_{xx}(\xi(t), t) b^2(t) \right] dt \\ &+ f_x(\xi(t), t) b(t) dW(t). \end{aligned}$$

Notice that if $w(t)$ is continuously differentiable on $t$, then (by the standard calculus formula for total derivatives), the term $\frac{1}{2} f_{xx} b^2 dt$ will not appear.

## 3. Dealing with System of FSDEs

In this section, we aim to recall the system of SDE as well as the system of FSDEs. This would pave the way to propose our approximate numerical solutions for these systems.

**Definition 3.1.** Let $t$ be nonnegative real numbers, then the system of SDEs is defined as:

$$dX_i(t) = f_i(t, \mathbf{X}(t))dt + g_i(t, \mathbf{X}(t))dW(t), \tag{3.1}$$

with initial condition $\mathbf{X_0} = \mathbf{X}(t_0)$, for $i = 1, 2, \cdots, n$ where $t \geq t_0$ is a time, $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is the vector state, $f_i(t, \mathbf{X}), g_i(t, \mathbf{X})$ are called respectively the drift and diffusion terms of the SDE and $W$ is the stochastic Wiener process.

**Definition 3.2.** Let $t$ be nonnegative real numbers, then the system of FSDEs is defined as:

$$D^\alpha X_i(t) = f_i(t, \mathbf{X}(t))dt^\alpha + g_i(t, \mathbf{X}(t))dW(t), \tag{3.2}$$

with initial condition

$$\mathbf{X_0} = \mathbf{X}(t_0),$$

where $\alpha \in (0, 1]$ and $t, x, f_i(t, \mathbf{X}(t)), g_i(t, \mathbf{X}(t))$, are defined above.

With the aim of illustrating how we can apply the MFEM on a system of FSDEs given in (3.2), we rewrite this system as follows:

$$\begin{aligned}
D^{\alpha}X_1(t) &= f_1(t,\mathbf{X}(t))dt^{\alpha} + g_1(t,\mathbf{X}(t))dW(t),\\
D^{\alpha}X_2(t) &= f_2(t,\mathbf{X}(t))dt^{\alpha} + g_2(t,\mathbf{X}(t))dW(t),\\
D^{\alpha}X_3(t) &= f_3(t,\mathbf{X}(t))dt^{\alpha} + g_3(t,\mathbf{X}(t))dW(t),\\
&\vdots\\
D^{\alpha}X_n(t) &= f_n(t,\mathbf{X}(t))dt^{\alpha} + g_n(t,\mathbf{X}(t))dW(t),
\end{aligned} \tag{3.3}$$

with initial conditions

$$X_1(0) = a_1, X_2(0) = a_2, X_3(0) = a_3, \cdots, X_n(0) = a_n, \tag{3.4}$$

where $a_i$ are constants, for $i = 1, 2, \cdots, n$. Now, to solve system (3.3), we apply MFEM to obtain the following formulas:

$$\begin{aligned}
X_1(t_{i+1}) &= X_1(t_i) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_1\Big(t_i + \phi(h,\alpha), \mathbf{X}(t_i) + \phi(h,\alpha) f_1(t_i, \mathbf{X}(t_i))\Big)\\
&\quad + g_1(t_i, \mathbf{X}(t_i))\Delta W_i,\\
X_2(t_{i+1}) &= X_2(t_i) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_2\Big(t_i + \phi(h,\alpha), \mathbf{X}(t_i) + \phi(h,\alpha) f_2(t_i, \mathbf{X}(t_i))\Big)\\
&\quad + g_2(t_i, \mathbf{X}(t_i))\Delta W_i,\\
X_3(t_{i+1}) &= X_3(t_i) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_3\Big(t_i + \phi(h,\alpha), \mathbf{X}(t_i) + \phi(h,\alpha) f_3(t_i, \mathbf{X}(t_i))\Big)\\
&\quad + g_3(t_i, \mathbf{X}(t_i))\Delta W_i,\\
&\vdots\\
X_n(t_{i+1}) &= X_n(t_i) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f_n\Big(t_i + \phi(h,\alpha), \mathbf{X}(t_i) + \phi(h,\alpha) f_n(t_i, \mathbf{X}(t_i))\Big)\\
&\quad + g_n(t_i, \mathbf{X}(t_i))\Delta W_i,
\end{aligned} \tag{3.5}$$

where $\phi(h,\alpha) = \frac{h^{\alpha}}{2\Gamma(\alpha+1)}$.

