Global and Stochastic Analysis
Vol. 8 No. 3 (December, 2021)
Special Issue: Modern Stochastic Models and Problems of Actuarial Mathematics

PROBABILISTIC MODELS FOR NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS FOR A SYSTEM OF ELLIPTIC EQUATIONS

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Abstract. The first boundary value problem (Dirichlet's problem) for a system of elliptic equations is considered. Under the assumption of the existence of a solution, a special system of integral equations is written that connects the values of the functions with their integrals over spheres and balls. Based on the obtained system of integral equations, a probabilistic representation of the solution of the system is obtained. A random process is constructed that is consistent with the probability representation. On the trajectories of a random process, an unbiased estimate of a solution with finite variance is constructed. For the constructed estimate algorithms, numerical experiments were carried out and their results are presented in the table. A similar system was considered in paper [5]. In contrast to the computational algorithm proposed in [5], in this paper, the equation number is modelled not equal probability, but in according to the constant or variable coefficients c_{ij} of the system under consideration. For the system equations of parabolic type were constructed such kind of stochastic algorithms in [7].

1. The Description of the Problem

Let be D a bounded domain in R^3 with a regular boundary Γ. Consider the Dirichlet's problem for the following system of elliptic equations:

$$\begin{cases} -\Delta u_1(x) + c_{11}u_1(x) = c_{12}u_2(x) + \ldots + c_{1n}u_n(x) + f_1(x), \\ -\Delta u_2(x) + c_{22}u_2(x) = c_{21}u_1(x) + c_{23}u_3(x) + \ldots + c_{2n}u_n(x) + f_2(x), \\ \vdots \\ -\Delta u_n(x) + c_{nn}u_n(x) = c_{n1}u_1(x) + \ldots + c_{(n-1)n}u_{n-1}(x) + f_n(x), \end{cases} \tag{1}$$

for $x \in D$ with boundary conditions:

$$u_i(x) = \varphi_i(x), \quad x \in \Gamma, \quad i = \overline{1,n}. \tag{2}$$

Here $c_{ii} > 0$, $i = \overline{1,n}$. Assume that the functions $f_i(x)$, $\varphi_i(x)$ and for the beginnig constant coefficients c_{ij}, $(i,j = \overline{1,n})$ such that there exists [3] the only continuous solution to the problem $u_i(x) \in C(\overline{D}) \cap C^2(\overline{D})$, $(i = \overline{1,n})$. We construct a statistical algorithm for solving problem (1)-(2) at some arbitrary point $x \in D$. To do this, we will use some well-known facts from the course of classical equations

2000 *Mathematics Subject Classification.* Primary 65C05; Secondary 82M31.
Key words and phrases. Statistical modeling, Monte Carlo method, unbiased estimates, random process, random walk trajectory, Markov chains.

of mathematical physics and obtain an integral representation of the solution to the problem.

2. Transformation of the system and obtaining a probabilistic representation

As is known [5], the following representation is valid for solving the equation $-\Delta u(x) + c\,u(x) = f(x)$:

$$u(x) = \frac{R\sqrt{c}}{4\pi R^2\,\mathrm{sh}\,(R\sqrt{c})}\int\limits_{S_R} u(y)dy + \int\limits_{K_R} \frac{\mathrm{sh}\,[(R-|x-y|)\sqrt{c}]}{4\pi|x-y|\mathrm{h}\,(R\sqrt{c})} f(y)\,dy. \tag{3}$$

Here $R = d(x) = \min\limits_{y\in\Gamma}|x-y|$ is the distance from the point to the boundary, K_R is the ball of radius R centered at the point x, S_R is the corresponding sphere.

The first integral in (3) is the integral over the surface of the sphere S_R. So since the surface area of the sphere is equal $4\pi R^2$, after the introduction of this expression under the sign of the first integral, it becomes three-dimensional, representing a uniformly distribution on the sphere. In view of the foregoing, we rewrite this expression as follows

$$u(x) = q\int\limits_{S_R} u(y)d\omega + (1-q)\int\limits_{K_R} p(x,y)\tfrac{f(y)}{c}dy, \tag{4}$$

here we used the following notation: $q = \frac{R\sqrt{c}}{\mathrm{sh}(R\sqrt{c})}$, ω is the uniform distribution on S_R, $p(x,y)$ - transition density from x in y $(x,y\in K_R)$,

$$p(x,y) = \frac{\mathrm{sh}\,[(R-|x-y|)\,\sqrt{c}]}{4\pi|x-y|\,(\mathrm{sh}\,(R\sqrt{c}) - R\sqrt{c})}.$$

