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A NOVEL TECHNIQUE TO SOLVE FRACTIONAL ORDER INITIAL VALUE PROBLEMS

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Abstract. We generally use numerical approximations like ADM, DTM to solve fractional order initial value problems. This paper deals with the concept of α- fractional Laplace transform as an analytic development of new directions in the theory. The model generalizes classical Laplace transform along with some suitable kernel, using the theory of the UD derivative. Moreover, the illustrative numerical examples are also included to demonstrate the validity and applicability of the proposed α- fractional Laplace transform.

1. Introduction

The basic properties and history of fractional calculus may be searched in [14, 15, 16, 17] and still, the theory is under the developing stage. There are so many definitions of fractional derivatives available with some properties. But we see that even finding a fractional higher-order derivative of a function is a difficult task. Though many mathematicians have presented the solution of fractional differential equation [3, 10] of the concerned problem through appropriate modeling with the help of Mittag-Leffler function, Reimann-Liouville integral, or Caputo integral. But It should be noted that some approximation methods like ADM or DTM have been used for the computation of the complex operations which causes error [1, 2, 8, 9]. And to obtain some error-free results, an analytic method has been searched. When the theory of conformable derivative [11, 12] has taken place, it has minimized the difficulty of computation so that many important properties like product rule, quotient rule, chain rule, fundamental theorems, Taylor series, power series, etc. have been studied for a fractional derivative and a number of applications have been produced [4, 5, 18].
As differential equations describe the quantities of interest vary over time along with some initial or boundary conditions. Laplace transform has seemed like a powerful technique to solve differential equations. It converts an initial value problem to algebraic equations and using inverse operator we get the desired solution. Although, authors use fractional Laplace transform in sense of Caputo derivative to solve respective initial value problems but it is not convenient to apply and therefore, we require a suitable α- fractional Laplace transform [13]. This is the motivation of the authors to work forward.
Recently theory of The UD derivative has been taken. It is easy to apply as well

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as to compute the fractional derivative of the desired order. To brief study of the UD derivative, we refer [6, 7].

2. Preliminaries

Definition 2.1. (The UD derivative) Following differential operator D^α is called UD derivative of $g(t)$.

$$D^\alpha g(t) = (1-\alpha)g(t) + \alpha g'(t); \alpha \in [0,1] \tag{2.1}$$

which satisfies the condition of being Conformable $D^0 g(t) = g(t)$ and $D^1 g(t) = g'(t)$

Note: If we take $\beta = 1-\alpha$ then $D^\alpha g(t) = \beta g(t) + \alpha g'(t)$.

Definition 2.2. (The UD Integral) Let $\alpha \in (0,1]$ and $G(t)$ be the anti- α derivative (in UD sense) of $g(t)$ then

$$G(t) = \frac{1}{\alpha} e^{\left(\frac{\alpha-1}{\alpha}\right)t} \int e^{\frac{1-\alpha}{\alpha}t} g(t)dt + Ce^{-\left(\frac{\alpha-1}{\alpha}\right)t}. \tag{2.2}$$

where C is constant.

Definition 2.3. (Laplace transform) Let function $g(t)$ is piece wise continuous and is of exponential order. Then following improper integral is known as Laplace transform with kernel $k(s,t) = e^{-st}$ and denoted by $L(g(t))$

$$L(g(t)) = \int_0^\infty e^{-st} g(t)dt = G(s) \tag{2.3}$$

where $s > 0$ is the parameter.

3. α-Fractional Laplace transform

Definition 3.1. Let $\alpha \in (0,1]$ and $f(t)$ be a real valued piece wise continuous function for $t > 0$ which is of the exponential order. Then α- Laplace transform of $f(t)$ of order α is defined and denoted as:

$$\begin{aligned} L_\alpha(f(t)) &= \int_0^\infty k(s,t)f(t)dt \\ &= \frac{1}{\alpha} e^{-\frac{\beta s}{\alpha}} \int_0^\infty e^{-\left(s-\frac{\beta}{\alpha}\right)t} f(t)dt = F_\alpha(s), \end{aligned}$$

where $k(s,t) = \frac{1}{\alpha} e^{-\frac{\beta}{\alpha}(s-t)} e^{-st}$ is the kernel and $\beta = 1-\alpha$.

