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SOLUTIONS OF BOUNDARY VALUE PROBLEMS BY FIXED POINT METHODS IN CLIFFORD ANALYSIS

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Abstract. Boundary value problems for elliptic equations of type $Lu = F$ in higher dimensions, can be reduced to fixed-point problems for a suitable defined operator. This defined operator involves a fundamental solution of the equation $Lu = 0$. We consider the case when the right-hand side depends also on the function u itself and on its derivatives $\partial_j u$ for $j = 1, \ldots, n$. As L, we consider operators in the framework of Clifford analysis as the generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$. To solve the equivalent fixed-point problem, we apply the Contraction Mapping Principle and Schauder type estimates.

1. Introduction

Clifford algebras can be defined as the quotient $\dfrac{\mathcal{R}[X_1, ..., X_n]}{\mathcal{I}}$, where $\mathcal{R}[...]$ is the ring of special polynomials in X_1, ...,X_n (actually an $\mathcal{R}-$ algebra, see [5]) and $\mathcal{I}$ is the ideal of $\mathcal{R}[X_1, ..., X_n]$ spanned by polynomials of the form

$$X_j^2 + 1 \quad \text{or} \quad X_iX_j + X_jX_i \ . \tag{1.1}$$

Considering the Euclidean space $\mathbb{R}^{n+1}$ whose basis is $e_0 = 1$, e_1, ..., e_n and denoting X_j by e_j, $j = 1, ..., n$, the structure polynomials (1.1) imply the well-known rules of the usual Clifford algebra $\mathcal{A}_n$:

$$e_j^2 = -1, \quad j = 1, ..., n, \quad \text{and} \quad e_ie_j = -e_je_i \quad \text{for } i \neq j.$$

We recall that the Cauchy-Riemann operator D is defined by $D = \partial_0 + \sum_{i=1}^n e_i\partial_i$ and a (continuously differentiable) Clifford-algebra-valued function is said to be left monogenic if it satisfies the Cauchy-Riemann equation $Du = 0$.

Now we consider the equation:

$$Du = F(x, u, \partial_1 u, \ldots, \partial_n u), \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x_i}. \tag{1.2}$$

The solutions of this system are called generalized monogenic functions, provided that the system is of elliptic type.

In this paper we show how boundary value problems for elliptic equations of type $Lu = F(x, u, \partial_i u), i = 1, \ldots, n$ in higher dimensions, can be reduced to fixed-point problems for a suitable defined operator. This defined operator involves a fundamental solution of the equation $Lu = 0$. In order to illustrate the method

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we consider as L, the generalized Cauchy-Riemann operator D. To solve the equivalent fixed-point problem, we apply the Contraction Mapping Principle and Schauder type estimates.

2. A Cauchy-Pompeiu Integral Formula

As we know the function

$$E(x,\xi) = \frac{1}{\omega_{n+1}} \cdot \frac{\bar{x} - \bar{\xi}}{|x-\xi|^{n+1}},$$

where ω_{n+1} is the surface measure of the unit sphere in $\mathbb{R}^{n+1}$, is a fundamental solution for the equation $Du = 0$ with singularity at ξ (see [3]).

Now we recall the Green Integral Formula for the Cauchy-Riemann operator. Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with sufficiently smooth boundary. Let u, v Clifford-algebra-valued functions being continuously differentiable in $\bar{\Omega}$, then the Green Integral Formula for the Cauchy-Riemann operator has the form

$$\int_{\Omega} \Big(vD \cdot u + v \cdot Du \Big) dx = \int_{\partial\Omega} v \cdot d\sigma \cdot u, \tag{2.1}$$

where $d\sigma = \sum_{j=0}^{n} e_j N_j d\mu$ is the surface element with values in $\mathcal{A}_n$ and where in turn $(N_0, N_1, \ldots, N_n)$ is the outer unit normal and $d\mu$ is the measure element on the boundary. In order to apply that Green Integral Formula with $u = E(x,\xi)$, one has to omit the (isolated) singularity ξ. Define $\Omega_\epsilon = \Omega \setminus \bar{U}_{\epsilon(\xi)}$, where $U_\epsilon(\xi)$ is the ϵ-neighbourhood of ξ. Since $E(x,\xi)$ has a weak singularity at ξ and carrying out the limiting process $\epsilon \to 0$, we get the following Cauchy-Pompeiu Integral Formula:

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with sufficiently smooth boundary and suppose v is a continuously differentiable function in $\bar{\Omega}$ with values in the Clifford algebra $\mathcal{A}_n$. Then for points ξ in Ω, we have*

$$v(\xi) = \int_{\partial\Omega} v \cdot d\sigma \cdot E(x,\xi) - \int_{\Omega} vD \cdot E(x,\xi) \cdot dx. \tag{2.2}$$

Remark 2.2. In [8] there is a similar representation when the Green Integral Formula for the Cauchy-Riemann operator (2.1) is applied with $v = E(x,\xi)$ instead of $u = E(x,\xi)$.

2.1. Distributional solution for the inhomogeneous equation $Du = h$.

A distributional solution of inhomogeneous equation $Du = h$ is given by the following theorem whose proof follows easily taking into account (2.2) and using Fubini's Theorem for weakly singular integral (see [10]).

Theorem 2.3. *Let $E(x,\xi)$ a fundamental solution of $Du = 0$ with singularity in ξ and suppose that h is an integrable $\mathcal{A}_n$-valued function in the bounded domain Ω, then the function u defined by*

$$u(x) = \int_{\Omega} E(x,\xi) \cdot h(\xi) d\xi$$

is a distributional solution of inhomogeneous equation $Du = h$.

2.2. Monogenic functions in the distributional sense.

An integrable function u is a monogenic function in the distributional sense if it satisfies the following relation:

$$\int_\Omega \varphi D \cdot u dx = 0, \text{ for each test function } \varphi.$$

it can be proved that a monogenic function in the distributional sense is continuously differentiable and satisfies $Du = 0$ pointwise. We called this result Weyl Lemma for monogenic functions. Its proof follows in a form analogous to that of the complex case (see [10]).

3. Hölder spaces and Schauder estimates

Let Ω be a bounded domain in $\mathbb{R}^{n+1}$. We denote by $C^\lambda(\overline{\Omega})$ the set of real valued Hölder continuous functions in $\overline{\Omega}$ with λ as Hölder exponent. Through the functional

$$||f||_{C^\lambda(\overline{\Omega})} = \max\left(\sup_{\overline{\Omega}} |f|, \sup_{\zeta_1 \neq \zeta_2} \frac{|f(\zeta_1) - f(\zeta_2)|}{|\zeta_1 - \zeta_2|^\lambda}\right) \tag{3.1}$$

we define a norm called the *Hölder norm* and so $C^\lambda(\overline{\Omega})$ with the *Hölder norm* is a Banach space (see [10]). For $\mathcal{A}_n-$valued functions u we define the Hölder norm as

$$||u||_{C^\lambda(\overline{\Omega})} = \max_A \left\{||u_A||_{C^\lambda(\overline{\Omega})}\right\}.$$

Now we consider $C^{1,\lambda}(\overline{\Omega})$ as a Banach space where the norm for any real valued function $f \in C^{1,\lambda}(\overline{\Omega})$ is defined as follow

$$||f||_{C^{1,\lambda}(\overline{\Omega})} := \max\left(\sup_{\overline{\Omega}} |f|, \sup_{\overline{\Omega}} |\partial_i f|, \sup_{\zeta_1 \neq \zeta_2} \frac{|\partial_i f(\zeta_1) - \partial_i f(\zeta_2)|}{|\zeta_1 - \zeta_2|^\lambda}\right). \tag{3.2}$$

Then we define the norm $|| \cdot ||_{1,\lambda}$ de una función $u(x) = \sum_{A \in \Gamma_n} u_A(x) e_A$ with values in $\mathcal{A}_n$ as the maximun of the $|| \cdot ||_{1,\lambda}$-norms of its 2^n real components, i.e.,

$$||u||_{C^{1,\lambda}(\overline{\Omega})} = \max_A \left\{||u_A||_{C^{1,\lambda}(\overline{\Omega})}\right\}.$$

Schauder estimates play a very important role in the theory of the existence of solutions of linear and non-linear elliptic Partial Differential Equations. These estimates guarantee that the Hölder bound for the solutions of Partial Differential Equations is controlled, in general, by the Hölder norm of the boundary data (see [2]).