Applying relation (4) to each of the equations, we obtain the following system of integral equations:

$$\begin{cases} u_1(x) = q_1\int\limits_{S_R} u_1(y)\,d\omega + \\ \quad +(1-q_1)\int\limits_{K_R} p_1(x,y)\frac{1}{c_{11}}\left(c_{12}u_2(y) + \cdots + c_{1n}u_n(y) + f_1(y)\right)\,dy \\ \vdots \\ u_k(x) = q_k\int\limits_{S_R} u_k(y)\,d\omega + \\ \quad +(1-q_k)\int\limits_{K_R} p_k(x,y)\frac{1}{c_{kk}}\left(\sum\limits_{i=\overline{1,n};\ i\neq k} c_{ki}u_i(y) + f_k(y)\right) \\ \vdots \\ u_n(x) = q_n\int\limits_{S_R} u_n(y)\,d\omega + \\ \quad +(1-q_n)\int\limits_{K_R} p_n(x,y)\frac{1}{c_{nn}}\left(c_{n1}u_1(y) + \cdots + c_{n(n-1)}u_{n-1}(y) + f_n(y)\right)\,dy. \end{cases} \tag{5}$$

Here

$$q_k = \frac{R\sqrt{c_{kk}}}{\mathrm{sh}\left(R\sqrt{c_{kk}}\right)}, \quad p_k(x,y) = \frac{\mathrm{sh}\left[(R-|x-y|)\,\sqrt{c_{kk}}\right]}{4\pi|x-y|\left(\mathrm{sh}\left(R\sqrt{c_{kk}}\right) - R\sqrt{c_{kk}}\right)}.$$

When constructing estimates of the Monte Carlo method for solving systems of integral equations:

$$u_i(x) = \sum_{j=1}^{k} \int_{\Omega} k_{ij}(x,y)u_j(y)dy + h_i(x)$$

or in vector form $U = \mathrm{K}U + H$, it is assumed that the spectral radius $\rho(\mathrm{K}) < 1$ (see [6]).

Under this condition, the solution of the system of integral equations can be represented in the form of a convergent Neumann series $U = \sum\limits_{i}^{\infty} \mathrm{K}^i F$. Let K_{ij} an integral operator with a kernel $k_{ij}(x,y)$. The following theorem belongs to G. Mikhailov.

Theorem 2.1. [6] *If K_{ij} are bounded operators, then*

$$\rho(\mathrm{K}) \le \max_i \rho(K_{ii}) = \rho_0.$$

If $k_{ij}(x,t;y,\tau) \ge 0$, then $\rho(\mathrm{K}) = \rho_0$.

According to this theorem 2.1, it suffices to show that $\rho_0 = \max\limits_i \rho(K_{ii}) < 1$. Since the spectral radius $\rho(\mathrm{K})$ is determined by the relation $\rho(\mathrm{K}) = \lim \|\mathrm{K}^n\|^{1/n} = \inf \|\mathrm{K}^n\|^{1/n}$, then condition $|\mathrm{K}| < 1$ is equivalent to the inequality $\rho(\mathrm{K}) < 1$. From representation (5) it follows that

$$K_{ii}u = \frac{R\sqrt{c_{ii}}}{4\pi R^2 \mathrm{sh}\left(R\sqrt{c_{ii}}\right)} \int_{S_R} u(y)dy.$$

From following representation

$$\|K_{ii}\| = \sup_{x\in\overline{D}} \int_{S_R} \left| \frac{R\sqrt{c_{ii}}}{4\pi R^2 \ \mathrm{sh}\left(R\sqrt{c_{ii}}\right)} \right| \ dy = \sup_{x\in\overline{D}} q_i = \sup_{x\in\overline{D}} \frac{R(x)\sqrt{c_{ii}}}{\mathrm{sh}R\left((x)\sqrt{c_{ii}}\right)} < 1,$$

we show that $\max\limits_i \|K_{ii}\| < 1$. Further, in accordance with representation (5), a random process in D is constructed and a simulation algorithm is proposed.