Here it is clear that if L denotes the usual Laplace transform such that

$$L(f(t)) = F(s),$$

then

$$L_\alpha\left(f(t)\right) = \frac{1}{\alpha}e^{-\frac{\beta s}{\alpha}}F\left(s-\frac{\beta}{\alpha}\right) = F_\alpha(s)$$

or

$$L_\alpha\left(f(t)\right) = \frac{1}{\alpha}e^{-\frac{\beta}{\alpha}s}L\left(e^{\frac{\beta}{\alpha}t}f(t)\right)$$

4. α-Fractional Laplace Transform

Definition 4.1. Let $\alpha \in (0,1]$ and $p(t)$ be a real valued piece wise continuous function for $t > 0$ which is of the exponential order. Then α-fractional Laplace Transform of $p(t)$ is defined and denoted as:

$$\begin{aligned} L_\alpha\left(p(t)\right) &= \int_0^\infty k(s,t)p(t)dt \\ &= \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt \\ &= P_\alpha(s) \end{aligned}$$

where $k(s,t) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}(s-t)}e^{-st}$ is the kernel

Here it is clear that if L denotes the usual Laplace Transform such that

$$L\left(p(t)\right) = P(s),$$

then

$$L_\alpha\left(p(t)\right) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}F\left(s-\frac{(1-\alpha)}{\alpha}\right) = P_\alpha(s) \tag{4.1}$$

or

$$L_\alpha\left(p(t)\right) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}L\left(e^{\frac{(1-\alpha)}{\alpha}t}p(t)\right). \tag{4.2}$$

5. Some Results on α-Fractional Laplace Transform

Theorem 5.1. *(Existence theorem of α-fractional Laplace Transform)*
Let p be a piece wise continuous function in $[0,\infty)$ and is of exponential order, then α- fractional Laplace Transform F_α exists for $s > b$ where b is real.

Proof. Consider a, M, b such that

$$|p(t)| \leqslant Me^{bt} \ \forall \ t \geqslant a,$$

now consider

$$I = \int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt = \int_0^a e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt + \int_a^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt = I_1 + I_2,$$

existence of I_1 is obvious and for $I_2 = \int\limits_a^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt$ we have

$$\left|e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)\right| \leqslant Me^{-\left(s-\frac{(1-\alpha)}{\alpha}-b\right)t}$$

therefore

$$\int_a^\infty \left| e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t} p(t) \right| dt \leqslant M \int_a^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}-b\right)t} dt = \frac{M}{s-\frac{(1-\alpha)}{\alpha}-b}.$$

Thus I_2 converges absolutely for $s > b+\frac{(1-\alpha)}{\alpha}$ and hence I exists for $s > b+\frac{(1-\alpha)}{\alpha}$.

Theorem 5.2. *If $L_\alpha(p(t)) = P_\alpha(s)$ then $P_\alpha(s) \to 0$ as $s \to \infty$.*

Proof. Since $p(t)$ is of exponential order, there exist M, b, a such that

$$|p(t)| \leqslant M_1 e^{bt}, t \geqslant a$$

and $p(t)$ is piece wise continuous function too in $[0.a]$, so we have $|p(t)| \leqslant M_2 e^{ct}$ for $0 \leqslant t \leqslant a$ except at the finite points where $p(t)$ is not defined.

Assume that $M = max\{M_1, M_2\},\ \theta = max\{b, c\}$ we get

$$\int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t} |p(t)|\, dt \leqslant M \int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}-\theta\right)t} dt = \frac{M}{s-\frac{(1-\alpha)}{\alpha}-\theta},$$

where $s > \theta + \frac{(1-\alpha)}{\alpha}$
hence

$$P_\alpha(s) = \frac{1}{\alpha} e^{-\frac{(1-\alpha)s}{\alpha}} \int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t} p(t) dt \to 0$$

as $s \to \infty$.