As a matter of fact, system (3.5) represents the numerical solution of the system of FSDEs given in (3.2). It should be noted here that the system of FSDEs (Like System 3.1) can be converted into a deterministic system by eliminating the stochastic terms $g_i(t,\mathbf{X}(t))dW(s)$ as well as assuming $\alpha = 1$, for $i = 1, 2, \cdots, n$. Accordingly, system (3.5) is also regarded as a numerical solution to the deterministic system once $\alpha = 1$ and $g_i(t,\mathbf{X}(t))dW(s)$ is eliminated, for $i = 1, 2, \cdots, n$.

## 4. An SIR Model

The SIR model is a fundamental epidemiological tool used to study the spread of infectious diseases within a population. It provides a mathematical framework for understanding how the number of susceptible, infected, and recovered individuals changes over time during an epidemic [20, 21, 22]. The model's simplicity and effectiveness have made it a cornerstone of infectious disease modeling and control strategies. In the SIR model, the population is divided into three compartments: Susceptible ($S$), Infected ($I$), and Recovered ($R$). Susceptible individuals can become infected when they come into contact with infected individuals, and infected individuals can recover from the disease and gain immunity, moving to the recovered compartment. The model assumes that once recovered, individuals cannot be infected again. Mathematically, the SIR model is described by a system of ordinary differential equations, as follows:

$$\begin{aligned} &\frac{dS(t)}{dt} = \delta R(t) - \beta \frac{S(t)}{N(t)} S(t)I(t), \\ &\frac{dI}{dt}(t) = -(\gamma+\sigma)I(t) + \beta \frac{S(t)}{N(t)} S(t)I(t) \\ &\frac{dR}{dt}(t) = -\delta R(t) + \gamma I(t), \end{aligned} \tag{4.1}$$

where $\beta$ is the transmission rate, $\gamma$ is the recovery rate, $\sigma$ is the disease mortality rate and $\delta$ is the waning immunity rate. It should be mentioned that $N(t) = S(t) + I(t) + R(t)$ such that $N(t)$ is not random. Also, it is important to know that $R(0) = 0$ and $S(0), I(0) > 0$. In view of the above discussion, we operate the Caputo differentiator on system (4.1) to get the following fractional system:

$$\begin{aligned} &D^\alpha S(t) = \delta R(t) - \beta \frac{S(t)}{N(t} S(t)I(t) \\ &D^\alpha I(t) = -(\gamma+\sigma)I(t) + \beta \frac{S(t)}{N(t)} S(t)I(t) \\ &D^\alpha R(t) = -\delta R(t) + \gamma I(t). \end{aligned} \tag{4.2}$$

As a result, if we add the white noise terms that satisfy the properties of the Wiener process, we obtain a system of FSDEs that represents a fractional stochastic version of the SIR model. This model would be of the form:

$$\begin{aligned} D^\alpha S(t) &= f_1(t, S(t), I(t), R(t))dt^\alpha + g_1(t, S(t), I(t), R(t))dW_1(t) \\ &\qquad - g_2(t, S(t), I(t), R(t))dW_2(t), \\ D^\alpha I(t) &= f_2(t, S(t), I(t), R(t))dt^\alpha + g_2(t, S(t), I(t), R(t))dW_2(t) \\ &\qquad - g_3(t, S(t), I(t), R(t))dW_3(t) - g_4(t, S(t), I(t), R(t))dW_4(t), \\ D^\alpha R(t) &= f_3(t, S(t), I(t), R(t))dt^\alpha - g_1(t, S(t), I(t), R(t))dW_1(t) \\ &\qquad + g_3(t, S(t), I(t), R(t))dW_3(t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} f_1(t, S(t), I(t), R(t)) &= \delta R(t) - \beta \frac{S(t)}{N(t} S(t)I(t), \\ f_2(t, S(t), I(t), R(t)) &= -(\gamma+\sigma)I(t) + \beta \frac{S(t)}{N(t)} S(t)I(t), \\ f_3(t, S(t), I(t), R(t)) &= -\delta R(t) + \gamma I(t), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} g_1(t,S(t),I(t),R(t)) &= \sqrt{\delta R(t)}, \\ g_2(t,S(t),I(t),R(t)) &= \sqrt{\beta \frac{S(t)}{N(t} S(t)I(t)}, \\ g_3(t,S(t),I(t),R(t)) &= \sqrt{\gamma I(t)}, \\ g_4(t,S(t),I(t),R(t)) &= \sqrt{\sigma I(t)}, \end{aligned} \tag{4.5}$$