3. The construction of a random process, consistent with the probabilistic representation

Define in D a random process as follows. We define $A(x)$ the transition probability matrix:

$$A(x) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(n+1)} \\ \alpha_{21} & \alpha_{21} & \cdots & \alpha_{2(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{n(n+1)} \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where

$$\alpha_{ii} = q_i, \quad M_i = \sum_{j=\overline{1,n};j\neq i} |c_{ij}|, \quad \alpha_{i(n+1)} = (1-q_i)/n, \ \ (i = \overline{1,n}),$$

$$\alpha_{ij} = (1-q_i)\frac{(n-1)}{n}\frac{|c_{ij}|}{M_i}, \quad (i,j=\overline{1,n};\ i\neq j),$$

and transition density matrix $P(x,y)$:

$$P(x,y) = \begin{pmatrix} p_{11}(x,y) & p_{12}(x,y) & \cdots & p_{1(n+1)}(x,y) \\ p_{21}(x,y) & p_{22}(x,y) & \cdots & p_{2(n+1)}(x,y) \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1}(x,y) & p_{n2}(x,y) & \cdots & p_{n(n+1)}(x,y) \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $p_{ii}(x,y) = I_{S_R}(y)\frac{1}{4\pi R^2}$, $(i=\overline{1,n})$ is the transition density corresponding to a uniform distribution on the sphere $S_R(x)$, $p_{ij}(x,y) = I_{K_R}(y)p_i(x,y)$, i.e.

$$p_{ij}(x,y) = I_{K_R}(y)\frac{\operatorname{sh}\left[(R-|x-y|)\sqrt{c_{ii}}\right]}{4\pi|x-y|\left(\operatorname{sh}\left(R\sqrt{c_{ii}}\right)-R\sqrt{c_{ii}}\right)}, \quad (j=\overline{1,n};\ j\neq i).$$

Next, we fix the number of the equation $i_0 \in \{1,2,\ldots,n\}$ and the starting point $x_0 = x$. Let at the initial moment there is a particle at a point $x_0 = x$. In one step, a transition is performed $i_k \to i_{k+1}$ in accordance with the matrix of transition probabilities $A(x_k)$ and a transition $x_k \to x_{k+1}$ in accordance with the $P(x_k,y)$ matrix of transition probability densities, i.e. with probability $\alpha_{i_k,i_{k+1}}(x_k)$, the particle also passes from a point x_k to a point x_{k+1} having a distribution density $p_{i_k i_{k+1}}(x_k,y)$. The probability of a process termination at a point x_n is:

$$g(x_n) = \begin{cases} 1, & x_n \in \Gamma; \\ \alpha_{i_{n-1}n}(x_{n-1}), & x_n \in D. \end{cases}$$

A random process defined in this way is a modified spherical process, i.e. the next point is chosen with probability q_i uniformly distributed on the sphere S_R with the maximum radius contained in D, or distributed with probability $1-q_i$ in a ball K_R with density $p_i(x,y)$. The trajectories of the spherical process with probability 1 converges to the boundary of the domain (see, for example, [2]).

Next, we describe a method for modelling a transition with a transition density $p_i(x,y)$. If the distribution density of the point x_{n+1} is equal $p_i(x_n,y)$ to fixed x_n, then for modelling it is advisable to use the "acceptance-rejection" method:
A) Modelled random variables α_0, α_1, α_2 uniformly distributed over in $(0;1)$; Then random variable $\xi = -\ln(\alpha_1\alpha_2)/\sqrt{c_{ii}}$ is determined;
B) If $\xi > R$, then the A) is repeated, otherwise, a random variable $\zeta = \alpha_0\,\xi\exp(-\sqrt{c_{ii}}\xi)$ is selected;
C) If $\zeta \geq \xi\operatorname{sh}\left[(R-\xi)\sqrt{c_{ii}}\right]/\operatorname{sh}\left(R\sqrt{c_{ii}}\right)$, then go to A, otherwise $x_{n+1} = \xi$.
An estimation of the problem solution is constructed on the trajectories of the random process described below.

4. Construction of unbiased and ε - biased estimates of the solution.

Let $\{(i_k,x_k)\}_k$ be the trajectory of a random process and

$$\Theta_0 = 1, \quad \Theta_n = \Theta_{n-1}\times V_{i_{n-1}i_n}, \tag{6}$$

where V_{ij} are defined as follows:

$$V_{ii} = 1, \quad V_{ij} = \frac{nM_i \operatorname{sgn}(c_{ij})}{(n-1)c_{ii}}, \quad V_{i(n+1)} = \frac{n}{c_{ii}}, \quad (i, j = \overline{1,n};\ \ i \neq j).$$