Proposition 5.3. *(Relation between α-fractional Laplace Transform and ordinary Laplace Transform)*
From the definition of ordinary Laplace Transform

$$L(p(t)) = \int_0^\infty e^{-st} p(t) dt = P(s),$$

consider Heaviside function for $\alpha \in (0, 1]$

$$H(t) = \begin{cases} f\left(t-\frac{(1-\alpha)}{\alpha}\right), t > \frac{(1-\alpha)}{\alpha} \\ 0, t < \frac{(1-\alpha)}{\alpha} \end{cases}$$

so that

$$\begin{aligned}
L\left(e^{\frac{(1-\alpha)}{\alpha}\left(t-\frac{(1-\alpha)}{\alpha}\right)}H(t)\right) &= \int_0^\infty e^{-st} e^{\frac{(1-\alpha)\left(t-\frac{(1-\alpha)}{\alpha}\right)}{\alpha}} H(t)dt \\
&= \int_{\frac{(1-\alpha)}{\alpha}}^\infty e^{-st} e^{-\frac{(1-\alpha)}{\alpha}\left(t-\frac{(1-\alpha)}{\alpha}\right)} f\left(t-\frac{(1-\alpha)}{\alpha}\right) dt \\
&= e^{-\frac{(1-\alpha)}{\alpha}s} \int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)y} f(y)dy;\ y = t-\frac{(1-\alpha)}{\alpha} \\
&= e^{-\frac{(1-\alpha)}{\alpha}s} F\left(s-\frac{(1-\alpha)}{\alpha}\right).
\end{aligned}$$

Therefore,

$$L_\alpha(p(t)) = \frac{1}{\alpha} L\left(e^{\frac{(1-\alpha)}{\alpha}\left(t-\frac{(1-\alpha)}{\alpha}\right)}H(t)\right). \tag{5.1}$$

We may also write

$$L_\alpha(p(t)) = \frac{1}{\alpha} L\left[e^{\frac{(1-\alpha)}{\alpha}\left(t-\frac{(1-\alpha)}{\alpha}\right)}\left(p\left(t-\frac{(1-\alpha)}{\alpha}\right)U\left(t-\frac{(1-\alpha)}{\alpha}\right)\right)\right], \tag{5.2}$$

where $U(t)$ is the Unit step function.

Proposition 5.4. *Followings are the α-fractional Laplace Transform of some elementary functions:*

(a) $L_\alpha(1) = \frac{e^{-\frac{(1-\alpha)}{\alpha}}}{\alpha s-(1-\alpha)}$,
(b) $L_\alpha(t^n) = \alpha^n e^{-\frac{(1-\alpha)}{\alpha}s} \frac{\Gamma(n+1)}{(\alpha s-(1-\alpha))^{n+1}}$,
(c) $L_\alpha(e^{at}) = \frac{e^{-\frac{(1-\alpha)}{\alpha}s}}{(s-a)\alpha-(1-\alpha)}$,
(d) $L_\alpha(\sin at) = \frac{a\alpha e^{\frac{-(1-\alpha)}{\alpha}s}}{(\alpha s-(1-\alpha))^2+\alpha^2 a^2}$,
(e) $L_\alpha(\cos at) = \frac{s\alpha e^{\frac{-(1-\alpha)}{\alpha}s}}{(\alpha s-(1-\alpha))^2+\alpha^2 a^2}$,
(f) $L_\alpha(\sinh at) = \frac{a\alpha e^{\frac{-(1-\alpha)}{\alpha}s}}{(\alpha s-(1-\alpha))^2-\alpha^2 a^2}$,
(g) $L_\alpha(\cosh at) = \frac{s\alpha e^{\frac{-(1-\alpha)}{\alpha}s}}{(\alpha s-(1-\alpha))^2-\alpha^2 a^2}$.

where $\alpha \in (0,1]$.