and where $W_i(t)$ are independent Wiener process, for $i = 1, 2, \cdots, n$. Note that if $S(0) \approx N(0)$ such that $N(0)$ is huge, then we have $\frac{S(t)}{N(t)} \approx 1$. Now, if we aim to solve system (4.3) with the use of the solution's formula (3.5), we get:

$$\begin{aligned} S(t_{i+1}) &= S(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1\Big(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1(t_i, \mathbf{X}(t_i))\Big) \\ &\quad + g_1(t_i, \mathbf{X}(t_i))dW_1(t_i) - g_2(t_i, \mathbf{X}(t_i))dW_2(t_i), \\ I(t_{i+1}) &= I(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2\Big(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2(t_i, \mathbf{X}(t_i))\Big) \\ &\quad + g_2(t_i, \mathbf{X}(t_i))dW_2(t_i) - g_3(t_i, \mathbf{X}(t_i))dW_3(t_i) - g_4(t_i, \mathbf{X}(t_i))dW_4(t_i), \\ R(t_{i+1}) &= R(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_3\Big(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_3(t_i, \mathbf{X}(t_i))\Big) \\ &\quad - g_1(t_i, \mathbf{X}(t_i))dW_1(t_i) + g_3(t_i, \mathbf{X}(t_i))dW_3(t_i), \end{aligned} \tag{4.6}$$

for $i = 1, 2, \cdots, n$, where $\mathbf{X}(t)) = (S(t), I(t), R(t))$. Note that if we suppose $\alpha = 1$ and eliminate all Wiener process terms, then the deterministic model will be yielded.

In what follows, we will consider the data reported in Table 1 for the purpose of performing several numerical simulations related to our proposed fractional stochastic model.

| Parameters | Values |
|---|---|
| $\beta$ | 0.3 |
| $\gamma$ | 0.1 |
| $\sigma$ | 0.05 |
| $\delta$ | 0.01 |
| $I(0)$ | 2 |
| $R(0)$ | 0 |
| $S(0)$ | $N(0) - 2$ |

TABLE 1. The parameters and initial values of the SIR model

With the use of solutions formulas (4.5), we plot several figures that represent the dynamics of the fractional stochastic SIR model. In particular, one can see in Figure 1 the infected cases in their fractional stochastic state take the same behavior as the same cases in their deterministic state. Also, we plot Figure 2 that represents the recovered cases and their fractional stochastic and deterministic

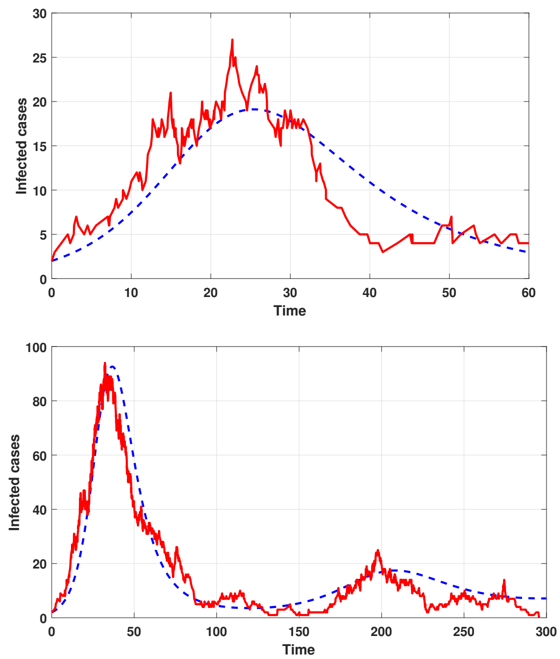


FIGURE 1. Dynamics of the infected cases: First subfigure when $N(0) = 100$ and second subfigure when $N(0) = 500$.

states. In addition, Figure 3 depicts such cases as well, but for susceptible cases. It should be noted here that in these figures, the dashed blue color represents the Deterministic state, while the red color represents the fractional stochastic state.

For more illustrations and in order to show the states dynamics of the model (4.2), we plot Figure 4 which shows all these states in accordance with different fractional-order values, ($\alpha = 0.8, 0.9, 1$).

## 5. Conclusion

A new numerical method for fractional stochastic differential equation systems is proposed. As an illustration of a system of fractional stochastic differential equations, the susceptible infected recovered model (SIR model) is used. To compare the completed results, a similar model is solved using the MFEM in both deterministic and stochastic cases.