Let's define a sequence of random variables on the trajectory of a random process $\{\eta_n(i_0, x_0)\}_{n=0}^{\infty}$:

$$\eta_n(i_0, x_0) = \Theta_n \times F(x_n) = \Theta_n \times \begin{cases} u_j(x_n),\ i_n = j,\ j \neq n+1; \\ f_{i_{n-1}}(x_n),\ i_n = n+1. \end{cases} \tag{7}$$

If there was a break at a time n, then we put

$$\eta_{n+k}(i_0, x_0) = \eta_n(i_0, x_0), \quad (i_{n+k}, x_{n+k}) = (i_n, x_n),\ k = 1, 2, \ldots.$$

Let $\Re_n$ – σ be an algebra generated by random variables $\{\omega_k\}_{k=0}^{n-1}$, $\{\alpha_{0k}\}_{k=0}^{n-1}$, $\{\alpha_{1k}\}_{k=0}^{n-1}$, $\{\alpha_{2k}\}_{k=0}^{n-1}$. The following theorem holds

Theorem 4.1. *Sequence $\{\eta_n(i_0, x_0)\}_{n=1}^{\infty}$ forms a martingale relative to $\Re_n$. If $M_i < (n-1)c_{ii}/n$ and $\max\limits_{x \in D} |f_i(x)| \leq c_0$, ($c_0 = const$, $i = \overline{1,n}$), then $\{\eta_n(i_0)\}$ is a uniformly integrable martingale.*

Proof. By definition, $\{\eta_n(i_0, x_0)\}$ $\Re_n$ is measurable.

$$E(\eta_{n+1}(i_0, x_0)/\Re_n) = E\left(\Theta_{n+1} \times F(x_{n+1})/\Re_n\right) = E\left(\Theta_n V_{i_n i_{n+1}} F(x_{n+1})/\Re_n\right) =$$

$$\Theta_n E\left(V_{i_n i_{n+1}} \times F(x_{n+1})\right) = \Theta_n \left(\sum_{j=1}^{n} \alpha_{i_n j} \int_{\overline{K_R}} V_{i_n j} p_{i_n j}(x_n, y) u_j(y) dy + \right.$$

$$\left. + \alpha_{i_n(n+1)} \int_{\overline{K_R}} V_{i_n(n+1)} p_{i_n(n+1)}(x_n, y) f_{i_n}(y) dy \right) = \eta_n(i_0, x_0).$$

And so the sequence $\{\eta_n(i_0, x_0)\}$ is a martingale. To prove the uniform integrability of $\eta_n(i_0, x_0)$, it is sufficient to show that $|\eta_n(i_0, x_0)| < \infty$. Since $u_i(x) \in C(\overline{D}) \cap C^2(\overline{D})$ and $\overline{D}$ bounded area, then $|u_i(x)| \leq const$ for $x \in \overline{D}$, $i = \overline{1,n}$. By virtue of the conditions of the theorem $|\Theta_n| \leq 1$, and hence $|\eta_n(i_0, x_0)| \leq const$. So the sequence $\{\eta_n(i_0, x_0)\}$ is uniformly integrable. The theorem is proved. □

Consider a slightly modified process with trajectories of shorter length and estimates on it with an arbitrarily small offset. Take ε - small enough and consider the inner ε- neighborhood of the boundary Γ_ε. Let N_1 is the moment of the rupture process inside the region and N_ε is the moment of first contact with Γ_ε. $N = \min\{N_1, N_\varepsilon\}$- the time the process stops. Then the probability of breaking the trajectory at the point will be equal to:

$$g(x_n) = \begin{cases} 1, & if \quad x_n \in \Gamma_\varepsilon; \\ \alpha_{i_{n-1}n}(x_{n-1}), & if \quad x_n \in \overline{D} \backslash \Gamma_\varepsilon. \end{cases}$$

As is known, for spherical processes, the average number of EN transitions in the sphere walk chain does not exceed $\nu(\varepsilon)$ [1]. For a wide class of boundaries Γ, we obtain a logarithmic estimate $EN \leq C|\ln(\varepsilon)|$, which is known to hold for convex regions.