6. The Basic Properties of α-Fractional Laplace Transform

(a) **Linear property:** If $P_\alpha(s), G_\alpha(s)$ represents the α-fractional Laplace Transform of $p(t), g(t)$ respectively. Then for the constants c_1, c_2 we have

$$L_\alpha(c_1 p(t) + c_2 g(t)) = c_1 P_\alpha(s) + c_2 G_\alpha(s). \tag{6.1}$$

(b) **First shifting property:** Let $P_\alpha(s)$ be the α-fractional Laplace Transform of $p(t)$ that is $L_\alpha\left(p(t)\right) = P_\alpha(s)$. Then

$$L_\alpha\left(e^{at}p(t)\right) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}P\left(s - \frac{(1-\alpha)}{\alpha} - a\right) = P_\alpha(s-a), \forall \alpha \in (0,1]. \quad (6.2)$$

(c) **Second shifting property:** Let $P_\alpha(s)$ be the α-fractional Laplace Transform of $p(t)$ that is $L_\alpha\left(p(t)\right) = P_\alpha(s)$. Then for

$$g(t) = \begin{cases} p(t-a); & t > a \\ 0; & t < 0 \end{cases},$$

$$\begin{aligned} L_\alpha\left(g(t)\right) &= e^{-a\left(s-\frac{(1-\alpha)}{\alpha}\right)}\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}P\left(s - \frac{(1-\alpha)}{\alpha}\right) \\ &= e^{-a\left(s-\frac{(1-\alpha)}{\alpha}\right)}P_\alpha(s) \quad \forall\ \alpha\ \in (0,1] \end{aligned}$$

(d) **Change of scale property:** Let $P_\alpha(s)$ be the α-fractional Laplace Transform of $p(t)$ that is $L_\alpha\left(p(t)\right) = P_\alpha(s)$. Then

$$L_\alpha\left(p(at)\right) = \frac{1}{a\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}P\left(\frac{1}{a}\left(s - \frac{(1-\alpha)}{\alpha}\right)\right) \ \forall\ \alpha \in (0,1]. \quad (6.3)$$

Theorem 6.1. *If $p(t)$ be continuous function and its derivative $p'(t)$ is a function of class A that is $p'(t)$ is piece wise continuous and is of the exponential order in any interval $t \in [0,c]$. Then for $\alpha \in [0,1)$*

$$L_\alpha\left(p'(t)\right) = \left(s - \frac{(1-\alpha)}{\alpha}\right)L_\alpha(p(t)) - \frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}p(0).$$

Proof. Let $c_1, c_2, ...c_n$ are the discontinuities of $p'(t)$ in $[0,c]$. Then we have

$$\begin{aligned} \int_0^c e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p'(t)dt &= \int_0^{c_1} e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p'(t)dt + \int_{c_1}^{c_2} e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p'(t)dt + \cdots + \int_{c_n}^{c} e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p'(t)dt \\ &= e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)\ |_0^{c_1} + e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)\ |_{c_1}^{c_2} + ... + e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)\ |_{c_n}^{c} \\ &\quad + \left(s - \frac{(1-\alpha)}{\alpha}\right)\int_0^c e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt \\ &= \left(e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)c}p(c) - p(0)\right) + \left(s - \frac{(1-\alpha)}{\alpha}\right)\int_0^c e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt. \end{aligned}$$

As $c \to \infty, e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)c}p(c) \to 0$,
so we get

$$L_\alpha\left(p'(t)\right) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}\int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p'(t)dt = \left(s - \frac{(1-\alpha)}{\alpha}\right)L_\alpha(p(t)) - \frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}p(0). \blacksquare$$

Replacing $p(t)$ by $p'(t)$ we may get

$$\begin{aligned} L_\alpha(p''(t)) &= \left(s-\frac{(1-\alpha)}{\alpha}\right)L_\alpha(p'(t))-\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}p'(0) \\ &= \left(s-\frac{(1-\alpha)}{\alpha}\right)\left(\left(s-\frac{(1-\alpha)}{\alpha}\right)L_\alpha(p(t))-\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}p(0)\right)-\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}p'(0) \\ &= \left(s-\frac{(1-\alpha)}{\alpha}\right)^2 L_\alpha(p(t))-\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}\left(s-\frac{(1-\alpha)}{\alpha}\right)f(0)-\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}p'(0). \end{aligned}$$