Schauder's Interior Estimates provide us the bounds for the up to second-order derivatives of the solution and their Hölder continuities in any compact subset of the domain, that is, for the solution of a boundary-value problem of the type

$$\begin{cases} \mathcal{L}u = F(\cdot, u, \partial_i u) \;\; in \;\; \Omega \\ \quad u = \phi \;\; in \;\; \partial\Omega \end{cases} \tag{3.3}$$

we have the following bound

$$||u||^*_{C^{2,\lambda}(\Omega)} \leq C\left(||F||_{C^{0,\lambda}(\Omega)} + ||u||_{C^0(\Omega)}\right). \tag{3.4}$$

The symbol * represents the weighted norm inside the domain that is at a positive distance from the boundary and the Constant C depends on other constants such as the exponent λ and the dimension of the space among others (see [4]).

In [2, 1] we found Schauder estimates for the Hölder continuity of the first derivative of the solution up to the boundary which are given by

$$||u||_{C^{1,\lambda}(\overline{\Omega})} \leq C\left(||F||_{C^{0,\lambda}(\Omega)} + ||u||_{C^{0}(\Omega)} + ||\varphi||_{C^{1,\lambda}(\partial\Omega)}\right) \tag{3.5}$$

and if the solution satisfies a certain maximum principle, the middle term can be dropped or it can be estimated by the Hölder norm of the boundary data.

4. Boundary value problems for monogenic functions

A monogenic function $u = \sum_A u_A e_A$ solves $Du = 0$. Its real valued components are solutions of $\Delta u = 0$. As in the plane case, we can not arbitrarily prescribe all components on $\partial\Omega$.

For example in case $n = 2$ ($\mathbb{R}^{n+1} = \mathbb{R}^3$): Consider a monogenic function $u = u_0 + u_1e_1 + u_2e_2 + u_{12}e_{12}$ given in the closure of Ω, where Ω is a cylindrical domain in x_0- direction

$$\Omega = \{(x_0, x_1, x_2) : \Psi_1(x_1, x_2) < x_0 < \Psi_2(x_1, x_2), (x_1, x_2) \in \Omega_0\},$$

where Ω_0 is a simply connected domain in the x_1, x_2- plane. Then u is completely determined if u_1, u_2 are arbitrarily given on the whole boundary, u_{12} is given on the basis of the cylindrical domain:

$$S_0 = \{(x_0, x_1, x_2) : x_0 = \Psi_1(x_1, x_2), (x_1, x_2) \in \overline{\Omega_0}\}$$

and $u_0(p_0) = c$, where p_0 is an arbitrarily point of Ω and c is a constant.

To show that, we use the system $Du = 0$ and the fact that the all components u_A of u satisfy the Laplace equation (see [7]).

For arbitrary n, one has to consider domains which can be decomposed into μ-dimensional fibres. These fibres are defined by so called distinguishing $(1+n-\mu)$-dimensional parts of the boundary (see [6]). Therefore, according with this decomposition, the Cauchy-Riemann system can be decomposed into μ-dimensional subsystems for particular components and these subsystems turn out to be completely integrable, so that the corresponding components can be calculated from their values in the distinguishing part of the boundary.

5. Reduction of boundary value problems to fixed-point problems

Now we consider a boundary value problem that can be solved for monogenic functions and define the operator

$$T[u](x) = u_h + u_c + \int_{\Omega} E(x,\xi)F(\xi, u(\xi), \partial_1 u(\xi), \ldots, \partial_n u(\xi))d\xi, \tag{5.1}$$

where u_h is a monogenic function solving the boundary value problem and u_c is a second monogenic function which compensates the boundary value of the integral in (5.1) to zero.

For example, in case $n = 2$, we can take u_h as the monogenic function solution of the boundary value problem given in section 4. On the other hand, we take u_c as the monogenic function which compensates the boundary values of $\int_\Omega E(x,\xi)F(\xi,u(\xi),\partial_1 u(\xi),\dots,\partial_n u(\xi))d\xi := I^T$, i.e.,

$$\begin{aligned} u_{c_i} &= -I_i^T \qquad \text{on } \partial\Omega, \quad i=1,2 \\ u_{c_{12}} &= -I_{12}^T \qquad \text{on } S_0 \\ u_{c_0} &= -I_0^T(p_0), \quad p_0 \in \Omega. \end{aligned}$$

The choice of the boundary values of u_h and u_c implies that $T[u](x)$ satisfies the given boundary conditions. Also, since $Du_h = 0$ and $Du_c = 0$, we get $DT[u](x) = F(x,u,\partial_1 u,\dots,\partial_n u)$.