Theorem 4.2. *Let $\eta_n(i_0)$ is a uniformly integrable martingale. Then $\eta_N(i_0, x_0)$ is an unbiased estimate for $u_{i_0}(x_0)$ with finite variance.*

Proof. Since $\eta_n(i_0, x_0)$ is a uniformly integrable martingale and N is a Markov moment, then according to the Dub free-choice transformation theorem [8] for the martingale $\eta_n(i_0, x_0)$ get $E\ \eta_N(i_0, x_0) = E\ \eta_1(i_0, x_0)$. From the definition of $\eta_1(i_0, x_0)$, it follows that $E\eta_1(i_0, x_0) = u_{i_0}(x_0)$. By virtue of the conditions of theorem 4.1. $E\left(\eta_N(i_0, x_0)\right)^2 < \infty$ is valid, and hence its variance is finite. The theorem is proved. □

Next, from $\eta_N(i_0, x_0)$, we construct a standard method of biased, but practically implemented on a computer estimate $\eta_N^*(i_0, x_0)$. Let $\Psi_i(x) = \varphi_i(x)$ for $x \in \Gamma$, x^* is the nearest point of the Γ boundary to x. η_N^* is obtained by replacing $u_i(x_N)$ in $\eta_N(i_0, x_0)$ with $\Psi_i(x_N^*)$.

5. Estimates of the solution of the Dirichlet problem for a system of elliptic equations with variable coefficients

The constructed model can be generalized to the case of variable coefficients $c_{ij} = c_{ij}(x)$ and get similar statements. Let in equation (1) the coefficients c_{ij} $(i, j = \overline{1,n};\ \ i \neq j)$ depend on x, i.e. $c_{ij} = c_{ij}(x)$ and we consider the Dirichlet problem for the following system of elliptic equations:

$$\begin{cases} -\Delta u_1(x) + c_{11}u_1(x) = c_{12}(x)u_2(x) + \ldots + c_{1n}(x)u_n(x) + f_1(x) \\ -\Delta u_2(x) + c_{22}u_2(x) = c_{21}(x)u_1(x) + c_{23}(x)u_3(x) + \ldots + c_{2n}(x)u_n(x) + f_2(x) \\ \vdots \\ -\Delta u_n(x) + c_{nn}u_n(x) = c_{n1}(x)u_1(x) + \ldots + c_{(n-1)n}(x)u_{n-1}(x) + f_n(x) \end{cases}$$

for $x \in D$.

By analogy with (5), the probabilistic representation of the solution in this case has the form $(k = \overline{1,n})$:

$$\begin{aligned} u_k(x) = q_k \textstyle\int_{S_R} u_k(y)d\omega + \\ +(1-q_k)\textstyle\int_{K_R} p_k(x,y)\frac{1}{c_{kk}}\left(\sum\limits_{i=\overline{1,n},i\neq k} c_{ki}(y)u_i(y) + f_k(y)\right)dy. \end{aligned} \tag{8}$$

Let us $c_{ij}^* \geq \max\limits_{x\in D}|c_{ij}(x)|$. In estimates $\eta_N(i_0, x_0)$, $\eta_N^*(i_0, x_0)$ coefficients c_{ij} is replaced with $c_{ij}(x)$.

In this case, the elements of the transition probability matrix $A(x)$ are defined as follows:

$$\alpha_{ii} = q_i, \quad M_i^* = \sum_{j=\overline{1,n};j\neq i} c_{ij}^*, \quad \alpha_{i(n+1)} = \frac{1-q_i}{n}, (i = \overline{1,n}),$$

$$\alpha_{ij} = (1-q_i)\frac{n-1}{n}\frac{c_{ij}^*}{M_i^*}, \quad (i,, j = \overline{1,n};\ i \neq j).$$

Sequences of random variables $\{\Theta_n\}$ and $\{\eta_n(i_0, x_0)\}$ are defined by formulas (6)-(7) with the exception of:

$$V_{ij} = \frac{nM_i^* c_{ij}}{(n-1)c_{ii}c_{ij}^*}, \quad (i, j = \overline{1,n};\ \ i \neq j).$$

The following theorem holds

Theorem 5.1. A. *The sequence* $\{\eta_n(i_0, x_0)\}_{n=1}^{\infty}$ *forms a martingale relative to* $\Re_n$. *If* $M_i^* < (n-1)c_{ii}/n$ *and* $\max\limits_{x\in D}|f_i(x)| \leq c_0$, $(c_0 = const,\ i = \overline{1,n})$, *then* $\{\eta_n(i_0, x_0)\}$ - *evenly integrated martingale.*
B. *If* N *is the moment when the process stops, then* $\eta_N(i_0, x_0)$ *is an unbiased estimate for* $u_{i_0}(x_0)$ *with a finite variance.*