Similarly one may get

$$\begin{aligned} L_\alpha\left(p'''(t)\right) &= \left(s-\frac{(1-\alpha)}{\alpha}\right)^3 L_\alpha(p(t))-\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s} \\ &\left(\left(s-\frac{(1-\alpha)}{\alpha}\right)^2 p(0)+\left(s-\frac{(1-\alpha)}{\alpha}\right)p'(0)+p''(0)\right). \quad (6.4) \end{aligned}$$

Using the above results we have

$$L_\alpha(D^\alpha p)=L_\alpha((1-\alpha)p+\alpha Dp)=\alpha sL_\alpha[p]-e^{-\frac{(1-\alpha)}{\alpha}s}p(0), \quad (6.5)$$

$$\begin{aligned} L_\alpha(D^\alpha D^\alpha p) &= L_\alpha((1-\alpha)^2p+\alpha^2D^2f+2\alpha(1-\alpha)Dp) \\ &= \alpha^2s^2L_\alpha[p]-(\alpha s+(1-\alpha))\,e^{-\frac{(1-\alpha)s}{\alpha}}p(0)-\alpha e^{-\frac{(1-\alpha)s}{\alpha}}p'(0). \end{aligned}$$

With the same process we have

$$L_\alpha[(\underbrace{D^\alpha D^\alpha\cdots D^\alpha}_{n\ times})p(t)]=L\left[((1-\alpha)+\alpha D)^n\,p(t)\right]. \quad (6.6)$$

Theorem 6.2. *If $L(p(t))=P(s)$,then for $\alpha\in(0,1]$*

$$L_\alpha\left(t^np(t)\right)=\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}(-1)^n\left[\frac{d^n}{ds^n}P(S)\right]_{S=s-\frac{(1-\alpha)}{\alpha}}.$$

Proof. Since

$$L_\alpha\left(p(t)\right)=P_\alpha(s)=\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt$$

$$\begin{aligned} \frac{d}{ds}P_\alpha(s) &= \frac{-(1-\alpha)}{\alpha}\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt-\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty te^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt \\ &= -L_\alpha\left[\left(\frac{(1-\alpha)}{\alpha}+t\right)p(t)\right]. \end{aligned}$$

$$\begin{aligned}\frac{d^2}{ds^2}P_\alpha(s) &= \left(\frac{(1-\alpha)}{\alpha}\right)^2 \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt + \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty t^2 e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt \\ &\quad + \frac{2(1-\alpha)}{\alpha}\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\int_0^\infty t e^{-\left(s-\frac{(1-\alpha)}{\alpha}\right)t}p(t)dt \\ &= (-1)^2 L_\alpha\left[\left(\frac{(1-\alpha)}{\alpha}+t\right)^2 p(t)\right].\end{aligned}$$

Similarly we get

$$\frac{d^n}{ds^n}P_\alpha(s) = (-1)^n L_\alpha\left[\left(\frac{(1-\alpha)}{\alpha}+t\right)^n p(t)\right], \tag{6.7}$$

we may also have

$$L_\alpha\left(t^n p(t)\right) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}(-1)^n\left[\frac{d^n}{ds^n}P(S)\right]_{S=s-\frac{(1-\alpha)}{\alpha}}. \tag{6.8}$$

As

$$L_\alpha\left(p(t)\right) = \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}P\left(s-\frac{(1-\alpha)}{\alpha}\right).$$

Theorem 6.3. *Let*

$$g(t) = \int_0^t p(t)dt$$

and

$$L_\alpha(p(t)) = P_\alpha(s)$$

then for $\alpha \in (0,1]$

$$L_\alpha(g(t)) = \frac{1}{\left(s-\frac{(1-\alpha)}{\alpha}\right)}L_\alpha[p(t)].$$