Therefore we have proved the following Theorem:

Theorem 5.1. *A fixed point of the operator (5.1) is a solution of the boundary value problem for the equation* $Du = F(x,u(x),\partial_1 u(x),\dots,\partial_n u(x))$.

The existence and uniqueness of this problem can be showed using the Contraction mapping principle. By the Contraction principle the Lipschitz condition with respect to the functions u_A and their derivatives on F is necessary. Since the right hand side involves the first order derivatives of the desired function u, a convenient underlying function space is the space of the Hölder continuously differentiable functions. Also the monogenic functions u_h and u_c can be estimated by Schauder estimates, which give the estimates in terms of the boundary values in the Hölder spaces as we showed in section 3. Since there are derivatives $\partial_i u$ in the integrand of I^T, it es necessary to differentiate $T(u(x))$ with respect to the x_i, $i=0,1,\dots,n$. Therefore the following strongly singular integrals appear

$$\int_\Omega \Big(\partial_{x_i}(\frac{\bar{x}-\bar{\xi}}{|x-\xi|^{n+1}})\Big)F(\xi,u(\xi),\partial_1 u(\xi),\dots,\partial_n u(\xi))d\xi.$$

Substituting $F(\xi,u(\xi),\partial_1 u(\xi),\dots,\partial_n u(\xi))$ by $\mathcal{F}(\xi)$, this integral can be rewritten as

$$\int_\Omega \Big(\partial_{x_i}(\frac{\bar{x}-\bar{\xi}}{|x-\xi|^{n+1}})\Big)(\mathcal{F}(\xi)-\mathcal{F}(x))d\xi + \int_\Omega \partial_{x_i}(\frac{\bar{x}-\bar{\xi}}{|x-\xi|^{n+1}})d\xi\,\mathcal{F}(x). \tag{5.2}$$

Now assuming that the function u is Hölder continuously differentiable with the exponent $0 < \lambda < 1$ and taking into account the Lipschitz condition on $\mathcal{F}$, the first integral becomes weakly singular.

Next we consider the domain $\Omega_\epsilon = \Omega \setminus \bar{U}_{\epsilon(\xi)}$, where $U_\epsilon(\xi)$ is the ball $|x-\xi| \le \epsilon$, then the Gauss Integral Theorem leads to

$$\int_{\Omega_\epsilon} \partial_{x_i}(\frac{\bar{x}-\bar{\xi}}{|x-\xi|^{n+1}})d\xi = \int_{\partial\Omega} \frac{\bar{x}-\bar{\xi}}{|x-\xi|^{n+1}}N_j d\mu + \int_{|x-\xi|=\epsilon} \frac{\bar{x}-\bar{\xi}}{|x-\xi|^{n+1}}N_j d\mu,$$

where $(N_0,N_1,\dots,N_n)$ is the outer unit normal and $d\mu$ is the measure element on the boundary. Since $|\bar{x}-\bar{\xi}| = \epsilon$ and $d\mu = \epsilon^n d\mu_1$, where $d\mu_1$ is the surface element of the $(n+1)-$dimensional unit sphere, the second integral on the right hand side does not depend on ϵ and then its limit exists as ϵ tends to 0. Therefore, the second integral in (5.2) exists as a principal value and it can be represented by a Cauchy type integral over the boundary $\partial\Omega$.

Note that the solution can be obtained by construction, using the method of successive approximations, i.e., the solution is the limit of the sequence $\{u_n\}_{n=0}^{\infty}$ defined by:

$$u_{n+1}(x) = u_0(x) + \int_{\Omega} E(x,\xi)F(\xi, u_n(\xi), \partial_1 u_n(\xi), \dots, \partial_n u_n(\xi))d\xi, \qquad (5.3)$$

where $u_0(x) = u_h + u_c$.

We can solve boundary value problems in the framework of more general Clifford algebras called parameter-depending Clifford algebras (see [7, 8, 9]) or we can change the operator L for another different from the operator D. In both cases, we need a fundamental solution of equation $Lu = 0$ to define the operator (5.1). On the other hand, we must bear in mind that when estimating the corresponding integral operator, strongly singular integrals appear which can be difficult to handle.

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