Proof. The proof is similar to the proof of theorems 4.1 and 4.2. By definition, $\{\eta_n(i_0, x_0)\}$ $\Re_n$ is measurable. $E(\eta_{n+1}(i_0, x_0)/\Re_n) = \eta_n(i_0, x_0)$. Further, if the conditions of the theorem are met, it is easy to show $|\Theta_n| \leq 1$, and hence $|\eta_n(i_0, x_0)| \leq const$. This implies uniform integrability of the sequence $\{\eta_n(i_0, x_0)\}$. Since $\eta_n(i_0, x_0)$ is a uniformly integrable martingale and N is a Markov moment, then according to the Dub free choice transformation theorem [8] for the martingale $\{\eta_n(i_0, x_0)\}$ we get $E\eta_N(i_0, x_0) = E\eta_1(i_0, x_0)$. From the definition of $\eta_1(i_0, x_0)$, it follows that $E\eta_1(i_0, x_0) = u_{i_0}(x_0)$. By virtue of the conditions of the theorem $E(\eta_N(i_0, x_0))^2 < \infty$ and hence its variance is finite. The theorem is proved. □

Next, $\eta_N(i_0, x_0)$ is used to construct a standard biased but practically feasible estimate of $\eta_N^*(i_0, x_0)$.

6. Computational experiment

The results of numerical experiments based on the proposed estimates of the solution of the initial boundary value problem for system (1) for the case of constant coefficients are presented below.

For each considered problem,the following cases were considered: the domain $\Omega = D \times [0, T]$, where D is a ball of radius R centered at the origin and D is a unit cube $D = \{(x_1, x_2, x_3) : 0 \leq x_1, x_2, x_3 \leq 1\}$. The results of the computational experiment for the case of a system of three equations are shown in the table 1.

Conclusion. The results of a computational experiment show that the algorithm can be used to build estimates that are effectively implemented on a computer. In the course of calculations, a 99.7% confidence interval is estimated, i.e. with a probability approximately equal to 0.997, the exact value of the solution will be in the range $(\bar{\xi} - 3\sigma, \bar{\xi} + 3\sigma)$. Since the exact solutions are known in the selected examples, we can make sure that all estimates fall within the confidence interval (see table 1).

Explanation of table 1.
D– the area where the problem is considered;
i_0 is the number of equations;
x_0– the point where the problem solution is located;
$U_e x(i_0)$– exact solution at x_0: $U_{ex}(i_0) = u_{i_0}(x_0)$;

TABLE 1. The results of the computational experiment.

D	i0	x0	Uex(i0)	ES	err	3sig
ball	1	(0,25;0,25;0,25)	0,20449	0,20608	0,00159	0,01553
ball	2	(0,25;0,25;0,25)	0,00781	0,01455	0,00674	0,01337
ball	3	(0,25;0,25;0,25)	0,29535	0,29939	0,00404	0,01062
ball	4	(0,25;0,25;0,25)	0,63510	0,64508	0,00998	0,02511
cube	1	(0,3;0,55;0,75)	0,29987	0,29912	0,00075	0,01018
cube	2	(0,3;0,55;0,75)	0,06172	0,06160	0,00012	0,01369
cube	3	(0,3;0,55;0,75)	0,26508	0,26666	0,00158	0,01343
cube	4	(0,3;0,55;0,75)	0,14859	0,15038	0,01786	0,03649
ball	1	(-0,45;-0,36;0,41)	-0,11683	-0,09660	0,02022	0,03842
ball	2	(-0,45;-0,36;0,41)	0,03319	0,04156	0,00837	0,03291
ball	3	(-0,45;-0,36;0,41)	0,28072	0,27690	0,00382	0,03127
shar	4	(-0,45;-0,36;0,41)	0,20110	0,20421	0,00312	0,03737
cube	1	(0,5;0,6;0,1)	0,27961	0,27977	0,00016	0,00776
cube	2	(0,5;0,6;0,1)	0,01500	0,01634	0,00134	0,00802
cube	3	(0,5;0,6;0,1)	0,29113	0,29240	0,00126	0,00784
cube	4	(0,5;0,6;0,1)	0,99604	0,98361	0,01242	0,02215

ES– Monte Carlo estimation, $ES = \bar{\xi} = \frac{1}{N_t}(\xi_1 + \xi_2 + \cdots + \xi_{N_t})$, where $\xi = \eta_{N^*}(i_0)$, ξ_i - independent implementations of the random evaluation of the solution ξ;
err– difference between the exact solution and the estimate $err = |u_{i_0}(x_0) - ES|$;
sig– statistical estimate of the value $\sqrt{Var(\xi)/N_t}$, where $Var(\xi)$ is the variance of the problem solution estimate;
$3sig$– confidence interval for estimation.

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