Proof. We have $g'(t) = p(t)$ and $g(0) = 0$

$$\begin{aligned}L_\alpha[g'(t)] &= \left(s-\frac{(1-\alpha)}{\alpha}\right)L_\alpha[g(t)] - \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}g(0) \\ &= \left(s-\frac{(1-\alpha)}{\alpha}\right)L_\alpha[g(t)].\end{aligned}$$

Therefore,

$$L_\alpha\left[\int_0^t p(t)dt\right] = \frac{1}{\left(s-\frac{(1-\alpha)}{\alpha}\right)}L_\alpha[p(t)]. \tag{6.9}$$

Proposition 6.4. ***(Convolution theorem)***
If $\alpha \in (0,1]$,

$$L_\alpha^{-1}\left(\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}P(s)\right) = L^{-1}\left(P(S)\right),$$

and

$$L_{\alpha}^{-1}\left(\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}G(s)\right)=L^{-1}\left(G(S)\right)$$

then

$$L^{-1}\left(P(S)G(S)\right)=f*g=\int_{0}^{t}p(x)g(t-x)dx. \tag{6.10}$$

where $L^{-1}(P(S))=p(t), L^{-1}(G(S))=g(t)$ *and* $S=s-\frac{(1-\alpha)}{\alpha}$.

7. Numerical Examples

The section contains some examples to evaluate the solution of the problems concerned.

Example 7.1. Consider

$$D^{1/3}(ye^{t})=t;\ y(0)=1/4.$$

The problem is equivalent to solve $Dy+3y=3te^{-t}$ with the condition $y(0)=1/4$, using ordinary Laplace Transform we get the following solution

$$y=e^{-3t}+\frac{3}{2}\left(t-\frac{1}{2}\right)e^{-t}.$$

Here we are interested to get the same solution using α-fractional Laplace Transform.
So taking α-fractional Laplace Transform both sides of $D^{1/3}(ye^{t})=t$, we get,

$$\alpha sL_{\alpha}[Y]-e^{-\frac{(1-\alpha)s}{\alpha}}Y(0)=\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\frac{1}{\left(s-\frac{(1-\alpha)}{\alpha}\right)^{2}}$$

where

$$Y=e^{t}y, L_{\alpha}(t)=\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\left(L(t)\right)_{s=s-\frac{(1-\alpha)}{\alpha}}$$

and

$$L_{\alpha}(D^{\alpha}f)=\alpha sL_{\alpha}[f]-e^{-\frac{(1-\alpha)s}{\alpha}}f(0)$$

As $\alpha=1/3,(1-\alpha)=2/3$ we have

$$\frac{1}{3}sL_{\alpha}[Y]-\frac{e^{-2s}}{4}Y(0)=3e^{-2s}\frac{1}{(s-2)^{2}}$$

$$\Rightarrow L_{\alpha}[Y]=\frac{9e^{-2s}}{s(s-2)^{2}}+\frac{3e^{-2s}}{4s}$$

taking Inverse α-fractional Laplace Transform

$$\begin{aligned} Y &= L_{\alpha}^{-1}\left(\frac{9e^{-2s}}{s(s-2)^{2}}+\frac{3e^{-2s}}{4s}\right) \\ &= L^{-1}\left[\frac{3}{(S+2)S^{2}}+\frac{1}{4(S+2)}\right]; S=s-2 \\ &= \frac{1}{4}\left(4e^{-2t}-3+6t\right) \end{aligned}$$

which implies

$$ye^t = e^{-2t} + \frac{1}{4}(6t-3)$$

and therefore,

$$y = e^{-3t} + \frac{3}{2}\left(t - \frac{1}{2}\right)e^{-t}.$$

Example 7.2. Consider the problem

$$(D^\alpha D^\alpha + 9)u(t) = \sin 2t; u(0) = 1, u'(0) = 0$$

where $\alpha \in (0,1]$.
Taking α-fractional Laplace Transform both sides, we get