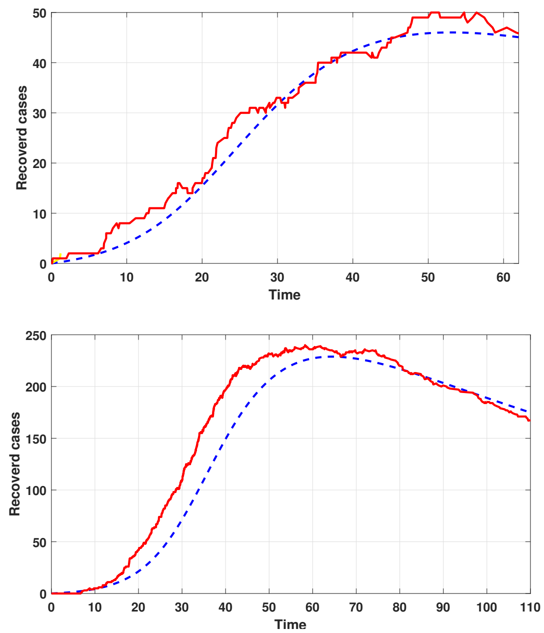


FIGURE 2. Dynamics of the recovered cases: First subfigure when $N(0) = 100$ and second subfigure when $N(0) = 500$.

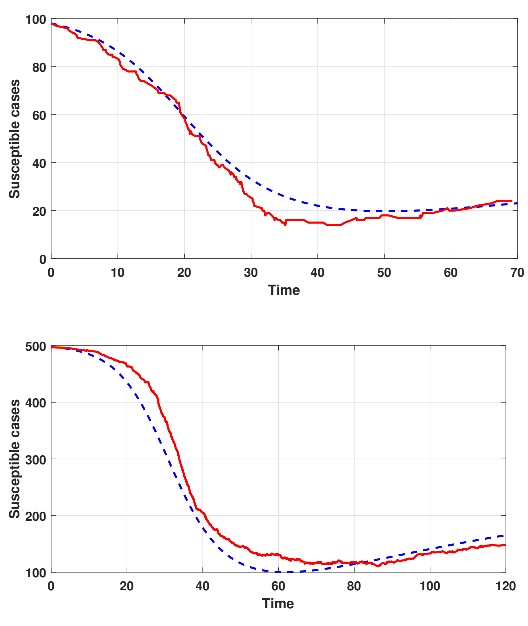


FIGURE 3. Dynamics of the susceptible cases: First subfigure when $N(0) = 100$ and second subfigure when $N(0) = 500$.

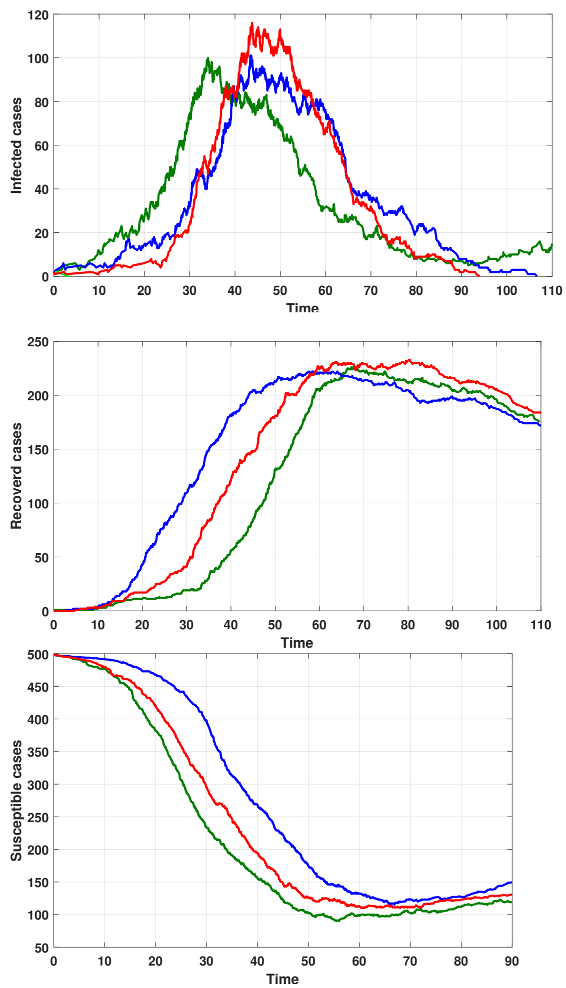


FIGURE 4. States' dynamics of model (4.2) when $N(0) = 100$ according to different values of $\alpha$ in which the green color is for $\alpha = 0.8$, the blue color is for $\alpha = 0.9$ and the red color is for $\alpha = 1$. The first subfigure represents the infected states, the second subfigure represents the recovered states, and the third subfigure represents the susceptible states.

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