$$\alpha^2 s^2 L_\alpha[u] - (\alpha s + (1-\alpha))e^{-\frac{(1-\alpha)s}{\alpha}}u(0) - \alpha e^{-\frac{(1-\alpha)s}{\alpha}}u'(0) + 9L_\alpha[u] = \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\frac{2}{\left(s - \frac{(1-\alpha)}{\alpha}\right)^2 + 4}$$

$$L_\alpha[u] = \frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}\frac{2}{\left(\left(s - \frac{(1-\alpha)}{\alpha}\right)^2 + 4\right)(\alpha^2 s^2 + 9)} + \frac{\alpha s + (1-\alpha)}{(\alpha^2 s^2 + 9)}e^{-\frac{(1-\alpha)s}{\alpha}}$$

implies,

$$u = L^{-1}\left[\frac{2}{(S^2+4)\left(\alpha^2\left(S + \frac{(1-\alpha)}{\alpha}\right)^2 + 9\right)} + \frac{\alpha\left(\alpha\left(\left(S + \frac{(1-\alpha)}{\alpha}\right)\right) + (1-\alpha)\right)}{\left(\alpha^2\left(S + \frac{(1-\alpha)}{\alpha}\right)^2 + 9\right)}\right].$$

As $L_\alpha^{-1}\left(\frac{1}{\alpha}e^{-\frac{(1-\alpha)s}{\alpha}}f(s)\right) = L^{-1}(f(S))$ where $s = S + \frac{(1-\alpha)}{\alpha}$.

Using Convolution theorem for Inverse α-fractional Laplace Transform

$$L^{-1}\frac{2}{(S^2+4)\left(\alpha^2\left(S + \frac{(1-\alpha)}{\alpha}\right)^2 + 9\right)} \quad = \quad \int_0^t e^{-\frac{(1-\alpha)}{\alpha}x}\sin\frac{3x}{\alpha}\sin 2(t-x)dx$$

$$= \frac{\alpha}{6}\left(\frac{e^{-\frac{(1-\alpha)t}{\alpha}}\left(-\frac{(1-\alpha)}{\alpha}\cos\frac{3}{\alpha}t + \left(\frac{3}{\alpha}+2\right)\sin\frac{3}{\alpha}t\right) + \left(\frac{(1-\alpha)}{\alpha}\cos 2t + \left(\frac{3}{\alpha}+2\right)\sin 2t\right)}{\left(\frac{(1-\alpha)}{\alpha}\right)^2 + \left(\frac{3}{\alpha}+2\right)}\right)$$

$$-\frac{\alpha}{6}\left(\frac{e^{-\frac{(1-\alpha)t}{\alpha}}\left(-\frac{(1-\alpha)}{\alpha}\cos\frac{3}{\alpha}t + \left(\frac{3}{\alpha}-2\right)\sin\frac{3}{\alpha}t\right) + \left(\frac{(1-\alpha)}{\alpha}\cos 2t - \left(\frac{3}{\alpha}-2\right)\sin 2t\right)}{\left(\frac{(1-\alpha)}{\alpha}\right)^2 + \left(\frac{3}{\alpha}-2\right)}\right) = A(say)$$

Also

$$L^{-1}\left[\frac{\alpha\left(S + \frac{(1-\alpha)}{\alpha}\right) + (1-\alpha)}{\alpha^2\left(S + \frac{(1-\alpha)}{\alpha}\right)^2 + 9}\right] = \alpha e^{-\frac{(1-\alpha)}{\alpha}t}\left(\cos\frac{3}{\alpha}t + \frac{(1-\alpha)}{3}\sin\frac{3}{\alpha}t\right) = B(say).$$

Therefore, complete solution is

$$u(t) = \frac{1}{\alpha}\left(\frac{1}{6}A + B\right).$$

It should be noted that when $\alpha = 1$ above equation reduces to the following

$$\left(D^2 + 9\right) u(t) = \sin 2t,$$

with the same initial conditions and having solution

$$u(t) = \cos 3t + \frac{1}{15}(3\sin 2t - 2\sin 3t)$$

and the solution coincides with this one.

Example 7.3. Consider

$$(D^{1/4}D^{1/4} + 2D^{1/4} + 1)y = 3te^{-7t}; y(0) = 4, y'(0) = 2.$$

Taking α-fractional Laplace Transform

$$\alpha^2 s^2 L_\alpha[y] - (\alpha s + (1-\alpha))e^{-\frac{(1-\alpha)}{\alpha}s}y(0) - \alpha e^{-\frac{(1-\alpha)}{\alpha}s}y'(0) + 2\left(\alpha s L_\alpha[y] - e^{-\frac{(1-\alpha)}{\alpha}s}y(0)\right)$$
$$+L_\alpha[y] = \frac{3}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}\left(-\frac{d}{ds}\left(\frac{1}{s+7}\right)\right)_{s=s-\frac{(1-\alpha)}{\alpha}}$$

where $\alpha = 1/4, (1-\alpha) = 3/4$

$$\left(\frac{s^2+8s+16}{16}\right) L_{1/4}[y] - e^{-3s}(s+11) - \frac{1}{2}e^{-3s} = 12\frac{e^{-3s}}{(s+4)^2}$$

$$L_{1/4}[y] = 16e^{-3s}\left[\frac{12}{(s+4)^4} + \frac{s}{(s+2)^2} + \frac{23}{2(s+4)^2}\right]$$

$$\begin{aligned} y &= 4L^{-1}\left(\frac{12}{(S+7)^4} + \frac{1}{(S+7)} + \frac{15}{2(S+7)^2}\right); S = s-3 \\ &= 4e^{-7t}\left(2t^3 + \frac{15t}{2} + 1\right). \end{aligned}$$

Example 7.4. Consider

$$(D^{1/2}D^{1/2} + 2D^{1/2} + 5)y = \frac{1}{3}e^{-3t}\sin t\ ; y(0) = 0, y'(0) = 1.$$

Taking α-fractional Laplace Transform of both sides

$$\frac{1}{4}s^2 L_{1/2}[y] - \left(\frac{1}{2}s + \frac{1}{2}\right)e^{-s}y(0) - \frac{1}{2}e^{-s}y'(0) + 2\left[\frac{1}{2}sL_{1/2}[y] - e^{-s}y(0)\right]$$
$$+5L_{1/2}[y] = 2e^{-s}\frac{1}{3}\left(\frac{3}{(s+3)^2+9}\right)_{s=s-1}$$

$$\frac{1}{4}\left(s^2 + 4s + 20\right) L_{1/2}[y] - 0 - \frac{1}{2}e^{-s} = 2e^{-s}\frac{1}{(s+2)^2+9}$$

$$
\begin{aligned}
L_{1/2}[y] &= 8e^{-s}\left[\frac{1}{(s^2+4s+13)(s^2+4s+20)}\right]+\frac{2e^{-s}}{s^2+4s+20}\\
&= 2e^{-s}\left[\frac{1}{7}\left(\frac{4}{(s+2)^2+9}-\frac{3}{(s+2)^2+16}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
y &= \frac{1}{7}L^{-1}\left[\left(\frac{4}{(S+3)^2+9}-\frac{3}{(S+3)^2+16}\right)\right]; s=S+1\\
&= \frac{e^{-3t}}{84}\left(16\sin 3t-9\sin 4t\right).
\end{aligned}
$$

8. Conclusion

Work is motivated by the concept of ordinary Laplace transform via the UD derivative approach. We tried to produce some properties of α-Laplace transforms and observed that these all coincide with the classical properties of Laplace transform at $\alpha = 1$. α-Laplace transform of engineering functions like Unit step, Dirac-delta, etc. can also be obtained in the same manner. Another Integral transform may be established by choosing another sufficient kernel of the integral. The authors hope that the work would be meaningful to other researchers in the future.

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